Exercise sheet 1

Exercise 1. *Let* $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle$*. Compute* $\rho_A = Tr_B|\psi_{AB}\rangle\langle\psi_{AB}|$ *and* $\rho_B = Tr_A|\psi_{AB}\rangle\langle\psi_{AB}|$*. Write the result in matrix form.*

Exercise 2.

1. *Consider any* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha^2+\beta^2 = 1$*, as well as 2 states* $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ *and* $|\psi_1\rangle = \beta|0\rangle - \alpha|1\rangle$*. Show that:*
$$\frac{1}{2}\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|\right) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$
2. *Show that the state* $\rho = a|0\rangle\langle 0| + (1-a)|1\rangle\langle 1|$ *with* $0 < a < 1$ *admits an infinite amount of descriptions of the form* $\sum_i p_i|e_i\rangle\langle e_i|$*. You can have more than 2 terms in the sum, but the* p_i *must remain non-negative.*

Exercise 3. *We have 2 states* ρ_0, ρ_1*. Bob is given* ρ_b *for a randomly chosen* $b \in \{0,1\}$ *and his goal is to guess* b*. Give the optimal measurement as well as his probability of success for the following states.*

1. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$.
2. $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = |+\rangle\langle +|$.
3. $\rho_0 = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$, $\rho_1 = |+\rangle\langle +|$. *In this case, give only the success probability and not the optimal measurement.*

Exercise 4. *We consider the following bit commitment between Alice and Bob, with a parameter* α*. In order to commit to a bit* b*, Alice chooses 2 random bits* c_1, c_2 *st.* $c_1 \oplus c_2 = b$*, creates*

$$\left|\psi^1_{A_1B_1}(c_1)\right\rangle = \sqrt{\alpha}\,|c_1\rangle\,|c_1\rangle + \sqrt{1-\alpha}\,|\overline{c_1}\rangle\,|\overline{c_1}\rangle$$
$$\left|\psi^2_{A_2B_2}(c_2)\right\rangle = \sqrt{\alpha}\,|c_2\rangle\,|c_2\rangle + \sqrt{1-\alpha}\,|\overline{c_2}\rangle\,|\overline{c_2}\rangle.$$

and sends registers B_1, B_2 *to Bob. At the reveal phase, Alice reveals* c_1, c_2*, and the registers* A_1, A_2*. Bob checks that he has the state* $\left|\psi^1_{A_1B_1}(c_1)\right\rangle$ *in registers* A_1, B_1 *and the state* $\left|\psi^2_{A_2B_2}(c_2)\right\rangle$ *in registers* A_2, B_2*.*

1. *Let* $\rho_z = Tr_{A_1}|\psi^1_{A_1,B_1}(z)\rangle\langle\psi^1_{A_1,B_1}(z)| = Tr_{A_2}|\psi^2_{A_2,B_2}(z)\rangle\langle\psi^2_{A_2,B_2}(z)|$*. Give the expression of* ρ_0 *and* ρ_1 *as a function of* α*.*

2. *Write a description of the following states:* $\rho_0 \otimes \rho_0$, $\rho_0 \otimes \rho_1$, $\rho_1 \otimes \rho_0$, $\rho_1 \otimes \rho_1$.

3. *Let* ξ_b *be the state that Bob receives from Alice after the commit phase. Show that*

$$\xi_0 = \frac{1}{2}(\alpha^2 + (1-\alpha)^2)\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) + (\alpha(1-\alpha))\left(|01\rangle\langle 01| + |10\rangle\langle 10|\right)$$
$$\xi_1 = \frac{1}{2}(\alpha^2 + (1-\alpha)^2)\left(|01\rangle\langle 01| + |10\rangle\langle 10|\right) + (\alpha(1-\alpha))\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

4. *Assume Bob wants to guess* b *from* ξ_b *after the commit phase. What is his optimal probability* P_B^* *of guessing* b*? Give a measurement that achieves this optimal probability.*

5. *Recall that Alice's optimal cheating probability is* $P_A^* = \frac{1}{2} + \frac{1}{2}F(\xi_0, \xi_1)$. *Compute this cheating probability. For what value of* α *do we have* $P_A^* = P_B^*$*?*

6. *Find a strategy for Alice that allows her to reveal both* $b = 0$ *and* $b = 1$*, each with probability* P_A^*