QII Exercise sheet 3

Exercise 1. *For a matrix* $\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|$ *in spectral decomposition, we define* $\rho^2 = \sum_i \lambda_i^2 |e_i\rangle\langle e_i|$. *Show that* ρ *is a pure state iff.* $tr(\rho^2) = 1$.

Exercise 2. *Consider the POVM* $F = \{F_0, F_1, F_2, F_3\}$ *with each* $F_i = M_i M_i^\dagger$ *and*

$$M_0 = \frac{1}{\sqrt{2}}|0\rangle\langle 0|$$
$$M_1 = \frac{1}{\sqrt{2}}|+\rangle\langle +|$$
$$M_2 = \frac{1}{\sqrt{2}}|-\rangle\langle -|$$
$$M_3 = \frac{1}{\sqrt{2}}|1\rangle\langle 1|$$

1. *Write* F_0. *Show that* F *is a POVM.*
2. *For a given state* $|\phi\rangle$, *what is the probability of getting each outcome* i *and what is the resulting state after this measurement?*
3. *Start with the state* $|0\rangle$ *and perform the POVM* F. *Is it possible that the resulting state is* $|1\rangle$*?*
4. *Start with the state* $|0\rangle$ *and perform the POVM* F. *You get some outcome* i *and the resulting state is* $|\psi_i\rangle$. *Now perform the POVM* F *again on* $|\psi_i\rangle$ *to get a new outcome* j *and resulting state* $|\psi_{ij}\rangle$. *Compute the probability* p_{ij} *to get outcomes* i *and* j *this way. Is it possible that the resulting state becomes* $|1\rangle$ *after these two measurements? If yes, with what probability does this happen?*

Exercise 3. *Consider* 2 *quantum states* ρ, σ, *and an* m*-outcome POVM* $F = \{F_1, \ldots, F_m\}$ *where each* $F_i = M_i M_i^\dagger$ *and* $\sum_i F_i = I$. *We define*

$$p_i = tr(F_i\rho) \quad \text{and } q_i = tr(F_i\sigma).$$

Our goal is to show that

$$\Delta(\rho, \sigma) \geq \Delta(p, q).$$

with $\Delta(p, q) = \frac{1}{2}\sum_i |p_i - q_i|$.

1. *To what correspond the values* p_i *and* q_i*?*

2. *We perform the spectral decomposition* $\rho - \sigma = \sum_i \lambda_i |e_i\rangle\langle e_i|$. *We define* $Q = \sum_{i:\lambda_i \geq 0} \lambda_i |e_i\rangle\langle e_i|$ *and* $S = \sum_{i:\lambda_i < 0} -\lambda_i |e_i\rangle\langle e_i|$, *so that* $\rho - \sigma = Q - S$ *and* $|\rho - \sigma| = Q + S$. *Show that for each* $i \in [m]$. $|p_i - q_i| \leq tr(F_i(Q+S))$.

3. *Conclude that* $\Delta(\rho, \sigma) \geq \Delta(p, q)$.

Exercise 4. *Assume Alice has 2 states* ρ_0 *and* ρ_1 *and sends to Bob* ρ_b *for a randomly chosen* $b \in \{0, 1\}$. *Use the previous exercise to show the result given in class that Bob can guess* b *wp. at most* $\frac{1}{2} + \frac{\Delta(\rho_0,\rho_1)}{2}$. *(Hint: any strategy for Bob can be expressed as a 2-outcome POVM* $\{F_0, F_1\}$ *where outcome* i *corresponds to his guess. Express his winning probability as a function of* $tr(F_0\rho_0)$ *and* $tr(F_1\rho_1)$*).*