Exercise sheet 1

Notations. $|+\rangle := \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$.

Exercise 1. For each of the states $|\psi_{AB}\rangle$ shared between Alice and Bob, write the reduced density matrices $\rho_A, \rho_B$ of Alice and Bob.

1. $|\psi_{AB}\rangle = \sqrt{\frac{1}{3}} |00\rangle + \sqrt{\frac{2}{3}} |11\rangle$.
2. $|\psi_{AB}\rangle = \frac{1}{\sqrt{3}} (|01\rangle + |12\rangle + |22\rangle)$.
3. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |1+\rangle)$.

Solution:

1. $\rho_A = \rho_B = \frac{1}{3} |0\rangle \langle 0| + \frac{2}{3} |1\rangle \langle 1|$. 
2. $\rho_A = \frac{1}{3} |0\rangle \langle 0| + \frac{2}{3} |12\rangle \langle 12|$ with $|12\rangle = \frac{1}{\sqrt{2}} |1\rangle + |2\rangle$. $\rho_B = \frac{1}{3} |1\rangle \langle 1| + \frac{2}{3} |2\rangle \langle 2|$. 
3. $\rho_B = \frac{1}{3} |0\rangle \langle 0| + \frac{1}{3} |+\rangle \langle +|$. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$ so $\rho_A = \frac{3}{4} |\phi_1\rangle \langle \phi_1| + \frac{1}{4} |1\rangle \langle 1|$ with $|\phi_1\rangle = \sqrt{\frac{3}{2}} |0\rangle + \sqrt{\frac{2}{3}} |1\rangle$.

Exercise 2. We have 2 states $\rho_0, \rho_1$. Bob is given $\rho_b$ for a randomly chosen $b \in \{0, 1\}$ and his goal is to guess $b$. Give the optimal measurement as well as his probability of success for the following states.

1. $\rho_0 = |0\rangle \langle 0|$, $\rho_1 = \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|)$.
2. $\rho_0 = |0\rangle \langle 0|$, $\rho_1 = |+\rangle \langle +|$. 
3. $\rho_0 = \frac{2}{3} |0\rangle \langle 0| + \frac{1}{3} |1\rangle \langle 1|$, $\rho_1 = |+\rangle \langle +|$. In this case, give only the success probability and not the optimal measurement.

Solution:

1. $\Delta(\rho_0, \rho_1) = \frac{1}{2}$. The optimal measurement is the one in the computational basis and succeeds wp. $\frac{3}{4}$.
2. As seen in class, the optimal strategy is to measure in the $\{|v\rangle, |v^\perp\rangle\}$ basis with $|v\rangle = \cos(\pi/8) |0\rangle - \sin(\pi/8) |1\rangle$ and $|v^\perp\rangle = \sin(\pi/8) |0\rangle + \cos(\pi/8) |1\rangle$. This strategy succeeds wp. $\cos^2(\pi/8)$.
3. $M = \rho_0 - \rho_1 = \begin{pmatrix} 1/6 & -1/2 \\ -1/2 & -1/6 \end{pmatrix}$. Moreover

$$(\rho_0 - \rho_1)(\rho_0 - \rho_1)^\dagger = \begin{pmatrix} 10/36 & 0 \\ 0 & 10/36 \end{pmatrix}$$

so $\Delta(\rho_0, \rho_1) = \sqrt{10/6}$ and the success probability is $1/2 + \sqrt{10/12}$.

Analysis around the fingerprint state

Consider the state $|\psi_{x_1 x_2 x_3 x_4}\rangle = \frac{1}{2} ((-1)^{x_1} |00\rangle + (-1)^{x_2} |01\rangle + (-1)^{x_3} |10\rangle + (-1)^{x_4} |11\rangle)$ that depends on 4 bits $x_1, x_2, x_3, x_4$.

Exercise 3. We assume $x_2, x_3, x_4 = 0$. Consider the states $|\phi_b\rangle = |\psi_{0000}\rangle$. This is 2-qubit state on some registers $AB$ where $A$ is the register corresponding to the first qubit and $B$ is the register corresponding to the second qubit.

1. Give an expression for $|\phi_0\rangle$ and $|\phi_1\rangle$.

2. Compute $\rho^A_0 = tr_B |\phi_b\rangle\langle \phi_b|$ for both $b = 0$ and $b = 1$.

3. Compute $\Delta(\rho^A_0, \rho^A_1)$. Give a measurement that, given $\rho^A_b$, outputs $b$ with probability $P_b$ with $\frac{1}{2} (P_0 + P_1) = \frac{3}{4}$ and argue that this measurement is optimal.

Solution:

1. $\rho^A_0 = \langle + | + \rangle, \rho^A_1 = \frac{1}{2} \langle - | - \rangle + \frac{1}{2} \langle + | + \rangle$.

2. $\Delta(\rho^A_0, \rho^A_1) = \frac{1}{2}$ so there is a measurement that succeeds in distinguishing $\rho^A_0$ from $\rho^A_1$ wp. $\frac{1}{2} + \frac{\Delta(\rho^A_0, \rho^A_1)}{2} = \frac{3}{4}$. This can be done by measuring in the $\{|+, -\rangle\}$ basis.

Exercise 4. We still assume $x_2, x_3, x_4 = 0$. Our goal is to analyze what is the probability of recovering $x_1$ assuming we now have access to the full state $|\phi_{x_1}\rangle$ (so Alice and Bob are together here).

1. Compute $\langle \phi_0 | \phi_1 \rangle$. Argue that there is a measurement that, given $|\phi_b\rangle$ for $b \in \{0, 1\}$, outputs $b$ with probability $P_b$ with $\frac{1}{2} (P_0 + P_1) = \frac{1}{2} + \sqrt{7}/4$. 


2. Find the measurement that distinguishes $|\phi_0\rangle$ and $|\phi_1\rangle$ wp. $\frac{1}{2} + \frac{\sqrt{3}}{4}$. One can use without proof $\frac{1}{2} + \frac{\sqrt{3}}{4} = \cos^2(\pi/12)$.

Solution:

1. 
   
   $|\phi_0\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) = |+\rangle|+\rangle$
   
   $|\phi_1\rangle = \frac{1}{2} (-|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (-|0\rangle + |+\rangle |1\rangle)$

2. $\langle \phi_0|\phi_1\rangle = \frac{1}{2}$. For pure states, we have $\Delta(|\phi_0\rangle\langle \phi_0|, |\phi_1\rangle\langle \phi_1|) = \sqrt{1 - |\langle \phi_0|\phi_1\rangle|^2} = \frac{\sqrt{3}}{2}$. So there is a measurement that succeeds in guessing $b$ wp. $\frac{1}{2} + \frac{\sqrt{3}}{4}$.

3. $|\phi_1\rangle = \cos(\pi/3) |\phi_0\rangle + \sin(\pi/3) |\phi_0^\perp\rangle$ for some state $|\phi_0^\perp\rangle$ orthogonal to $|\phi_0\rangle$. We define

   $|\zeta_0\rangle = \cos(\pi/12) |\phi_0\rangle - \sin(\pi/12) |\phi_0^\perp\rangle$
   
   $|\zeta_1\rangle = \sin(\pi/12) |\phi_0\rangle + \cos(\pi/12) |\phi_0^\perp\rangle$

We measure in the basis $\{|\zeta_0\rangle, |\zeta_1\rangle\}$. One can check that

   $|\langle \zeta_0|\phi_0\rangle|^2 = \cos^2(\pi/12)$
   
   $|\langle \zeta_1|\phi_1\rangle|^2 = (\cos(\pi/3) \sin(\pi/12) + \sin(\pi/3) \cos(\pi/12))^2 = \sin^2(5\pi/12) = \cos^2(\pi/12)$

which gives the result.

\[ \square \]

Exercise 5. We now don’t have $x_2, x_3, x_4 = 0$ anymore. Assume we are in one of the two following cases

1. $x_1 = x_2 = x_3 = x_4$.

2. $x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0$, but the 4 bits are not all equal.

Give a measurement on $|\psi_{x_1 x_2 x_3 x_4}\rangle$ that determines with certainty in which case we are.
Exercise 6 (Unambiguous state discrimination). Assume we have 2 qubits $|\phi_0\rangle = |0\rangle$ and $|\phi_1\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ with $\theta \in [0, \frac{\pi}{2}]$. Suppose Bob is given $|\phi_b\rangle$ for a random unknown $b \in \{0, 1\}$ and his goal is to guess $b$. We want a measurement that maybe succeeds with a smaller probability than the one seed in class but is always correct when it succeeds. More precisely, we want a measurement that has up to 3 outcomes: “0”, “1” and “2” st. the measurement always succeeds when measuring “0” or “1”. (the “2” outcome corresponds to unknown).

Let $|f_i\rangle = \sin(\theta)|0\rangle - \cos(\theta)|1\rangle$. We consider the 3 outcome POVM $F = \{F_0, F_1, F_2\}$ with $F_i = M_i M_i^\dagger$. We take $F_0 = \frac{1}{1 + \cos(\theta)}|f_1\rangle \langle f_1|$, $F_1 = \frac{1}{1 + \cos(\theta)}|1\rangle \langle 1|$, $F_2 = (I - F_0 - F_1)$.

1. Let $|w\rangle = -\sin(\theta/2)|0\rangle + \cos(\theta/2)|1\rangle$ and $|w^\perp\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$. Show that

$$\frac{1}{2} (|f_1\rangle \langle f_1| + |1\rangle \langle 1|) = \cos^2(\theta/2)|w\rangle \langle w| + \sin^2(\theta/2)|w^\perp\rangle \langle w^\perp|.$$

2. Show that $F_2 = (1 - \tan^2(\theta/2))|w^\perp\rangle \langle w^\perp|$ and that $(1 - \tan^2(\theta/2)) \geq 0$. From there, we easily have that $F_0, F_1, F_2$ are positive semi-definite and that $\{F_i\}$ is a valid POVM.

3. Show that this POVM satisfies our requirements. What is the probability of correctly guessing $b$ here? Compare with the optimal guessing probability seen in class.

Solution:

1. We have

$$\frac{1}{2} (|f_1\rangle \langle f_1| + |1\rangle \langle 1|) = \frac{1}{2} \begin{pmatrix} \sin^2(\theta) & -\sin(\theta) \cos(\theta) \\ -\sin(\theta) \cos(\theta) & 1 + \cos^2(\theta) \end{pmatrix}$$

Let $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$. We write

$$c^2|w\rangle \langle w| + s^2 |w^\perp\rangle \langle w^\perp| = \begin{pmatrix} c^2 s^2 & -c^3 s \\ -c^3 s & c^4 \end{pmatrix} + \begin{pmatrix} s^2 c^2 & s^3 c \\ s^3 c & s^4 \end{pmatrix}$$
Then we use trigonometric equalities
\[
\sin(\theta) = 2sc \\
\cos(\theta) = 2c^2 - 1 = c^2 - s^2 = 1 - 2s^2
\]
so we compare the 4 elements of each matrix.
\[
\frac{1}{2} \sin^2(\theta) = 2s^2c^2 \quad \text{(top left)} \\
-\frac{1}{2}(\sin(\theta) \cos(\theta)) = sc(-c^2 + s^2) \quad \text{(the two non-diagonal)} \\
\frac{1}{2}(1 + \cos^2(\theta)) = \frac{1}{2} (1 + 4c^4 - 4c^2 + 1) = \frac{1}{2} (2c^4 + 2(1 - c^2)^2) = c^4 + s^4 \quad \text{(bottom right)}
\]

2. Using the previous question, we can rewrite
\[
F_2 = I - \frac{2}{1 + \cos(\theta)} \left( \cos^2(\theta/2) |w\rangle \langle w| + \sin^2(\theta/2) |w^\perp\rangle \langle w^\perp| \right) \\
\left( 1 - \frac{2 \cos^2(\theta/2)}{1 + \cos(\theta)} \right) |w\rangle \langle w| + 1 - \frac{2 \sin^2(\theta/2)}{1 + \cos(\theta)} |w^\perp\rangle \langle w^\perp| \\
= (1 - \tan^2(\theta/2)) |w^\perp\rangle \langle w^\perp|
\]
We can conclude by noticing that $\theta/2 \in [0, \pi/4]$ hence $(1 - \tan^2(\theta/2)) \geq 0$.

3. We have
\[
\text{tr}(|\phi_0\rangle \langle \phi_0| F_0) = \frac{\sin^2(\theta)}{1 + \cos(\theta)} = 1 - \cos(\theta) \\
\text{tr}(|\phi_1\rangle \langle \phi_1| F_1) = \frac{\sin^2(\theta)}{1 + \cos(\theta)} = 1 - \cos(\theta)
\]
so $p = 1 - \cos(\theta)$ is the success probability. In the optimal setting, we have that the probability $p^*$ of guessing $b$ is
\[
p^* = \frac{1}{2} + \frac{\Delta(|\phi_0\rangle \langle \phi_0|, |\phi_1\rangle \langle \phi_1|)}{2} + \frac{1}{2} + \sqrt{\frac{1 - |\langle \phi_0 \mid \phi_1 \rangle|^2}{2}} = \frac{1}{2} + \frac{\sin(\theta)}{2}.
\]
We plot below $p^*$ (in blue) and $p$ (in green) and see the big difference between the 2.