Exercise sheet 1

Notations. $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Exercise 1. *For each of the states $|\psi_{AB}\rangle$ shared between Alice and Bob, write the reduced density matrices ρ_A, ρ_B of Alice and Bob.*

1. $|\psi_{AB}\rangle = \sqrt{\frac{1}{3}}\,|00\rangle + \sqrt{\frac{2}{3}}\,|11\rangle$.
2. $|\psi_{AB}\rangle = \frac{1}{\sqrt{3}}(|01\rangle + |12\rangle + |22\rangle)$.
3. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |1+\rangle)$.

Solution:

1. $\rho_A = \rho_B = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$.
2. $\rho_A = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|+_{12}\rangle\langle +_{12}|$ with $|+_{12}\rangle = \frac{1}{\sqrt{2}}\,|1\rangle + |2\rangle$. $\rho_B = \frac{1}{3}|1\rangle\langle 1| + \frac{2}{3}|2\rangle\langle 2|$.
3. $\rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{2}\,|10\rangle + \frac{1}{2}\,|11\rangle$ so $\rho_A = \frac{3}{4}|\phi_1\rangle\langle\phi_1| + \frac{1}{4}|1\rangle\langle 1|$ with $|\phi_1\rangle = \sqrt{\frac{2}{3}}\,|0\rangle + \sqrt{\frac{1}{3}}\,|1\rangle$.

□

Exercise 2. *We have 2 states ρ_0, ρ_1. Bob is given ρ_b for a randomly chosen $b \in \{0,1\}$ and his goal is to guess b. Give the optimal measurement as well as his probability of success for the following states.*

1. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.
2. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = |+\rangle\langle +|$.
3. $\rho_0 = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$, $\rho_1 = |+\rangle\langle +|$. *In this case, give only the success probability and not the optimal measurement.*

Solution:

1. $\Delta(\rho_0, \rho_1) = \frac{1}{2}$. The optimal measurement is the one in the computational basis and succeeds wp. $\frac{3}{4}$.
2. As seen in class, the optimal strategy is to measure in the $\{|v\rangle, |v^\perp\rangle\}$ basis with $|v\rangle = \cos(\pi/8)\,|0\rangle - \sin(\pi/8)\,|1\rangle$ and $|v^\perp\rangle = \sin(\pi/8)\,|0\rangle + \cos(\pi/8)\,|1\rangle$. This strategy succeeds wp. $\cos^2(\pi/8)$.

3. $M = \rho_0 - \rho_1 = \begin{pmatrix} 1/6 & -1/2 \\ -1/2 & -1/6 \end{pmatrix}$. Moreover

$$(\rho_0 - \rho_1)(\rho_0 - \rho_1)^\dagger = \begin{pmatrix} \frac{10}{36} & 0 \\ 0 & \frac{10}{36} \end{pmatrix}$$

so $\Delta(\rho_0, \rho_1) = \frac{\sqrt{10}}{6}$ and the success probability is $\frac{1}{2} + \frac{\sqrt{10}}{12}$.

□

Analysis around the fingerprint state

Consider the state $|\psi_{x_1x_2x_3x_4}\rangle = \frac{1}{2}\left((-1)^{x_1}|00\rangle + (-1)^{x_2}|01\rangle + (-1)^{x_3}|10\rangle + (-1)^{x_4}|11\rangle\right)$ that depends on 4 bits x_1, x_2, x_3, x_4.

Exercise 3. *We assume $x_2, x_3, x_4 = 0$. Consider the states $|\phi_b\rangle = |\psi_{b000}\rangle$. This is 2-qubit state on some registers AB where A is the register corresponding to the first qubit and B is the register corresponding to the second qubit.*

1. *Give an expression for $|\phi_0\rangle$ and $|\phi_1\rangle$.*
2. *Compute $\rho_b^A = tr_B|\phi_b\rangle\langle\phi_b|$ for both $b = 0$ and $b = 1$.*
3. *Compute $\Delta(\rho_0^A, \rho_1^A)$. Give a measurement that, given ρ_b^A, outputs b with probability P_b with $\frac{1}{2}(P_0 + P_1) = \frac{3}{4}$ and argue that this measurement is optimal.*

Solution:

1. $\rho_0^A = |+\rangle\langle+|$, $\rho_1^A = \frac{1}{2}|-\rangle\langle-| + \frac{1}{2}|+\rangle\langle+|$.
2. $\Delta(\rho_0^A, \rho_1^A) = \frac{1}{2}$ so there is a measurement that succeeds in distinguishing ρ_0^A from ρ_1^A wp. $\frac{1}{2} + \frac{\Delta(\rho_0^A, \rho_1^A)}{2} = \frac{3}{4}$. This can be done by measuring in the $\{|+\rangle, |-\rangle\}$ basis.

□

Exercise 4. *We still assume $x_2, x_3, x_4 = 0$. Our goal is to analyze what is the probability of recovering x_1 assuming we now have access to the full state $|\phi_{x_1}\rangle$ (so Alice and Bob are together here).*

1. *Compute $\langle\phi_0|\phi_1\rangle$. Argue that there is a measurement that, given $|\phi_b\rangle$ for $b \in \{0, 1\}$, outputs b with probability P_b with $\frac{1}{2}(P_0 + P_1) = \frac{1}{2} + \frac{\sqrt{3}}{4}$.*

2. *Find the measurement that distinguishes $|\phi_0\rangle$ and $|\phi_1\rangle$ wp. $\frac{1}{2}+\frac{\sqrt{3}}{4}$. One can use without proof $\frac{1}{2}+\frac{\sqrt{3}}{4}=\cos^2(\pi/12)$.*

Solution:

1.
$$|\phi_0\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = |+\rangle\,|+\rangle$$
$$|\phi_1\rangle = \frac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(-|-\rangle\,|0\rangle + |+\rangle\,|1\rangle\right)$$

2. $\langle\phi_0|\phi_1\rangle = \frac{1}{2}$. For pure states, we have $\Delta(|\phi_0\rangle\langle\phi_0|, |\phi_1\rangle\langle\phi_1|) = \sqrt{1-|\langle\phi_0|\phi_1\rangle|^2} = \frac{\sqrt{3}}{2}$. So there is a measurement that succeeds in guessing b wp. $\frac{1}{2}+\frac{\Delta(|\phi_0\rangle\langle\phi_0|,|\phi_1\rangle\langle\phi_1|)}{2} = \frac{1}{2}+\frac{\sqrt{3}}{4}$.

3. $|\phi_1\rangle = \cos(\pi/3)\,|\phi_0\rangle + \sin(\pi/3)\,\left|\phi_0^\perp\right\rangle$ for some state $\left|\phi_0^\perp\right\rangle$ orthogonal to $|\phi_0\rangle$. We define
$$|\zeta_0\rangle = \cos(\pi/12)\,|\phi_0\rangle - \sin(\pi/12)\,\left|\phi_0^\perp\right\rangle$$
$$|\zeta_1\rangle = \sin(\pi/12)\,|\phi_0\rangle + \cos(\pi/12)\,\left|\phi_0^\perp\right\rangle$$
We measure in the basis $\{|\zeta_0\rangle, |\zeta_1\rangle\}$. One can check that
$$|\langle\zeta_0|\phi_0\rangle|^2 = \cos^2(\pi/12)$$
$$|\langle\zeta_1|\phi_1\rangle|^2 = (\cos(\pi/3)\sin(\pi/12) + \sin(\pi/3)\cos(\pi/12))^2 = \sin^2(5\pi/12) = \cos^2(\pi/12)$$
which gives the result.

□

Exercise 5. *We now don't have $x_2, x_3, x_4 = 0$ anymore. Assume we are in one of the two following cases*

1. $x_1 = x_2 = x_3 = x_4$.

2. $x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0$, *but the 4 bits are not all equal.*

Give a measurement on $|\psi_{x_1x_2x_3x_4}\rangle$ that determines with certainty in which case we are.

Solution: $|\psi_{0000}\rangle = |+\rangle|+\rangle = -|\psi_{1111}\rangle$. $|\psi_{x_1x_2x_3x_4}\rangle \perp |\psi_{0000}\rangle$ when $x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0$ but x_1, x_2, x_3, x_4 are not all equal. We consider the measurement $M = \{|++\rangle\langle++|, I - |++\rangle\langle++|\} = \{|\psi_{0000}\rangle\langle\psi_{0000}|, I - |\psi_{0000}\rangle\langle\psi_{0000}|\}$ which perfectly distinguishes the 2 cases. □

Exercise 6 (Unambiguous state discrimination)**.** *Assume we have* 2 *qubits* $|\phi_0\rangle = |0\rangle$ *and* $|\phi_1\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ *with* $\theta \in [0, \frac{\pi}{2})$. *Suppose Bob is give* $|\phi_b\rangle$ *for a random unknown* $b \in \{0, 1\}$ *and his goal is to guess* b. *We want a measurement that maybe succeeds with a smaller probability than the one seed in class but is always correct when it succeeds. More precisely, we want a measurement that has up to* 3 *outcomes:* "0", "1" *and* "2" *st. the measurement always succeeds when measuring* "0" *or* "1". *(the* "2" *outcome corresponds to unknown).*

Let $|f_1\rangle = \sin(\theta)|0\rangle - \cos(\theta)|1\rangle$. *We consider the* 3 *outcome POVM* $F = \{F_0, F_1, F_2\}$ *with* $F_i = M_i M_i^\dagger$. *We take* $F_0 = \frac{1}{1+\cos(\theta)}|f_1\rangle\langle f_1|$, $F_1 = \frac{1}{1+\cos(\theta)}|1\rangle\langle 1|$, $F_2 = (I - F_0 - F_1)$.

1. *Let* $|w\rangle = -\sin(\theta/2)|0\rangle + \cos(\theta/2)|1\rangle$ *and* $|w^\perp\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$. *Show that*

$$\frac{1}{2}(|f_1\rangle\langle f_1| + |1\rangle\langle 1|) = \cos^2(\theta/2)|w\rangle\langle w| + \sin^2(\theta/2)|w^\perp\rangle\langle w^\perp|.$$

2. *Show that* $F_2 = (1 - \tan^2(\theta/2))|w^\perp\rangle\langle w^\perp|$ *and that* $(1 - \tan^2(\theta/2)) \geq 0$. *From there, we easily have that* F_0, F_1, F_2 *are positive semi-definite and that* $\{F_i\}$ *is a valid POVM.*

3. *Show that this POVM satisfies our requirements. What is the probability of correctly guessing* b *here? Compare with the optimal guessing probability seen in class.*

Solution:

1. We have

$$\frac{1}{2}(|f_1\rangle\langle f_1| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & 1 + \cos^2(\theta) \end{pmatrix}$$

Let $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$. We write

$$c^2|w\rangle\langle w| + s^2|w^\perp\rangle\langle w^\perp| = \begin{pmatrix} c^2s^2 & -c^3s \\ -c^3s & c^4 \end{pmatrix} + \begin{pmatrix} s^2c^2 & s^3c \\ s^3c & s^4 \end{pmatrix}$$

Then we use trigonometric equalities

$$\begin{aligned} sin(\theta) &= 2sc \\ \cos(\theta) &= 2c^2 - 1 = c^2 - s^2 = 1 - 2s^2 \end{aligned}$$

so we compare the 4 elements of each matrix.

$$\begin{aligned} \frac{1}{2}\sin^2(\theta) &= 2s^2c^2 \quad \text{(top left)} \\ -\frac{1}{2}(\sin(\theta)\cos(\theta)) &= sc(-c^2+s^2) \quad \text{(the two non-diagonal)} \\ \frac{1}{2}(1+\cos^2(\theta)) &= \frac{1}{2}\left(1+4c^4-4c^2+1\right) = \frac{1}{2}\left(2c^4+2(1-c^2)^2\right) = c^4+s^4 \quad \text{(bottom right)} \end{aligned}$$

2. Using the previous question, we can rewrite

$$\begin{aligned} F_2 &= I - \frac{2}{1+\cos(\theta)}\left(\cos^2(\theta/2)|w\rangle\langle w| + \sin^2(\theta/2)|w^\perp\rangle\langle w^\perp|\right) \\ &\left(1 - \frac{2\cos^2(\theta/2)}{1+\cos(\theta)}\right)|w\rangle\langle w| + 1 - \frac{2\sin^2(\theta/2)}{1+\cos(\theta)}|w^\perp\rangle\langle w^\perp| \\ &= (1-\tan^2(\theta/2))|w^\perp\rangle\langle w^\perp| \end{aligned}$$

We can conclude by noticing that $\theta/2 \in [0, \pi/4)$ hence $(1-\tan^2(\theta/2)) \geq 0$.

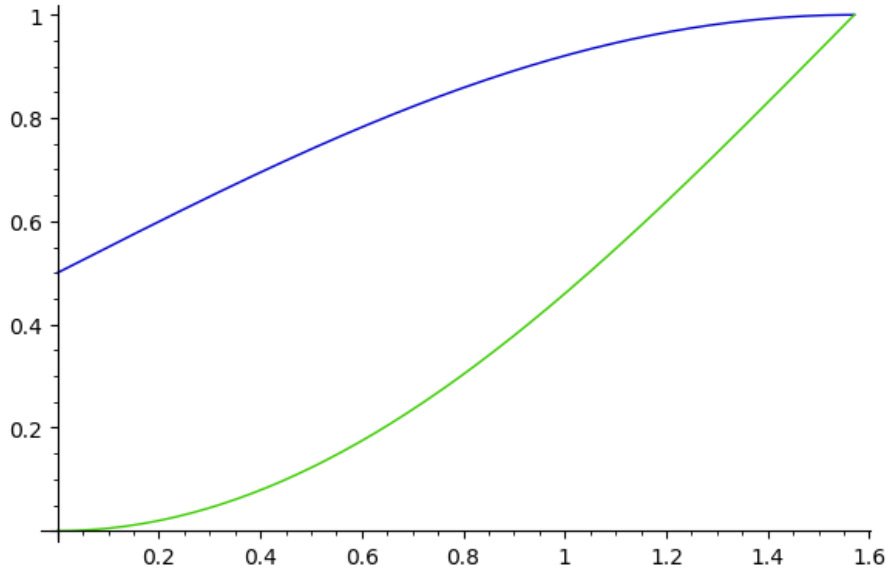
3. We have

$$\begin{aligned} tr(|\phi_0\rangle\langle\phi_0|F_0) &= \frac{\sin^2(\theta)}{1+\cos(\theta)} = 1-\cos(\theta) \\ tr(|\phi_1\rangle\langle\phi_1|F_1) &= \frac{\sin^2(\theta)}{1+\cos(\theta)} = 1-\cos(\theta) \end{aligned}$$

so $p = 1-\cos(\theta)$ is the success probability. In the optimal setting, we have that the probability p^* of guessing b is

$$p^* = \frac{1}{2} + \frac{\Delta(|\phi_0\rangle\langle\phi_0|, |\phi_1\rangle\langle\phi_1|)}{2} + \frac{1}{2} + \frac{\sqrt{1-|\langle\phi_0|\phi_1\rangle|^2}}{2} = \frac{1}{2} + \frac{\sin(\theta)}{2}.$$

We plot below p^* (in blue) and p (in green) and see the big difference between the 2.


1
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□