Exercise sheet 2

Notations. Every logarithm is in base 2.

Exercise 1. *Consider 2 discrete probability function* $p = (p_1, \ldots, p_n)$ *and* $q = (q_1, \ldots, q_m)$. *So we have* $p_i \geq 0, \sum_i p_i = 1$, *and* $q_i \geq 0, \sum_i q_i = 1$. *Consider the direct product distribution* $r = (r_{1,1}, \ldots, r_{n,m})$ *where* $r_{i,j} = p_i q_j$. *Show that*

$$H(r) = H(p) + H(q).$$

Exercise 2. *For each state* $|\psi_{AB}\rangle$, *give the reduced density matrices* $\rho_A = tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$ *and* $\rho_B = tr_A(|\psi_{AB}\rangle\langle\psi_{AB}|)$. *You can write your answers in Dirac's "ket,bra" notation or in matrix form. Compute also* $H(\rho_A)$ *in each case. You can use* $\log_2(3) \approx 1.585$.

1. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0-\rangle + |1+\rangle\right).$
2. $|\psi_{AB}\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right).$
3. $|\psi_{AB}\rangle = \sqrt{\frac{3}{8}}\,|00\rangle + \sqrt{\frac{3}{8}}\,|01\rangle - \sqrt{\frac{1}{8}}\,|10\rangle + \sqrt{\frac{1}{8}}\,|11\rangle.$

Exercise 3. *Consider 2 quantum mixed states* ρ *and* σ *which are orthogonal. This means we can write* ρ *and* σ *in their spectral decomposition*

$$\rho = \sum_i p_i |e_i\rangle\langle e_i|$$
$$\sigma = \sum_j q_j |f_j\rangle\langle f_j|$$

where each $p_i, q_j > 0$ *and* $\sum_i p_i = \sum_j q_j = 1$, *and the orthogonality constraint gives* $\forall i, j \ \ \langle e_i|f_j\rangle = 0$ *(or equivalently* $|e_i\rangle \perp |f_j\rangle$*). Let also* $I_\rho = \sum_i |e_i\rangle\langle e_i|$ *and* $I_\sigma = \sum_j |f_j\rangle\langle f_j|$

1. *Show that* $\rho \cdot \log(\sigma) = \rho \cdot I_\sigma = \mathbf{0}$ *where* $\mathbf{0}$ *is the all 0 matrix.*
2. *Let* $\xi = r\rho + (1-r)\sigma$ *with* $r \in [0,1]$. *Show that* $\log(\xi) = \log(r\rho) + \log((1-r)\sigma)$.
3. *Let* $r \geq 0$. *Show that* $\log(r\rho) = \log(\rho) + \log(r) I_\rho$.
4. *Let* $\xi = r\rho + (1-r)\sigma$ *with* $r \in [0,1]$. *Show that*

$$H(\xi) = H_2(r) + rH(\rho) + (1-r)H(\sigma)$$

where $H_2(r) = -r\log(r) - (1-r)\log(1-r)$.

Exercise 4. *Let* $\sigma_{AB} = r|0\rangle\langle 0|_A \otimes \rho_B^0 + (1-r)|1\rangle\langle 1|_A \otimes \rho_B^1$ *be a quantum mixed state on 2 registers. Using the previous question, show that*

$$I(A:B)_\sigma = H(r\rho_B^0 + (1-r)\rho_B^1) - \left(rH(\rho_B^0) + (1-r)H(\rho_B^1)\right)$$

Find matrices ρ_B^0, ρ_B^1 *st.* $I(A:B)_\sigma = 0$*. Find others st.* $I(A:B)_\sigma = H_2(r)$*.*