QCLG Exercise sheet 2

Notations

We use $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$.

$$CNOT = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

Exercise 1. *Write in matrix form the following* 1 *qubit unitaries* U_1, U_2, U_3 *st.*

1. $U_1|0\rangle = |0\rangle$, $U_1|1\rangle = -|1\rangle$.
2. $U_2|0\rangle = |+\rangle$, $U_2|1\rangle = |-\rangle$.
3. $U_3|0\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$, $U_3|1\rangle = \sin(\theta)|0\rangle - \cos(\theta)|1\rangle$ *for some* $\theta \in [0, 2\pi]$.

Exercise 2. *Let* $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

1. *Is* $|\psi\rangle$ *a valid* 2*-qubit state?*
2. *Find a* 1*-qubit state* $|\phi\rangle$ *st.* $|\psi\rangle = |\phi\rangle \otimes |\phi\rangle$.
3. *Compute* $(H \otimes H)|\psi\rangle$.

Exercise 3. *Let* $|\phi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$. *Show that* $|\phi\rangle$ *is entangled meaning that there doesn't exist any* 1*-qubit states* $|u\rangle, |v\rangle$ *satisfying* $|\phi\rangle = |u\rangle \otimes |v\rangle$.

Exercise 4.

1. *Compute* $CNOT|00\rangle, CNOT|01\rangle, CNOT|10\rangle, CNOT|11\rangle$.
2. *Construct a* 2*-qubit circuit* C *that uses* 1 *and* 2 *qubit operations such that* $C(|00\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Exercise 5 (Breidbart basis)**.** *Consider the* 2 *states* $|\phi_0\rangle = |0\rangle$ *and* $|\phi_1\rangle = |-\rangle$. *Suppose we have* 2 *players Alice and Bob. Alice picks a uniformly random bit* b *and sends* $|\phi_b\rangle$ *to Bob. Bob's goal is to determine* b.

1. *Find a strategy that succeeds wp.* $\frac{3}{4}$, *on average on* b.

2. *Find a (possibly randomized) strategy that succeeds wp.* $\frac{3}{4}$ *for any* b.

3. *Now assume Bob measures in the basis* $\{|v\rangle, |v^\perp\rangle\}$ *and gets outcome* 0 *or* 1. *Assume we can write* $|v\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ *and* $|v^\perp\rangle = \sin(\theta)|0\rangle - \cos(\theta)|1\rangle$ *with* $\theta \in [0, \pi/2]$.

   (a) *Compute the probability that Bob succeeds with this strategy, as a function of* θ.

   (b) *Find a strategy that will make him succeed wp.* $\cos^2(\pi/8)$ *for any* b. *The basis that achieves this is called the Breidbart basis and has application to non local games for instance.*

Exercice 6.[Efficient controlled operations]

Let U be a 1-qubit unitary that we would like to implement in a controlled way, i.e., we want to implement a map $|c\rangle|b\rangle \mapsto |c\rangle U^c|b\rangle$ for all $c, b \in \{0,1\}$. Suppose there exist 1-qubit unitaries A, B, and C, such that $ABC = I$ and $AXBXC = U$ (remember that X is the NOT-gate). Give a circuit that acts on two qubits and implements a controlled-U gate, using CNOTs and (uncontrolled) A, B, and C gates.