Exercise sheet 1

Exercise 1. *Let* $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle$. *Compute* $\rho_A = Tr_B|\psi_{AB}\rangle\langle\psi_{AB}|$ *and* $\rho_B = Tr_A|\psi_{AB}\rangle\langle\psi_{AB}|$. *Write the result in matrix form.*

Solution:

$$|\psi_{AB}\rangle = \frac{\sqrt{3}}{2}\left(\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle\right)|0\rangle + \frac{1}{2}|01\rangle$$

so $\rho_A = \frac{3}{4}|\psi_1\rangle\langle\psi_1| + \frac{1}{4}|0\rangle\langle 0|$ with $|\psi_1\rangle = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle$. This gives

$$\rho_A = \frac{3}{4}\begin{pmatrix} 2/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 1/3 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & 1/4 \end{pmatrix}.$$

Notice that $|\psi_{AB}\rangle$ is symmetric with respect to swapping the A and B registers so $\rho_A = \rho_B$.

We can also use the direct formula to prove this. We first write

$$|\psi_{AB}\rangle\langle\psi_{AB}| = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2\sqrt{2}}(|00\rangle\langle 01| + |00\rangle\langle 10| + |01\rangle\langle 00| + |10\rangle\langle 00|) + \frac{1}{4}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|).$$

We then write

$$\begin{aligned}
\rho_A &= \sum_{j\in\{0,1\}} (I_A \otimes \langle j|)|\psi_{AB}\rangle\langle\psi_{AB}|(I_A \otimes |j\rangle) \\
&= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2\sqrt{2}}(|0\rangle\langle 1| + |1\rangle\langle 0|) + \frac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&= \frac{3}{4}|0\rangle\langle 0| + \frac{1}{2\sqrt{2}}(|0\rangle\langle 1| + |1\rangle\langle 0|) + \frac{1}{4}|1\rangle\langle 1|
\end{aligned}$$

□

Exercise 2.

1. *Consider any* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha^2 + \beta^2 = 1$, *as well as 2 states* $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ *and* $|\psi_1\rangle = \beta|0\rangle - \alpha|1\rangle$. *Show that:*

$$\frac{1}{2}(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

2. *Show that the state* $\rho = a|0\rangle\langle 0| + (1-a)|1\rangle\langle 1|$ *with* $0 < a < 1$ *admits an infinite amount of descriptions of the form* $\sum_i p_i |e_i\rangle\langle e_i|$*. You can have more than* 2 *terms in the sum, but the* p_i *must remain non-negative.*

Solution:

1. We have

$$|\psi_0\rangle\langle\psi_0| = \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix} \tag{1}$$

$$|\psi_1\rangle\langle\psi_1| = \begin{pmatrix} \beta^2 & -\alpha\beta \\ -\alpha\beta & \alpha^2 \end{pmatrix} \tag{2}$$

so

$$\frac{1}{2}\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

2. Assume $a \leq (1-a)$ (the other case will work the same). Take any state $|\psi_0\rangle, |\psi_1\rangle$ as in question 1. We have

$$\rho = a|\psi_0\rangle\langle\psi_0| + a|\psi_1\rangle\langle\psi_1| + (1-2a)|1\rangle\langle 1|,$$

and this can be written for any $|\psi_0\rangle, |\psi_1\rangle$ as in question 1.

□

Exercise 3. *We have* 2 *states* ρ_0, ρ_1*. Bob is given* ρ_b *for a randomly chosen* $b \in \{0,1\}$ *and his goal is to guess* b*. Give the optimal measurement as well as his probability of success for the following states.*

1. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.
2. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = |+\rangle\langle +|$.
3. $\rho_0 = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$, $\rho_1 = |+\rangle\langle +|$. *In this case, give only the success probability and not the optimal measurement.*

Solution:

1. $\Delta(\rho_0, \rho_1) = \frac{1}{2}$. The optimal measurement is the one in the computational basis and succeeds wp. $\frac{3}{4}$.

2. Since we have pure states, we know that $\Delta(\rho_0, \rho_1) = \sqrt{1 - |\langle 0|+\rangle|^2} = \frac{1}{\sqrt{2}}$. The best strategy to guess b succeeds therefore w.p. $\frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2(\pi/8)$. The optimal strategy is to measure in the $\{|v\rangle, |v^\perp\rangle\}$ basis with $|v\rangle = \cos(\pi/8)|0\rangle - \sin(\pi/8)|1\rangle$ and $|v^\perp\rangle = \sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle$. This strategy succeeds indeed wp. $\cos^2(\pi/8)$.

3. $M = \rho_0 - \rho_1 = \begin{pmatrix} 1/6 & -1/2 \\ -1/2 & -1/6 \end{pmatrix}$. Moreover

$$(\rho_0 - \rho_1)(\rho_0 - \rho_1)^\dagger = \begin{pmatrix} \frac{10}{36} & 0 \\ 0 & \frac{10}{36} \end{pmatrix}$$

so $\Delta(\rho_0, \rho_1) = \frac{\sqrt{10}}{6}$ and the success probability is $\frac{1}{2} + \frac{\sqrt{10}}{12}$.

□

Exercise 4. *We consider the following bit commitment between Alice and Bob, with a parameter α. In order to commit to a bit b, Alice chooses 2 random bits c_1, c_2 st. $c_1 \oplus c_2 = b$, creates*

$$\left|\psi^1_{A_1B_1}(c_1)\right\rangle = \sqrt{\alpha}\,|c_1\rangle\,|c_1\rangle + \sqrt{1-\alpha}\,|\overline{c_1}\rangle\,|\overline{c_1}\rangle$$
$$\left|\psi^2_{A_2B_2}(c_2)\right\rangle = \sqrt{\alpha}\,|c_2\rangle\,|c_2\rangle + \sqrt{1-\alpha}\,|\overline{c_2}\rangle\,|\overline{c_2}\rangle.$$

and sends registers B_1, B_2 to Bob. At the reveal phase, Alice reveals c_1, c_2, and the registers A_1, A_2. Bob checks that he has the state $\left|\psi^1_{A_1B_1}(c_1)\right\rangle$ in registers A_1, B_1 and the state $\left|\psi^2_{A_2B_2}(c_2)\right\rangle$ in registers A_2, B_2.

1. *Let $\rho_z = Tr_{A_1}|\psi^1_{A_1,B_1}(z)\rangle\langle\psi^1_{A_1,B_1}(z)| = Tr_{A_2}|\psi^2_{A_2,B_2}(z)\rangle\langle\psi^2_{A_2,B_2}(z)|$. Give the expression of ρ_0 and ρ_1 as a function of α.*

2. *Write a description of the following states: $\rho_0 \otimes \rho_0$, $\rho_0 \otimes \rho_1$, $\rho_1 \otimes \rho_0$, $\rho_1 \otimes \rho_1$.*

3. *Let ξ_b be the state that Bob receives from Alice after the commit phase. Show that*

$$\xi_0 = \frac{1}{2}(\alpha^2 + (1-\alpha)^2)\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) + (\alpha(1-\alpha))\left(|01\rangle\langle 01| + |10\rangle\langle 10|\right)$$
$$\xi_1 = \frac{1}{2}(\alpha^2 + (1-\alpha)^2)\left(|01\rangle\langle 01| + |10\rangle\langle 10|\right) + (\alpha(1-\alpha))\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

4. *Assume Bob wants to guess b from ξ_b after the commit phase. What is his optimal probability P_B^* of guessing b? Give a measurement that achieves this optimal probability.*

5. *Recall that Alice's optimal cheating probability is $P_A^* = \frac{1}{2} + \frac{1}{2}F(\xi_0, \xi_1)$. Compute this cheating probability. For what value of α do we have $P_A^* = P_B^*$?*

6. *Find a strategy for Alice that allows her to reveal both $b = 0$ and $b = 1$, each with probability P_A^**

Solution:

1. $$\rho_z = \alpha|z\rangle\langle z| + (1-\alpha)|\overline{z}\rangle\langle\overline{z}|.$$

2. $$\rho_{z_1} \otimes \rho_{z_2} = \alpha^2|z_1 z_2\rangle\langle z_1 z_2| + \alpha(1-\alpha)\left(|z_1\overline{z_2}\rangle\langle z_1\overline{z_2}| + |\overline{z_1}z_2\rangle\langle\overline{z_1}z_2|\right) + (1-\alpha)^2|\overline{z_1 z_2}\rangle\langle\overline{z_1 z_2}|.$$

3. $\xi_0 = \frac{1}{2}(\rho_0 \otimes \rho_0) + \frac{1}{2}(\rho_1 \otimes \rho_1)$ and $\xi_1 = \frac{1}{2}(\rho_0 \otimes \rho_1) + \frac{1}{2}(\rho_1 \otimes \rho_0)$. We obtain the result by pluggin in the expression from the previous question.

4. $P_B^* = \frac{1}{2} + \frac{\Delta(\xi_0,\xi_1)}{2}$. We have
$$\Delta(\xi_0, \xi_1) = 2 \cdot \frac{1}{2}\left(\alpha^2 + (1-\alpha)^2 - 2\alpha(1-\alpha)\right) = (2\alpha - 1)^2.$$
and $P_B^* = \frac{1}{2} + \frac{1}{2}(2\alpha - 1)^2$. You can get this probability by measuring in the computational basis, and outputting $b = c_1 \oplus c_2$ where c_1, c_2 are the 2 outcomes.

5. $$F(\xi_0, \xi_1) = 4\sqrt{\frac{1}{2}\left((\alpha^2 + (1-\alpha^2)) \cdot (\alpha(1-\alpha))\right)}.$$
From there, we have
$$P_A^* = \frac{1}{2} + 2\sqrt{\frac{1}{2}\left((\alpha^2 + (1-\alpha^2)) \cdot (\alpha(1-\alpha))\right)}.$$

6. Didn't have time to write it, sorry :(.

□