QII Exercise sheet 4

Analysis around the fingerprint state

Consider the state $|\psi_{x_1x_2x_3x_4}\rangle = \frac{1}{2}\left((-1)^{x_1}|00\rangle + (-1)^{x_2}|01\rangle + (-1)^{x_3}|10\rangle + (-1)^{x_4}|11\rangle\right)$ that depends on 4 bits x_1, x_2, x_3, x_4.

Exercise 1. *We assume $x_2, x_3, x_4 = 0$. Consider the states $|\phi_b\rangle = |\psi_{b000}\rangle$. This is 2-qubit state on some registers AB where A is the register corresponding to the first qubit and B is the register corresponding to the second qubit.*

1. *Give an expression for $|\phi_0\rangle$ and $|\phi_1\rangle$.*
2. *Compute $\rho_b^A = tr_B|\phi_b\rangle\langle\phi_b|$ for both $b = 0$ and $b = 1$.*
3. *Compute $\Delta(\rho_0^A, \rho_1^A)$. Give a measurement that, given ρ_b^A, outputs b with probability P_b with $\frac{1}{2}(P_0 + P_1) = \frac{3}{4}$ and argue that this measurement is optimal.*

Exercise 2. *We still assume $x_2, x_3, x_4 = 0$. Our goal is to analyze what is the probability of recovering x_1 assuming we now have access to the full state $|\phi_{x_1}\rangle$ (so Alice and Bob are together here).*

1. *Compute $\langle\phi_0|\phi_1\rangle$. Argue that there is a measurement that, given $|\phi_b\rangle$ for $b \in \{0,1\}$, outputs b with probability P_b with $\frac{1}{2}(P_0 + P_1) = \frac{1}{2} + \frac{\sqrt{3}}{4}$.*
2. *Find the measurement that distinguishes $|\phi_0\rangle$ and $|\phi_1\rangle$ wp. $\frac{1}{2} + \frac{\sqrt{3}}{4}$. One can use without proof $\frac{1}{2} + \frac{\sqrt{3}}{4} = \cos^2(\pi/12)$.*

Exercise 3. *We now don't have $x_2, x_3, x_4 = 0$ anymore. Assume we are in one of the two following cases*

1. $x_1 = x_2 = x_3 = x_4$.
2. $x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0$*, but the 4 bits are not all equal.*

Give a measurement on $|\psi_{x_1x_2x_3x_4}\rangle$ that determines with certainty in which case we are.

Fidelity relation

Exercise 4. *Our goal is to show the following result given in class: for any quantum states ρ, σ, we have*

$$\max_{\zeta}\{\frac{1}{2}F^2(\rho,\zeta) + \frac{1}{2}F^2(\zeta,\sigma)\} = \frac{1}{2} + \frac{F(\rho,\sigma)}{2}. \tag{1}$$

1. *Show that for any angles $\alpha, \beta \in [0, \pi/2]$*

   $$\cos(\alpha+\beta) \geq \cos^2(\alpha) + \cos^2(\beta) - 1.$$

   (Hint: you can use the following inequality that comes from the concavity of the cos function on $[0,\pi]$:

   $$\forall x, y \in [0,\pi] : \cos(\frac{x+y}{2}) \geq \frac{1}{2}\left(\cos(x) + \cos(y)\right).$$

   as well as known trigonometric equalities)

2. *Using the angle distance, show that*

   $$\max_{\zeta}\{\frac{1}{2}F^2(\rho,\zeta) + \frac{1}{2}F^2(\zeta,\sigma)\} \leq \frac{1}{2} + \frac{F(\rho,\sigma)}{2}.$$

3. *For any states ρ, σ, show that there exists ζ st.*

   $$\frac{1}{2}F^2(\rho,\zeta) + \frac{1}{2}F^2(\zeta,\sigma) \geq \frac{1}{2} + \frac{F(\rho,\sigma)}{2}.$$

   (Hint: Consider purifications $|A\rangle, |B\rangle$ of ρ, σ from Ulhmann's theorem and look at the state " in between" $|A\rangle$ and $|B\rangle$.)