QCLG Exercise sheet 4

Notations

We use $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$.
$CNOT = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$. Recall that for $x \in \{0,1\}^n$, we have

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y}|y\rangle,$$

where $x \cdot y = \sum_{i=1}^n x_i y_i$ when writing $x = x_1 \dots x_n$ and $y = y_1 \dots y_n$.

Exercise 1.

1. *Construct* 4 *orthogonal* 2*-qubit states* $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle$ *st. measuring in the* $\{|0\rangle, |1\rangle\}$ *basis any qubit of any of these states gives outcome "0" wp.* $\frac{1}{2}$ *and outcome "1" wp.* $\frac{1}{2}$.

2. *Is it possible to construct* 2 *orthogonal* 1*-qubit states* $|\psi_1\rangle, |\psi_2\rangle$ *st. measuring in the* $\{|0\rangle, |1\rangle\}$ *basis any of these states gives outcome "0" wp.* 0.6 *and outcome "1" wp.* 0.4*? If yes, construct these states, if no, explain why it isn't possible.*

Exercise 2.

1. *Consider the operation on* 3 *qubits* $W_1 : |a\rangle|b\rangle|c\rangle = |a \oplus b\rangle|a \oplus c\rangle|b \oplus c\rangle$ *for each* $a, b, c \in \{0,1\}$*. Is* W_1 *a unitary? Justify your answer.*

2. *Consider the operation on* 3 *qubits* $W_2 : |a\rangle|b\rangle|c\rangle = |a \oplus b\rangle|a \oplus c\rangle|a \oplus b \oplus c\rangle$ *for each* $a, b, c \in \{0,1\}$*. Is* W_2 *a unitary? Justify your answer.*

3. *Construct the circuits for* W_1 *and/or* W_2 *if the corresponding operation is a unitary, using elementary* 1 *and* 2 *qubit gates.*

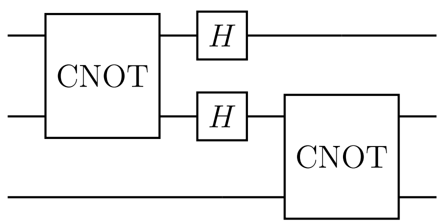
Exercise 3. *Let* $|\phi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$*. Show that* $|\phi\rangle$ *is entangled meaning that there doesn't exist any* 1*-qubit states* $|u\rangle, |v\rangle$ *satisfying* $|\phi\rangle = |u\rangle \otimes |v\rangle$.

Exercise 4. *Consider a function* $f : \{0,1\}^n \to \{0,1\}$*. Let*

$$|\psi_f\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle .$$

1. *Compute* $|\psi'\rangle = H^{\otimes n} |\psi_f\rangle$ *.*
2. *Let* $T_b = |\{x : f(x) = b\}|$*, so* $T_0 + T_1 = 2^n$*. Write the probability of measuring* 0^n *when having* $|\psi'\rangle$ *as a function of* T_0, T_1*.*
3. *Show how starting from* $|\psi_f\rangle$*, one can distinguish whether* f *is constant (i.e.* $T_0 = 0$ *or* $T_1 = 0$*) or whether* $\min\{T_0, T_1\} = \frac{1}{16} \cdot 2^n$ *with a constant probability with a single call to* $H^{\otimes n}$ *and a single measurement.*

Exercise 5. *Consider the following circuit* C*:*



(a) Compute $C|000\rangle$*.*

(b) Compute $C|111\rangle$*.*