

QII Exercise sheet 15

Notations. Every logarithm is in base 2.

Exercise 1. Consider 2 discrete probability function $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_m)$. So we have $p_i \geq 0, \sum_i p_i = 1$, and $q_i \geq 0, \sum_i q_i = 1$. Consider the direct product distribution $r = (r_{1,1}, \dots, r_{n,m})$ where $r_{i,j} = p_i q_j$. Show that

$$H(r) = H(p) + H(q).$$

Exercise 2. For each state $|\psi_{AB}\rangle$, give the reduced density matrices $\rho_A = \text{tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$ and $\rho_B = \text{tr}_A(|\psi_{AB}\rangle\langle\psi_{AB}|)$. You can write your answers in Dirac's "ket,bra" notation or in matrix form. Compute also $H(\rho_A)$ in each case. You can use $\log_2(3) \approx 1.585$.

1. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0-\rangle + |1+\rangle)$.
2. $|\psi_{AB}\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$.
3. $|\psi_{AB}\rangle = \sqrt{\frac{3}{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle - \sqrt{\frac{1}{8}}|10\rangle + \sqrt{\frac{1}{8}}|11\rangle$.

Exercise 3. Consider 2 quantum mixed states ρ and σ which are orthogonal. This means we can write ρ and σ in their spectral decomposition

$$\begin{aligned}\rho &= \sum_i p_i |e_i\rangle\langle e_i| \\ \sigma &= \sum_j q_j |f_j\rangle\langle f_j|\end{aligned}$$

where each $p_i, q_j > 0$ and $\sum_i p_i = \sum_j q_j = 1$, and the orthogonality constraint gives $\forall i, j \langle e_i | f_j \rangle = 0$ (or equivalently $|e_i\rangle \perp |f_j\rangle$). Let also $I_\rho = \sum_i |e_i\rangle\langle e_i|$ and $I_\sigma = \sum_j |f_j\rangle\langle f_j|$

1. Show that $\rho \cdot \log(\sigma) = \rho \cdot I_\sigma = \mathbf{0}$ where $\mathbf{0}$ is the all 0 matrix.
2. Let $\xi = r\rho + (1-r)\sigma$ with $r \in [0, 1]$. Show that $\log(\xi) = \log(r\rho) + \log((1-r)\sigma)$.
3. Let $r \geq 0$. Show that $\log(r\rho) = \log(\rho) + \log(r)I_\rho$.
4. Let $\xi = r\rho + (1-r)\sigma$ with $r \in [0, 1]$. Show that

$$H(\xi) = H_2(r) + rH(\rho) + (1-r)H(\sigma)$$

where $H_2(r) = -r \log(r) - (1-r) \log(1-r)$.

Exercise 4. Let $\sigma_{AB} = r|0\rangle\langle 0|_A \otimes \rho_B^0 + (1-r)|1\rangle\langle 1|_A \otimes \rho_B^1$ be a quantum mixed state on 2 registers. Using the previous question, show that

$$I(A : B)_\sigma = H(r\rho_B^0 + (1-r)\rho_B^1) - (rH(\rho_B^0) + (1-r)H(\rho_B^1))$$

Find matrices ρ_B^0, ρ_B^1 st. $I(A : B)_\sigma = 0$. Find others st. $I(A : B)_\sigma = H_2(r)$.