QII Exercise sheet 16

Hidden Matching Problem

Alice has a random string $x = (x_1, \dots, x_n)$ unknown to Bob with n even. Bob has a matching M which is a list of $\frac{n}{2}$ disjoint pairs (i, j) for $i, j \neq i \in \{1, \dots, n\}$. For example for $n = 6$, the list $((1,4),(2,3),(5,6))$ is a possible matching. For $n = 10$, the lists $((1,7),(2,9),(3,4),(5,10),(6,8))$ and $((1,4),(2,10),(3,5),(6,9),(7,8))$ are two possible matchings.

Alice and Bob cooperate and we are in the one-way communication model where Alice is allowed to send a single message to Bob. Bob outputs a triplet (i, j, b). They win iff. $(i, j) \in M$ and $x_i \oplus x_j = b$.

Exercise 1. *We consider the following quantum protocol for this problem*

- *Alice creates the state* $|\psi_x\rangle = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} (-1)^{x_l} |l\rangle$ *and sends it to Bob.*
- *Bob has a matching* $M = \big((i_1, j_1), \dots, (i_{n/2}, j_{n/2})\big)$. *Let* $\Pi = \{\Pi_i\}_{i \in \{1,\dots,n/2\}}$ *be a projective measurement with* $\Pi_k = |i_k\rangle\langle i_k| + |j_k\rangle\langle j_k|$. *Bob measures using this measurement. He uses the resulting outcome to output a triplet* (i, j, b) *in a way you will show in one of the questions.*

1. *What is the size of* $|\psi_x\rangle$*?*
2. *Show that* Π *is a valid quantum measurement.*
3. *For each* $k \in \{1, \dots, n/2\}$, *what is the probability that Bob outputs "k"?*
4. *Assume Bob gets outcome* k, *what is the remaining state* $\left|\psi_x^k\right\rangle$ *after the measurement?*
5. *Show how Bob can output* (i, j, b) *from his outcome "k" and the state* $\left|\psi_x^k\right\rangle$.

Exercise 2. *Show a randomized strategy where Alice sends a message of size* $\widetilde{O}(\sqrt{n})$ *that succeeds wp. at least* $\frac{2}{3}$. *Think of the birthday's paradox.*

This is actually optimal for classical one-way communication so it shows an exponential separation between classical and quantum one-way communication complexity. See [GKK+06] for more details.

Quantum Fingerprinting

We consider a different scenario here where Alice and Bob have respectively a string $x \in \{0,1\}^n$ and $y \in \{0,1\}^n$. They each send a message to a referee R that has to determine whether $x = y$ or not. Alice and Bob *do not have shared randomness.*

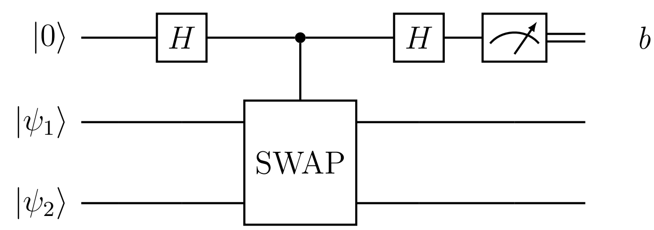
Alice, Bob and the referee cooperate and they win iff. R outputs "0" if $x = y$ and "1" if $x \neq y$.

Assume the referee has a procedure that given any 2 states $|\psi_1\rangle, |\psi_2\rangle$ outputs "0" wp $\frac{1}{2} + \frac{|\langle\psi_1|\psi_2\rangle|^2}{2}$.

Exercise 3. *Using states of the form* $|\psi_x\rangle = \frac{1}{\sqrt{n}} \sum_i |i\rangle |x_i\rangle$*, give a procedure where Alice and Bob send states of size* $\lceil \log_2(n) \rceil + 1$ *and where the referee always succeeds when* $x = y$ *and succeeds wp.* $\frac{1}{2} - \frac{\delta^2}{2}$ *where* $\delta = \frac{1}{n}|\{i : x_i = y_i\}|$.

Exercise 4. *For each* $c \in \mathbb{N}^*$*, there exists a function* $E : \{0,1\}^n \to \{0,1\}^{cn}$ *st.* $\forall x, y \neq x \in \{0,1\}^n$, $|\{i \in [cn] : E(x)_i = E(y)_i\}| \leq \frac{9}{10} + \frac{1}{15c}$*. This construction is known as Justesen codes. Show how to use this function in order to win the game with constant probability non zero when* $x \neq y$*, by sending* $O(\log(n))$ *qubits to the referee.*

Exercise 5. *For* 2 *states* $|\psi_1\rangle, |\psi_2\rangle$*, we consider the circuit*



where the second gate is a control-SWAP defined as

$$C - \mathrm{SWAP}\,|0\rangle|x\rangle|y\rangle = |0\rangle|x\rangle|y\rangle \;;\; C - \mathrm{SWAP}\,|1\rangle|x\rangle|y\rangle = |1\rangle|y\rangle|x\rangle.$$

for any $x, y \in \{0,1\}$*. Show that this circuit performs the procedure of the referee that outputs* 0 *wp.* $\frac{1}{2} + \frac{|\langle\psi_1|\psi_2\rangle|^2}{2}$.

References

[GKK+06] Dmytro Gavinsky, Julia Kempe, Iordanis Kerenidis, Ran Raz, and Ronald de Wolf. Exponential separations for one-way quantum communication complexity, with applications to cryptography. *arXiv, https://arxiv.org/abs/quant-ph/0611209*, 2006.