QII Exercise sheet 10

These exercices are taken from Ronald de Wolf's lecture notes.

Exercise 1.

Give a quantum circuit for the encoding of Shor's 9-qubit code, i.e., a circuit that maps $|00^8\rangle \mapsto |\overline{0}\rangle$ and $|10^8\rangle \mapsto |\overline{1}\rangle$. Explain why the circuit works.

Exercise 2.

Shor's 9-qubit code allows to *correct* a bit flip and/or a phase flip on one of its 9 qubits. Below we give a 4-qubit code which allows to *detect* a bitflip and/or a phaseflip. By this we mean that after the detection procedure we either have the original uncorrupted state back, or we know that an error occurred (though we do not know which one). The logical 0 and 1 are encoded as:

$$|\overline{0}\rangle = \frac{1}{2}(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$$
$$|\overline{1}\rangle = \frac{1}{2}(|00\rangle - |11\rangle) \otimes (|00\rangle - |11\rangle)$$

(a) Give a procedure (either as a circuit or as sufficiently-detailed pseudo-code) that detects a bitflip error on one of the 4 qubits of $\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle$.

(b) Give a procedure (either as a circuit or as sufficiently-detailed pseudo-code) that detects a phaseflip error on one of the 4 qubits of $\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle$.

(c) Does that mean that we can now detect *any* unitary 1-qubit error on one of the 4 qubits? Explain your answer.

Exercise 3. *Show that there cannot be a quantum error correcting code that encode one logical qubit into $2n$ physical qubits that can correct arbitrary error on n qubits.* Hint: think of the no cloning theorem.

Exercise 4.

Suppose we have a qubit $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ to which we would like to apply a $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$ gate, but for some reason we cannot. However, we have a second qubit available in state $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$, and we can apply a CNOT gate and an $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ gate.

(a) What state do we get if we apply a CNOT to the first and second qubit?

(b) Suppose we measure the second qubit in the computational basis. What are the probabilities of outcomes 0 and 1, respectively?

(c) Suppose the measurement yields 0. Show how we can get $T|\phi\rangle$ in the first qubit.

(d) Suppose the measurement yields 1. Show how we can get $T|\phi\rangle$ in the first qubit, up to an (irrelevant) global phase.

Comment: This way of implementing the T-gate is very helpful in fault-tolerant computing, where often CNOT and S are easy to do on encoded states but T is not. What this exercise shows is that we can prepare (encodings of) the so-called "magic state" $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$ beforehand (offline, assuming we can store them until we need them), and use those to indirectly implement a T-gate using only CNOT and S-gates.