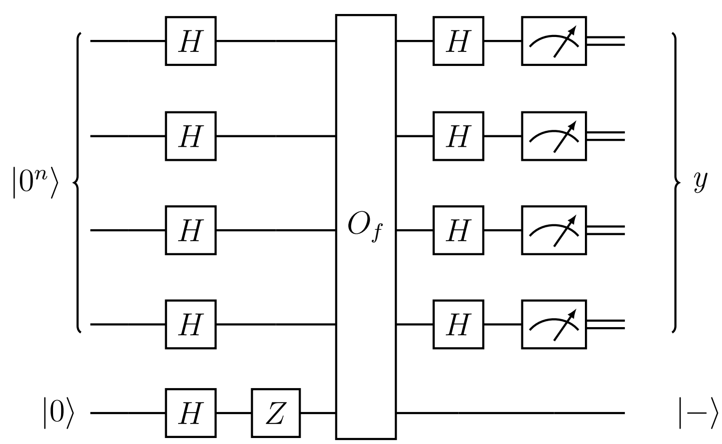
QCLG Exercise sheet 3

Exercise 1 (Writing Bell states in the Hadamard basis)**.** *Let* $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$ *and* $|\Phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - |11\rangle$. *Let also* $|\Psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + |10\rangle$ *and* $|\Psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - |10\rangle$.

1. *Show that* $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$.
2. *Show that* $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle - |{-}{-}\rangle)$.
3. *Show that* $|\Psi^-\rangle$ *and* $|\Psi^-\rangle$ *are of the form* $\alpha|{-}{+}\rangle + \beta|{+}{-}\rangle$ *and find the values of* α, β *in each case.*

Exercise 2 (Bernstein-Vazirani algorithm)**.** *Recall the circuit used in the Deutsch-Jozsa algorithm, where* f *is a function from* $\{0,1\}^n$ *to* $\{0,1\}$. *and* O_f *is the unitary st.* $O_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$. *Recall also that* $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.



1. *Let* $|\psi_3\rangle$ *the quantum state just before the final measurements. Redo the calculations from the lecture, i.e. prove without looking at your lecture notes that*

$$|\psi_3\rangle = \frac{1}{2^n}\sum_{y\in\{0,1\}^n}\left(\sum_{x\in\{0,1\}^n}(-1)^{f(x)+x\cdot y}\right)|y\rangle|-\rangle.$$

you can use without proof the equality $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{y\in\{0,1\}^n}(-1)^{x\cdot y}|y\rangle$ *for any* $x \in \{0,1\}^n$. *Also recall that we use* $x \cdot y = \sum_{i=1}^n x_i y_i \ mod(2)$ *for* $x = x_1 \dots x_n$ *and* $y = y_1 \dots y_n$.

2. *Assume our function f satisfies the following property: $\exists s \in \{0,1\}^n, f(x) = x \cdot s$. Show that the above algorithm always outputs $y = s$. This algorithm is known as the Bernstein-Vazirani algorithm, if we have the promise that the function f satisfies the property above, then this algorithm finds s with a single query to O_f.*

Exercise 3 (Constructing reflexions over a quantum state)**.** *Consider an n qubit state $|\psi\rangle$ and assume we have an efficiently computable unitary U st. $U|0^n\rangle = |\psi\rangle$. Our goal is to show we can efficient compute the reflexion $R_{|\psi\rangle}$ i.e. the unitary satisfying*

$$R_{|\psi\rangle}(|\psi\rangle) = |\psi\rangle\,,\ \forall\,|\phi\rangle\ \text{st.}\ |\psi\rangle \perp |\phi\rangle \quad R_{|\psi\rangle}(|\phi\rangle) = -\,|\phi\rangle\,.$$

1. *Show that $\forall\,|\phi\rangle$ st. $|\phi\rangle \perp |\psi\rangle$, we can write*

$$U^\dagger(|\phi\rangle) = \sum_{\substack{i\in\{0,1\}^n \\ i\neq 0^n}} \alpha_i\,|i\rangle\,.$$

2. *Argue, without writing the circuit, that one can efficiently compute the unitary V on $n+1$ qubits that satisfies*

$$V(|x\rangle\,|y\rangle) \to |x\rangle\,|y \oplus g(x)\rangle$$

where $g(x) = 0$ iff. $x = 0^n$ and $g(x) = 1$ otherwise.

3. *Construct using the previous unitaries and elementary gates the unitary W on n qubits with an extra auxiliary qubit such that*

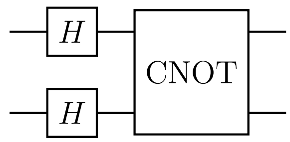
$$W\,|x\rangle\,|0\rangle = (-1)^{g(x)}\,|x\rangle\,|0\rangle\,.$$

There is a construction that uses only 2 calls to V or $V^\dagger$ and a phase flip gate Z. There is another construction that uses a single call to V and 2 calls to H or $H^\dagger$ and 2 calls to the bit flip X. Find at least one construction, can you find both?

4. *Show how to construct $R_{|\psi\rangle}$ (with an auxiliary qubit) with 2 calls to U or $U^\dagger$ and 1 call to W.*

Exercise 4.

1. *Write the unitary acting on 2 qubits corresponding to the following circuit in matrix form (in the* $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ *basis):*



2. *Write the unitary acting on 2 qubits corresponding to the following circuit in matrix form (in the* $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ *basis):*