QII Exercise sheet 9

Notations. We write $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Exercise 1 (On Pauli matrices)**.**

1. *Let* $M = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$*. Show that there exist* $\alpha, \beta \in \mathbb{C}$ *st.* $M = \alpha X + \beta Y$*.*
2. *Let* M *be any* 2×2 *complex matrix. Show that there exist* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *st.* $M = \alpha I + \beta X + \gamma Y + \delta Z$*.*
3. *Compute* XZ, XY *and* YZ*. Let* $P_1, P_2 \in \{I, X, Y, Z\}$*. Show that* $tr(P_1P_2) = 0$ *if* $P_1 \neq P_2$ *and* $tr(P_1P_2) = 2$ *if* $P_1 = P_2$*.*
4. *Let* U *be any unitary matrix on* 1 *qubit. We can hence write* $U = \alpha I + \beta X + \gamma Y + \delta Z$*. Show that*
$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.$$
Hint: what is $UU^\dagger$? What is $tr(UU^\dagger)$?

Exercise 2 (Shor's code on an example)**.**

1. *Write the* 1*-qubit Hadamard transform* $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *as a linear combination of the Pauli matrices.*
2. *Suppose that an* H*-error occurs on the first qubit of* $S_9(\alpha|0\rangle + \beta|1\rangle) = \alpha|\widetilde{0}\rangle + \beta|\widetilde{1}\rangle$ *using Shor's* 9*-qubit code. Give the steps of the error correction procedure that corrects this error.*

Exercise 3. *Show that there cannot be a quantum error correcting code that encode one logical qubit into* $2n$ *physical qubits that can correct arbitrary error on* n *qubits.* Hint: think of the no cloning theorem.