QII Exercise sheet 2

Trigonometric relations

$$\begin{aligned}
\cos^2(x) + \sin^2(x) &= 1 \\
\cos(x) &= \sin(\pi/2 - x) \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(x+y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y)
\end{aligned}$$

Exercise 1. *We have 2 states ρ_0, ρ_1. Bob is given ρ_b for a randomly chosen $b \in \{0,1\}$ and his goal is to guess b. Give the optimal measurement as well as his probability of success for the following states.*

1. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.
2. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = |+\rangle\langle +|$.
3. $\rho_0 = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$, $\rho_1 = |+\rangle\langle +|$. *In this case, give only the success probability and not the optimal measurement.*

Exercise 2. *Assume we have 2 qubits $|\phi_0\rangle = |0\rangle$ and $|\phi_1\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ with $\theta \in [0, \frac{\pi}{2})$. Suppose Bob is give $|\phi_b\rangle$ for a random unknown $b \in \{0,1\}$ and his goal is to guess b. We want a measurement that maybe succeeds with a smaller probability than the one seed in class but is always correct when it succeeds. More precisely, we want a measurement that has up to 3 outcomes: "0", "1" and "2" st. the measurement always succeeds when measuring "0" or "1". (the "2" outcome corresponds to unknown).*

Let $|f_1\rangle = \sin(\theta)|0\rangle - \cos(\theta)|1\rangle$. We consider the 3 outcome POVM $F = \{F_0, F_1, F_2\}$ with $F_i = M_i M_i^\dagger$. We take $F_0 = \frac{1}{1+\cos(\theta)}|f_1\rangle\langle f_1|$, $F_1 = \frac{1}{1+\cos(\theta)}|1\rangle\langle 1|$, $F_2 = (I - F_0 - F_1)$.

1. *Let $|w\rangle = -\sin(\theta/2)|0\rangle + \cos(\theta/2)|1\rangle$ and $|w^\perp\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$. Show that*

$$\frac{1}{2}(|f_1\rangle\langle f_1| + |1\rangle\langle 1|) = \cos^2(\theta/2)|w\rangle\langle w| + \sin^2(\theta/2)|w^\perp\rangle\langle w^\perp|.$$

2. *Show that* $F_2 = (1 - \tan^2(\theta/2))|w^\perp\rangle\langle w^\perp|$ *and that* $(1 - \tan^2(\theta/2)) \geq 0$. *From there, we easily have that* F_0, F_1, F_2 *are positive semi-definite and that* $\{F_i\}$ *is a valid POVM.*

3. *Show that this POVM satisfies our requirements. What is the probability of correctly guessing b here? Compare with the optimal guessing probability seen in class.*

Exercise 3. *Consider* 2 *quantum states* ρ, σ, *and an* m*-outcome POVM* $F = \{F_1, \ldots, F_m\}$ *where each* $F_i = M_i M_i^\dagger$ *and* $\sum_i F_i = I$. *We define*

$$p_i = tr(F_i \rho) \quad \text{and } q_i = tr(F_i \sigma).$$

Our goal is to show that

$$\Delta(\rho, \sigma) \geq \Delta(p, q).$$

with $\Delta(p, q) = \frac{1}{2} \sum_i |p_i - q_i|$.

1. *To what correspond the values* p_i *and* q_i*?*

2. *We perform the spectral decomposition* $\rho - \sigma = \sum_i \lambda_i |e_i\rangle\langle e_i|$. *We define* $Q = \sum_{i:\lambda_i \geq 0} \lambda_i |e_i\rangle\langle e_i|$ *and* $S = \sum_{i:\lambda_i < 0} -\lambda_i |e_i\rangle\langle e_i|$, *so that* $|\rho - \sigma| = Q + S$. *Show that for each* $i \in [m]$. $|p_i - q_i| \leq tr(F_i(Q + S))$.

3. *Conclude that* $\Delta(\rho, \sigma) \geq \Delta(p, q)$.

Exercise 4. *Assume Alice has* 2 *states* ρ_0 *and* ρ_1 *and sends to Bob* ρ_b *for a randomly chosen* $b \in \{0, 1\}$. *Use the previous exercise to show the result given in class that Bob can guess b wp. at most* $\frac{1}{2} + \frac{\Delta(\rho_0, \rho_1)}{2}$. *(Hint: any strategy for Bob can be expressed as a* 2*-outcome POVM* $\{F_0, F_1\}$ *where outcome i corresponds to his guess. Express his winning probability as a function of* $tr(F_0 \rho_0)$ *and* $tr(F_1 \rho_1)$*).*

Exercise 5. *Recall the Fuchs-van de Graaf inequalities*

$$(1 - F(\rho, \sigma)) \leq \Delta(\rho, \sigma) \leq \sqrt{1 - F^2(\rho, \sigma)}$$

1. *Give* 2 *quantum states* ρ, σ *st.* $\Delta(\rho, \sigma) = \frac{1}{2}$ *and* $(1 - F(\rho, \sigma)) = \Delta(\rho, \sigma)$.

2. *Give* 2 *quantum states* ρ, σ *st.* $\Delta(\rho, \sigma) = \frac{1}{2}$ *and* $\Delta(\rho, \sigma) = \sqrt{1 - F^2(\rho, \sigma)}$.