QCLG Exercise sheet 1

Exercise 1. *Determine whether the following states are qubits or not:*

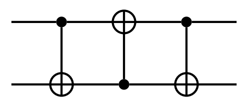
- $\frac{1}{3}|0\rangle - \frac{2}{3}|1\rangle$.
- $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|+\rangle$.
- $\frac{1}{2}|0\rangle + (1 - \frac{1}{\sqrt{2}})|+\rangle + \frac{1}{2}|1\rangle$.

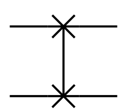
Exercise 2 (No cloning theorem)**.** *Our goal is to prove the no cloning theorem:*

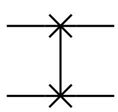
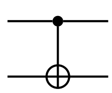
Theorem. *There is no unitary U on 2 qubits that on input $|\psi\rangle|0\rangle$ outputs $|\psi\rangle|\psi\rangle$ for all qubits $|\psi\rangle$.*

Prove this theorem. To do this, assume that such a unitary U exists and obtain a contradiction, for example by computing $U|\psi\rangle|0\rangle$ for $|\psi\rangle$ in the computational basis and in the Hadamard i.e.$\{|+\rangle, |-\rangle\}$ basis.

Exercise 3. *Show that the 2 circuits below compute the same unitary.*



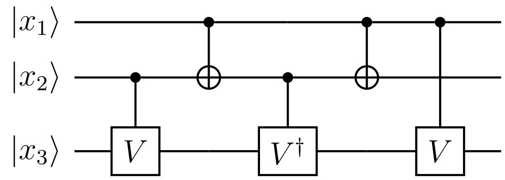


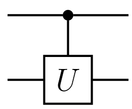
where  *is the* SWAP *gate satisfying* $\text{SWAP}(|x\rangle|y\rangle) = |y\rangle|x\rangle$ *for* $x, y \in \{0,1\}$*, and*  *is the* CNOT *gate satisfying* $\text{CNOT}(|x\rangle|y\rangle) = |x\rangle|y \oplus x\rangle$ *for* $x, y \in \{0,1\}$.

Exercise 4. *Determine whether the following matrices are unitary matrices:*

- $M_1 = |0\rangle\langle 0| + |+\rangle\langle 1|$.
- $M_2 = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$.
- $M_3 = |+\rangle\langle -| + |-\rangle\langle +|$

Exercise 5. *Consider the circuit*



where for any unitary U, 

is the ctrl-U *gate satisfying:*

$$\text{ctrl-}U(|0\rangle|y\rangle) = |x\rangle|y\rangle\,,\;\; \text{ctrl-}U(|1\rangle|y\rangle) = |1\rangle U|y\rangle\,,$$

which can be written concisely as ctrl-$U(|x\rangle|y\rangle) = |x\rangle U^x|y\rangle$ *for any* $x, y \in \{0, 1\}$.

1. *Let* U *be an arbitrary unitary over one qubit such that* $V^2 = U$. *Write the output of the circuit in terms of* U.
2. *Let* X *be the* NOT *gate. Find a unitary matrices* V *such that* $V^2 = X$ *and write also* $V^\dagger$ *(hint, use the fact that* $X|+\rangle = |+\rangle$ *and* $X|-\rangle = -|-\rangle$*).*
3. *Show how to construct the Toffoli gate: Toffoli*$(|x_1\rangle|x_2\rangle|x_3\rangle) = |x_1\rangle|x_2\rangle|x_3 \oplus (x_1 \wedge x_2)\rangle$ *with* 5 *two-qubit gates.*