Exercise sheet 3

Notations. Every logarithm is in base 2.

Exercise 1. *Consider 2 discrete probability function* $p = (p_1, \dots, p_n)$ *and* $q = (q_1, \dots, q_m)$. *So we have* $p_i \geq 0, \sum_i p_i = 1$, *and* $q_i \geq 0, \sum_i q_i = 1$. *Consider the direct product distribution* $r = (r_{1,1}, \dots, r_{n,m})$ *where* $r_{i,j} = p_i q_j$. *Show that*

$$H(r) = H(p) + H(q).$$

Solution: We write

$$\begin{aligned} H(r) &= \sum_{i,j} -r_{i,j} \log_2(r_{i,j}) \\ &= \sum_{i,j} -r_{i,j}(\log_2(p_i) + \log_2(q_j)) \\ &= -\sum_{i,j} r_{ij} \log_2(p_i) - \sum_{i,j} r_{ij} \log_2(q_j) \\ &= -\sum_i p_i (\sum_j q_j) \log_2(p_i) - \sum_j q_j (\sum_i p_i) \log_2(q_j) \\ &= -\sum_i p_i \log_2(p_i) - \sum_j q_j \log_2(q_j) = H(p) + H(q) \end{aligned}$$

□

Exercise 2. *For each state* $|\psi_{AB}\rangle$, *give the reduced density matrices* $\rho_A = tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$ *and* $\rho_B = tr_A(|\psi_{AB}\rangle\langle\psi_{AB}|)$. *You can write your answers in Dirac's "ket,bra" notation or in matrix form. Compute also* $H(\rho_A)$ *in each case. You can use* $\log_2(3) \approx 1.585$.

1. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|0-\rangle + |1+\rangle)$.
2. $|\psi_{AB}\rangle = \frac{1}{2} (|00\rangle - |01\rangle - |10\rangle + |11\rangle)$.
3. $|\psi_{AB}\rangle = \sqrt{\frac{3}{8}} |00\rangle + \sqrt{\frac{3}{8}} |01\rangle - \sqrt{\frac{1}{8}} |10\rangle + \sqrt{\frac{1}{8}} |11\rangle$.

Solution:

1. $\rho_A = \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|)$. $\rho_B = \frac{1}{2} (|-\rangle\langle -| + |+\rangle\langle +|) = \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|)$. $S(\rho_A) = 1$.

2. $|\psi_{AB}\rangle = |-\rangle\,|-\rangle$ so $\rho_A = \rho_B = |-\rangle\langle -|$. $S(\rho_A) = 0$.

3. We can rewrite $|\psi_{AB}\rangle = \sqrt{\frac{3}{4}}\,(|0\rangle\,|+\rangle) - \sqrt{\frac{1}{4}}\,(|1\rangle\,|-\rangle)$. So $\rho_A = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$ and $\rho_B = \frac{3}{4}|-\rangle\langle -| + \frac{1}{4}|+\rangle\langle +|$.

$$S(\rho_A) = \frac{3}{4}\log_2(4/3) + \frac{1}{4}\log_2(4) = \frac{3}{4}\,(2 - \log_2(3)) + \frac{1}{4}2 = 2 - \frac{3\log_2(3)}{4} \approx 0.811.$$

□

Exercise 3. *Consider 2 quantum mixed states ρ and σ which are orthogonal. This means we can write ρ and σ in their spectral decomposition*

$$\rho = \sum_i p_i |e_i\rangle\langle e_i|$$
$$\sigma = \sum_j q_j |f_j\rangle\langle f_j|$$

where each $p_i, q_j > 0$ and $\sum_i p_i = \sum_j q_j = 1$, and the orthogonality constraint gives $\forall i,j \;\; \langle e_i|f_j\rangle = 0$ (or equivalently $|e_i\rangle \perp |f_j\rangle$). Let also $I_\rho = \sum_i |e_i\rangle\langle e_i|$ and $I_\sigma = \sum_j |f_j\rangle\langle f_j|$

1. *Show that $\rho \cdot \log(\sigma) = \rho \cdot I_\sigma = \mathbf{0}$ where $\mathbf{0}$ is the all 0 matrix.*

2. *Let $\xi = r\rho + (1-r)\sigma$ with $r \in [0,1]$. Show that $\log(\xi) = \log(r\rho) + \log((1-r)\sigma)$.*

3. *Let $r \geq 0$. Show that $\log(r\rho) = \log(\rho) + \log(r)I_\rho$.*

4. *Let $\xi = r\rho + (1-r)\sigma$ with $r \in [0,1]$. Show that*

$$H(\xi) = H_2(r) + rH(\rho) + (1-r)H(\sigma)$$

where $H_2(r) = -r\log(r) - (1-r)\log(1-r)$.

Solution:

1. We write

$$\rho \cdot \log(\sigma) = \sum_i p_i |e_i\rangle\langle e_i| \cdot \sum_j \log(q_j)|f_j\rangle\langle f_j| = \sum_{i,j} p_i \log(q_j)\,|e_i\rangle\,\langle e_i|f_j\rangle\,\langle f_j| = \mathbf{0}.$$

since each $\langle e_i|f_j\rangle = 0$. The proof $\rho \cdot I_\sigma = \mathbf{0}$ follows the same line.

2. When we write $\xi = \sum_i rp_i|e_i\rangle\langle e_i| + \sum_j (1-r)q_j|e_j\rangle\langle e_j|$ this is actually the spectral decomposition of ξ. Indeed the $|e_i\rangle$ are pairwise orthogonal because ρ is in spectral decomposition, the $|f_j\rangle$ are pairwise orthogonal because σ is in spectral decomposition and we also have each $|e_i\rangle \perp |f_j\rangle$ because of the orthogonality constraint. So we have

$$\log(\xi) = \sum_i \log(rp_i)|e_i\rangle\langle e_i| + \sum_j \log((1-r)q_j)|e_j\rangle\langle e_j| = \log(r\rho) + \log((1-r)\sigma)$$

3. We write

$$\log(r\rho) = \sum_i \log(rp_i)|e_i\rangle\langle e_i| = \sum_i (\log(r) + \log(p_i))|e_i\rangle\langle e_i| = \log(r)I_\rho + \log(\rho).$$

4. We have

$$\begin{aligned} H(\xi) &= -Tr(\xi\log(\xi)) = Tr(\xi\log(r\rho + (1-r)\sigma)) = Tr(\xi\log(r\rho) + \xi\log((1-r)\sigma) \\ &= -Tr\Big(\xi(\log(r)I_\rho + \log(\rho)) + \xi(\log(1-r)I_\sigma + \log(\sigma))\Big) \end{aligned}$$

Then, we write

$$\begin{aligned} \xi\left(\log(r)I_\rho + \log(\rho)\right) &= (r\rho + (1-r)\sigma)\left(\log(r)I_\rho + \log(\rho)\right) \\ &= r\log(r)\rho + r\rho\log(\rho) \end{aligned}$$

where we use the previous questions as well as $\rho I_\rho = \rho$. Similarly, we obtain

$$\xi(\log(1-r)I_\sigma + \log(\sigma)) = ((1-r)\log(1-r))\,\sigma + (1-r)\sigma\log(\sigma).$$

Putting everything together, we obtain

$$\begin{aligned} H(\xi) &= -Tr\left(r\log(r)\rho + r\rho\log(\rho) + ((1-r)\log(1-r))\,\sigma + (1-r)\sigma\log(\sigma)\right) \\ &= -r\log(r) + rH(\rho) - (1-r)\log(1-r) + (1-r)\log(\sigma) \\ &= H_2(r) + rH(\rho) + (1-r)\log(\sigma) \end{aligned}$$

where we used $Tr(\rho) = Tr(\sigma) = 1$.

□

Exercise 4. *Let $\sigma_{AB} = r|0\rangle\langle 0|_A \otimes \rho_B^0 + (1-r)|1\rangle\langle 1|_A \otimes \rho_B^1$ be a quantum mixed state on 2 registers. Using the previous question, show that*

$$I(A:B)_\sigma = H(r\rho_B^0 + (1-r)\rho_B^1) - \left(rH(\rho_B^0) + (1-r)H(\rho_B^1)\right)$$

Find matrices ρ_B^0, ρ_B^1 st. $I(A:B)_\sigma = 0$. Find others st. $I(A:B)_\sigma = H_2(r)$.

Solution: Let $\sigma_A = Tr_B(\sigma_{AB})$ and $\sigma_B = Tr_A(\sigma_{AB})$. We write

$$I(A:B)_\sigma = H(\sigma_A) + H(\sigma_B) - H(\sigma)$$

We have $\sigma_1 = r|0\rangle\langle 0| + (1-r)|1\rangle\langle 1|$ so $H(\sigma_1) = H_2(r)$. Then, $\sigma_2 = r\rho^0 + (1-r)\rho^1$. Finally, we have $\sigma = r|0\rangle\langle 0|\rho^0 + (1-r)|1\rangle\langle 1|\rho^1$. We have $|0\rangle\langle 0|\rho^0 \perp |1\rangle\langle 1|\rho^1$ (for any ρ^0, ρ^1 since they are orthogonal already from the first register). From the previous exercice, we therefore have

$$H(\sigma) = H_2(r) + rH(|0\rangle\langle 0|\rho^0) + (1-r)H(|1\rangle\langle 1|\rho^1) = H_2(r) + rH(\rho^0) + (1-r)H(\rho^1).$$

Putting everything together, we have indeed

$$I(A:B)_\sigma = H_2(r) + H(r\rho^0 + (1-r)\rho^1) - \big(H_2(r) + rH(\rho^0) + (1-r)H(\rho^1)\big)$$

which gives the desired result. Take $\rho^0 = \rho^1 = |0\rangle\langle 0|$ to have $I(A:B)_\sigma = 0$. Take $\rho^i = |i\rangle\langle i|$ to have $I(A:B) = H_2(r)$. □