QII Exercise sheet 18 - Mockup exam

Exercise 1. *For a matrix $\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|$ in spectral decomposition, we define $\rho^2 = \sum_i \lambda_i^2 |e_i\rangle\langle e_i|$. Show that ρ is a pure state iff. $tr(\rho^2) = 1$.*

Exercise 2. *What is the entropy of the state $\rho = \frac{1}{4}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + |+\rangle\langle +| + |-\rangle\langle -|\right)$?*

Exercise 3. *Consider the POVM $F = \{F_0, F_1, F_2, F_3\}$ with each $F_i = M_i M_i^\dagger$ and*

$$M_0 = \frac{1}{\sqrt{2}}|0\rangle\langle 0|$$
$$M_1 = \frac{1}{\sqrt{2}}|+\rangle\langle +|$$
$$M_2 = \frac{1}{\sqrt{2}}|-\rangle\langle -|$$
$$M_3 = \frac{1}{\sqrt{2}}|1\rangle\langle 1|$$

1. *Write F_0. Show that F is a POVM.*
2. *For a given state $|\phi\rangle$, what is the probability of getting each outcome i and what is the resulting state after this measurement?*
3. *Start with the state $|0\rangle$ and perform the POVM F. Is it possible that the resulting state is $|1\rangle$?*
4. *Start with the state $|0\rangle$ and perform the POVM F. You get some outcome i and the resulting state is $|\psi_i\rangle$. Now perform the POVM F again on $|\psi_i\rangle$ to get a new outcom j and resulting state $|\psi_{ij}\rangle$. Compute the probability that $|\psi_{ij}\rangle = |1\rangle$.*

Exercise 4 (Qutrit encoding)**.** *Alice has 2 uniformly random btis x_1, x_2 unknown to Bob and wants to send as much information as possible to Bob by sending a single qutrit to Bob. More precisely, she constructs the state*

$$|\phi_{x_1,x_2}\rangle = \frac{1}{\sqrt{3}}\left((-1)^{x_1}|0\rangle + (-1)^{x_2}|1\rangle + (-1)^{x_1 \oplus x_2}|2\rangle\right).$$

and sends it to Bob. For $b \in \{0,1\}$, let ρ_b^1 be Bob's state conditioned on the fact that $x_1 = b$ and let ρ_b^2 be Bob's state conditioned on $x_2 = b$.

1. *Write the state $|\phi_{x_1,x_2}\rangle\langle\phi_{x_1,x_2}|$ in matrix form in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$ as a function of x_1, x_2. Write the density matrices $\rho_0^1, \rho_1^1, \rho_0^2, \rho_1^2$. What is the entropy of each of these states? You can use $\log_2(3) \approx 1.585$.*

2. *Assume Bob wants to learn* x_1 *from* $|\phi_{x_1,x_2}\rangle$. *Show that the optimal probability of success is* $\frac{5}{6}$ *and give a corresponding optimal strategy to learn* x_1.

3. *Assume Bob wants to learn* (x_1, x_2). *Give a simple strategy that succeeds wp.* $\frac{5}{12}$.

4. *Consider the POVM* $F = \{F_{00}, F_{01}, F_{10}, F_{11}\}$ *with* $F_{ij} = \frac{1}{N}|\phi_{i,j}\rangle\langle\phi_{i,j}|$, *where* N *is a normalizing factor.*

   (a) *What* N *should we take such that* F *is a POVM?*

   (b) *Assume Bob wants to learn* (x_1, x_2). *Give a strategy that succeeds wp.* $\frac{3}{4}$.

Exercise 5. *Assume Alice has 2 states* ρ_0 *and* ρ_1 *and sends to Bob* ρ_b *for a randomly chosen* $b \in \{0, 1\}$. *Use the previous exercise to show the result given in class that Bob can guess* b *wp. at most* $\frac{1}{2} + \frac{\Delta(\rho_0,\rho_1)}{2}$. *(Hint: any strategy for Bob can be expressed as a 2-outcome POVM* $\{F_0, F_1\}$ *where outcome* i *corresponds to his guess. Express his winning probability as a function of* $tr(F_0\rho_0)$ *and* $tr(F_1\rho_1)$*).*