The Algebraic Freelunch: Efficient Gröbner Basis Attacks Against Arithmetization-Oriented Primitives

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COSMIQ Seminar, Paris

Anemoi
Crypto23

Griffin
Crypto23

ArionHash
arXiv

Anemoi
Crypto23

~~Griffin~~
~~Crypto23~~

ArionHash
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Full-round break
of some instances

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Maybe full-round break?

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Full-round break
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~~arXiv~~

Full-round break
of some instances

Three main improvements on previous cryptanalysis:

1. Free Gröbner basis for some monomial orders.
2. Better approach to solving the system than generic FGLM variants.
3. Bypassing the first few rounds of Griffin and Arion with symmetric-like techniques.

INTRODUCTION TO GRÖBNER BASES

ARITHMETIZATION-ORIENTED PRIMITIVES

FREELUNCH SYSTEMS FOR FREE GRÖBNER BASES

SOLVING THE SYSTEM GIVEN A GRÖBNER BASIS

WHAT DO WE WANT?

Consider a multivariate polynomial ring $\mathbb{F}[x_1, x_2, \ldots, x_N]$.
We want to solve:

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ p_2(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

What do we want?

$$\begin{cases} m_{1,1}x_1 + \cdots + m_{1,N}x_N + a_1 = 0 \\ m_{2,1}x_1 + \cdots + m_{2,N}x_N + a_2 = 0 \\ \vdots \\ m_{k,1}x_1 + \cdots + m_{k,N}x_N + a_k = 0 \end{cases}$$

Polynomials of **degree 1**: Linear system ⇒ **Linear algebra**.

WHAT DO WE WANT?

$$\begin{cases} p_1(x_1) = 0 \\ p_2(x_1) = 0 \\ \qquad\vdots \\ p_k(x_1) = 0 \end{cases}$$

One variable: Univariate root finding ⇒ **Euclidian division** (for Berlekamp-Rabin algorithm).

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Several variables, high degree: **Linear algebra** + **Euclidian division** (F4/F5, FGLM, Fast-FGLM...).

The Problem with Multivariate

- Euclidian division on **integers**:

$$a = bq + r \ , \ \ 0 \leq r < b.$$

Division of 13 by 3:

$$13 = 4 \times 3 + 1.$$

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- Euclidian division on **univariate polynomials** ($\mathbb{F}[X]$):

$$A = BQ + R \ , \ \ \deg(R) < \deg(B).$$

Division of $X^3 + X + 1$ by X:

$$X^3 + X + 1 = (X^2 + 1)X + 1.$$

THE PROBLEM WITH MULTIVARIATE

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Division of x by $x + y$ in $\mathbb{F}[x, y]$:

$$x = 0 \cdot (x+y) + x$$

or

$$x = 1 \cdot (x+y) - y \; ?$$

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or

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Need to define a **monomial ordering**.

$\Longrightarrow$ Division steps determined by **leading monomials (LM)**.

Monomial orderings

In $\mathbb{F}[x, y, z]$:

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z \ , \quad x^{1000} \ ? \quad y$$

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- **Weighted Graded LEX:** Compare the **weighted sum** of degrees, then lex if equality. Examples for $x <_{\text{lex}} y <_{\text{lex}} z$ and $\mathbf{wt}(x) = 6, \mathbf{wt}(y) = 1, \mathbf{wt}(z) = 2$:

$$x \quad ? \quad yz^{2}$$

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$$x \;\; >_{\text{wglex}} yz^2 \text{ because } \mathbf{wt}(x) = 6 \text{ and } \mathbf{wt}(yz) = \mathbf{wt}(y) + 2\mathbf{wt}(z) = 5 \;.$$

$$x^2 \quad ? \quad z^6$$

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$$x^2 \;\; <_{\text{wglex}} z^6 \text{ because } \mathbf{wt}(x^2) = \mathbf{wt}(z^6) = 12 \text{ and } x^2 <_{\text{lex}} z^6 \; .$$

THE PROBLEM... STILL.

Consider a system $\{p_1, \ldots, p_k\}$.
$\implies$ Division of a polynomial p by $\{p_1, \ldots, p_k\}$ for some ordering: **final remainder can depend on the choice of divisors!**

The Problem... Still.

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Example: in $\mathbb{F}[x, y]$ with **lex** ordering ($x <_{lex} y$), divide y^2 by $\{y^2 - 1, y - x\}$.

$$\begin{array}{ll|ll} y^2 & & y^2 & \\ \Downarrow & \text{red. by } y^2 - 1 & \Downarrow & \text{red. by } y - x \\ 1 & & xy & \\ \Downarrow & \text{no further red.} & \Downarrow & \text{red. by } y - x \\ 1 & & x^2 & \end{array}$$

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The solution: Gröbner Bases.

WHAT IS A GRÖBNER BASIS?

Let $G = \{p_1, \ldots, p_k\}$ and $<$ a monomial ordering.

DEFINITION

G is a Gröbner basis iff reduction defined by $<$ of any polynomial P does not depend on the order chosen for the reductors.

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USEFUL PROPOSITION

If $LM_<(p_1), \ldots, LM_<(p_k)$ are pairwise ***coprime*** *(e.g. x^2 and y), then G is a Gröbner basis.*

Gröbner Basis - Examples

In $\mathbb{F}[x, y]$:

- $\{y^2 - 1, y - x\}$ is not a Gröbner basis for **lex** order with $x < y$ (previous example).

GRÖBNER BASIS - EXAMPLES

In $\mathbb{F}[x, y]$:

- $\{y^2 - 1, y - x\}$ is not a Gröbner basis for **lex** order with $x < y$ (previous example).
- However, it is a Gröbner basis for **lex** order with $x > y$. Proof: $\mathrm{LM}(y^2 - 1) = y^2$ and $\mathrm{LM}(y - x) = x$ are **coprime**.

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- $\{y^3 + x, y^3 + x^2\}$ is not a Gröbner basis for any **lex** or **deglex** order.

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- $\{y^3 + x, y^3 + x^2\}$ is not a Gröbner basis for any **lex** or **deglex** order.
- However, it is a Gröbner basis for **weighted degree** orders with $\mathbf{wt}(x) = 2$ and $\mathbf{wt}(y) = 1$, as then $\mathrm{LM}(y^3 + x) = y^3$ and $\mathrm{LM}(y^3 + x^2) = x^2$ are **coprime**.

Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

1. Define system

Generic System Solving

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4. Find the roots in $\mathbb{F}_q$ of g_N^* with univariate methods, etc.

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Remark: Steps 2 and 3 are both computationally costly, but not for the same reasons. For most AOPs, step 2 dominates, **but we can skip it**.

Introduction to Gröbner Bases

Arithmetization-Oriented Primitives

Freelunch Systems for Free Gröbner Bases

Solving the System given a Gröbner Basis

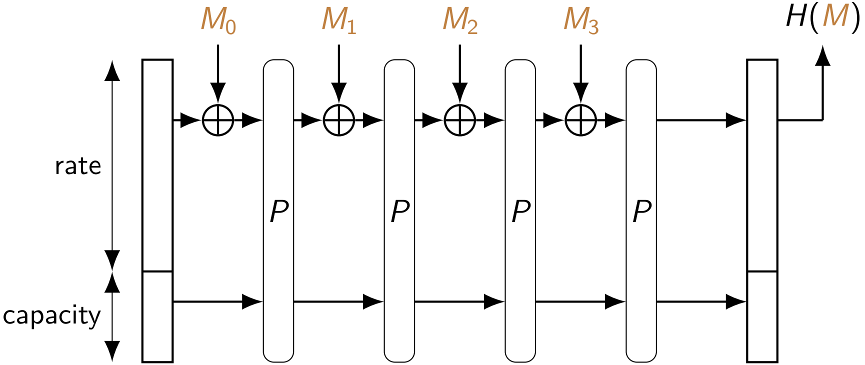
WHAT IS A HASH FUNCTION?

DEFINITION

A hash function is a function that maps an input of **any size** in $\mathbb{F}_q$ to an element of $\mathbb{F}_q^r$ for a **fixed** integer r.

- **collision resistance**: hard to find $x,\ y$ such that $H(x) = H(y)$.
- **preimage resistance**: given $y \in \mathbb{F}_q^r$, hard to find x such that $H(x) = y$.
- **second preimage resistance**: given x, hard to find x' such that $H(x) = H(x')$.

SPONGE HASH FUNCTIONS



A sponge construction, originally designed for the standard **SHA-3**.
P is, for example, a **fixed-key Block Cipher**.

CICO Problem

CICO Problem of size c (capacity of the sponge) for permutation P:

$$P(*, \ldots, *, \underbrace{0, \ldots, 0}_{c \text{ elements}}) = (*', \ldots, *', \underbrace{0, \ldots, 0}_{c \text{ elements}})$$

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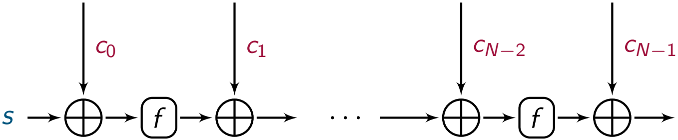
Solving CICO of size c gives collisions to the hash function.

⇒ Multivariate attack: solve CICO faster than brute-force attacks using a model of P.
⇒ We focus on $c = 1$.

$$P(x, *, \ldots, *, 0) = (*', \ldots, *', 0).$$

Block Cipher



The ever-popular Block Cipher construction.

ARITHMETIZATION-ORIENTED SYMMETRIC PRIMITIVES

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Classic	**Arithmetization-Oriented**
Binary operations	Arithmetic operations
Algebraically complex (for cheap)	Algebraically simple
Small field ($\mathbb{F}_{2^8}$)	Large field ($\mathbb{F}_q$, $q > 2^{64}$)
e.g. AES, SHA-3	e.g. Griffin, Anemoi

Arithmetization for Zero-Knowledge

- Implementation of ZK based on algebraic equations.
- **Low degree equations = Better performance**.

Function → Set of equations → Proof system

ARITHMETIZATION FOR ZERO-KNOWLEDGE

- Implementation of ZK based on algebraic equations.
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x^3	**AES** S-box	$x^{1/3}$
High ZK performance	**Low** ZK performance	**High** ZK performance
Low (algebraic) security	**High** (algebraic) security	**Good** (algebraic) security?
		(Low degree inverse: $y = x^\alpha$ vs $y^\alpha = x$)

QUICK OVERVIEW OF GRIFFIN, ARION, ANEMOI

Our targets:

Anemoi	**Griffin**	**ArionHash**
Crypto23	Crypto23	arXiv

- Griffin, ArionHash and AnemoiSponge are Arithmetization-Oriented families of hash functions.
- Based on Griffin-π, Arion-π and Anemoi families of permutations (all fixed-key block ciphers).

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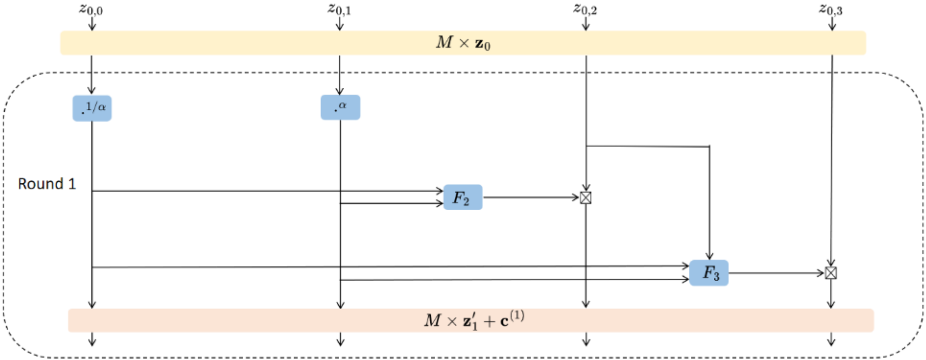
- Based on Griffin-π, Arion-π and Anemoi families of permutations (all fixed-key block ciphers).
- **Many** instances are defined: variable $\mathbb{F}_p$, number of branches, exponents for monomial permutations...

$\Longrightarrow$ We attack some instance better than others.

GRIFFIN-π - ROUND FUNCTION (4 BRANCHES)



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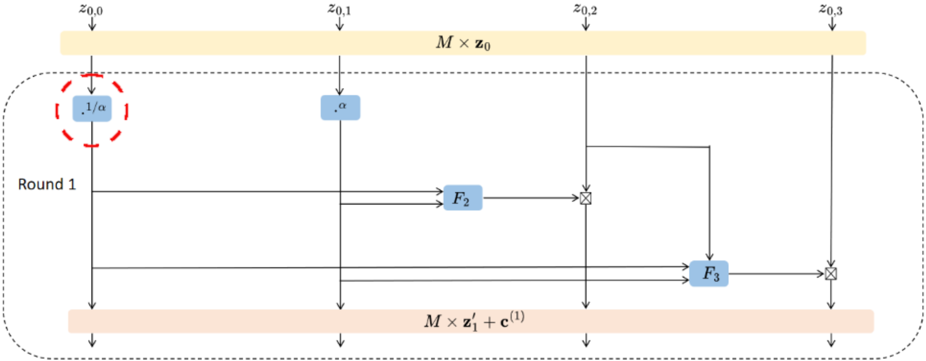

$\cdot^{1/\alpha}$ is the only high-degree operation $\implies$ add one variable per $\cdot^{1/\alpha}$.

Griffin-π - Model

- **CICO problem**: $\mathcal{G}_\pi(\cdots||0) = (\cdots||0)$.
  $\Longrightarrow$ One variable x_0 in the input. One equation for the output (last branch at 0).
- N_{rounds} equations of the form $x_i^\alpha = P_i(x_0, x_1, \ldots x_{i-1})$ $(\cdot^{1/\alpha}$ S-boxes).

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EXAMPLE ($\alpha = 3$, ONE ROUND)

$$x_1^3 = a x_0 + b$$

$$x_0^7 + c x_0^4 x_1 + d x_0 x_1^2 + \cdots = 0$$

GENERIC SYSTEM SOLVING

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1^*(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1}, x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

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2. Find a GB (F4/F5)
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Designers of Anemoi and Griffin base their security on the hardness of **Step 2**.

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2. ~~Find a GB (F4/F5)~~

$$\begin{cases} g_1^*(x_1,\ldots,x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1},x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

3. Change order to **lex** (FGLM)

4. Find the roots in $\mathbb{F}_q$ of g_N^* with univariate methods, etc.

Designers of Anemoi and Griffin base their security on the hardness of **Step 2**.

But we can skip it!

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Solving the System given a Gröbner Basis

GRIFFIN-π - MODEL

- **CICO problem**: $\mathcal{G}_\pi(\cdots||0) = (\cdots||0)$.
  $\Longrightarrow$ One variable x_0 in the input. One equation for the output (last branch at 0).
- N_{rounds} equations of the form $x_i^\alpha = P_i(x_0, x_1, \ldots x_{i-1})$ $\quad$ ($\cdot^{1/\alpha}$ S-boxes).

EXAMPLE ($\alpha = 3$, ONE ROUND)

$$x_1^3 = ax_0 + b$$

$$x_0^7 + cx_0^4x_1 + dx_0x_1^2 + \cdots = 0$$

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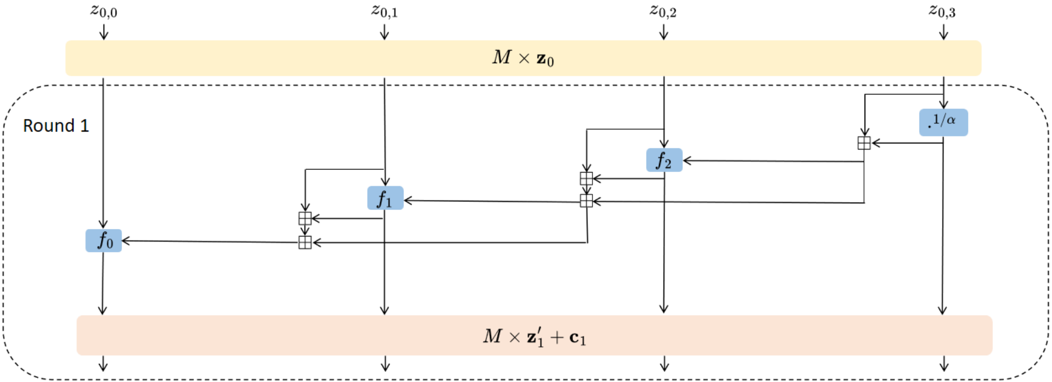
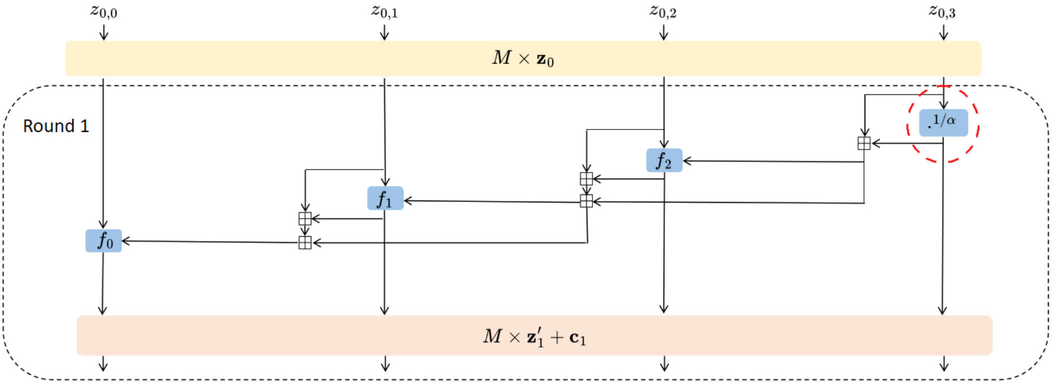
Observation: *x_1 has a lower degree than x_0 in the last equation.*
*$\implies$ In **grevlex**, the leading monomials are x_0^7 and x_1^3. $\implies$ **It's a Gröbner basis** ! (coprime leading monomials)*
*$\implies$ For more rounds, **grevlex** doesn't work. We need **weighted degree** orders, with* $\mathbf{wt}(x_0) = 1$ *and* $\mathbf{wt}(x_i) = 7^{i-1}$.

Arion-π - Round Function (4 branches)

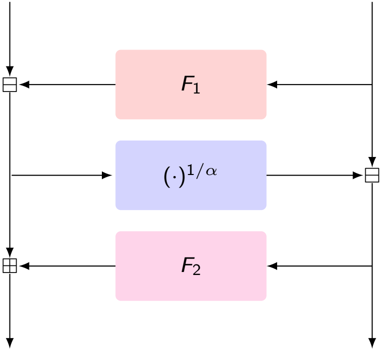


ARION-π - ROUND FUNCTION (4 BRANCHES)

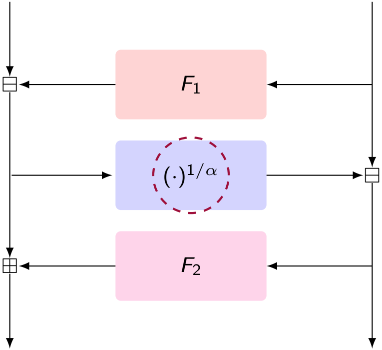


$\cdot^{1/\alpha}$ is the only high-degree operation $\implies$ add one variable per $\cdot^{1/\alpha}$.

Anemoi - Nonlinear layer (2 branches)



ANEMOI - NONLINEAR LAYER (2 BRANCHES)



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Anemoi - Model

Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0^2 + bx_0 + c$$

$$x_0x_1 + dx_1^2 + ex_0 + fx_1 + g = 0$$

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x_0^2 cancels out: this isn't a Gröbner basis for any order!

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Solution: multiply last equation by x_1^2 and reduce it by the first equation. We get:

$$p^*(x_0, x_1) = a x_0^3 + bd x_0^2 x_1 + \cdots$$

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$\implies$ The first equation and p^* are a Gröbner basis for some weighted order.
$\implies$ This adds a few parasitic solutions (corresponding to $x_1 = 0$), but not many.
$\implies$ This generalizes for more rounds (multiply the last polynomial by some of the x_i and reduce it). **Freelunch is saved**!

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FGLM IN A NUTSHELL

- Given a zero-dimensional ideal I, a Gröbner basis G_1 for I some ordering $<_1$, and an ordering $<_2$, FGLM computes a Gröbner basis G_2 for $<_2$ in $O(n_{var} D_I^3)$.
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- **This is interesting** because a GB in **lex** order **must have** a univariate polynomial in the smallest variable, which we can solve. (This corresponds to eliminating the other variables.)
- Free Gröbner basis, FGLM and symmetric techniques to bypass the first rounds is already enough to break some instances of Griffin and Arion.

FASTER CHANGE OF ORDER STRATEGY

- Idea from a 2022 paper by Jérémy Berthomieu, Vincent Neiger, Mohab Safey El Din.
- Strategy: for the smallest variable x, compute the characteristic polynomial χ of the linear operation $P \mapsto \mathrm{Red}_{<}(x \cdot P, G)$.
- $\chi(x) = 0$. Generically, this is **exactly** the univariate polynomial in x in the reduced GB of I in **lex** order.
- **Issue:** our systems **do not** verify an important property of the original paper.

COMPUTING THE MULTIPLICATION MATRIX

Step 1: Compute the matrix T of the linear operation in $\mathbb{F}[x_0, x_1, \ldots, x_N]$ that maps P to $x_0 \cdot P$.

- Need to reduce monomials of the form $x_0^{k_0+1} x_1^{k_1} \cdots x_N^{k_N}$. **We have no tight complexity estimate for this step**.

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- The matrix is sparse. If leading monomials are $x_0^{d_0}, \ldots, x_N^{d_N}$:

$$T_0 = \underbrace{\left(\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{array}\right.}_{(d_0-1)d_1\cdots d_N}\underbrace{\left|\begin{array}{c} * \\ \hline * \\ \vdots \\ \vdots \\ * \end{array}\right)}_{d_1\cdots d_N}$$

COMPUTING THE CHARACTERISTIC POLYNOMIAL

Step 2: Given T, compute $\det(XI - M)$.

$\implies$ T is sparse. With block matrix reasoning, this reduces to computing the determinant of a polynomial matrix of size $D_1 = d_1 \cdots d_N$.

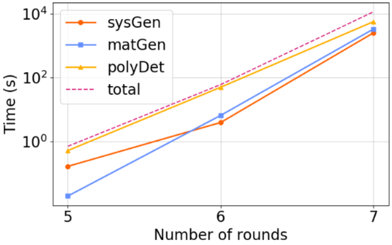
$\implies$ In Griffin and Arion, d_0 is by far the highest degree, so this reduces complexity by a lot.

$\implies$ This can be computed with fast linear algebra, in $\mathcal{O}(d_0 \log(d_0)^2 d_1^{\omega} \cdots d_N^{\omega})$.

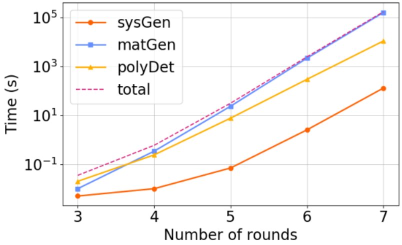
OUR FULL ALGORITHM

1. `sysGen`: Compute the Freelunch system and the order for a free Gröbner basis.
2. `matGen`: Compute the multiplication matrix T. **Complexity hard to evaluate**.
3. `polyDet`: Compute the characteristic polynomial χ of T.
   $\Longrightarrow$ **Longest step aside from** `matGen`.
4. `uniSol`: Find roots of χ with Berlekamp-Rabin in $\mathcal{O}(D_I \log(D_I) \log(pD_I))$.

Experimental Results



Complexity of Griffin
(7 out of 10 rounds, α=3)



Complexity of Anemoi
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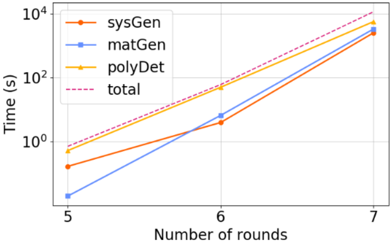

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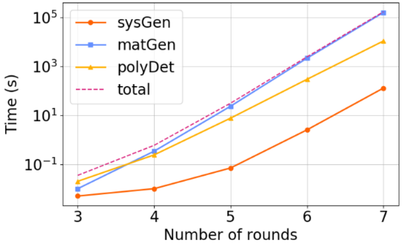

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$\Longrightarrow$ For Griffin, `polyDet` upper-bounds the others up to 7 rounds.

$\Longrightarrow$ For Anemoi, `matGen` is the bottleneck.

CONCLUSION

- Arithmetization-Oriented hash functions should not base their security on the complexity of finding a Gröbner basis (F4/F5).
- Instead, focus on the growth of D_I with the number of rounds (impacts the complexity of solving algorithms).
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- CICO on more than one branch?

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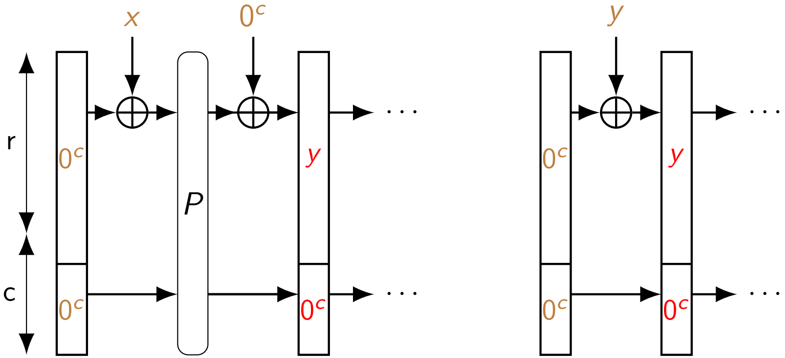
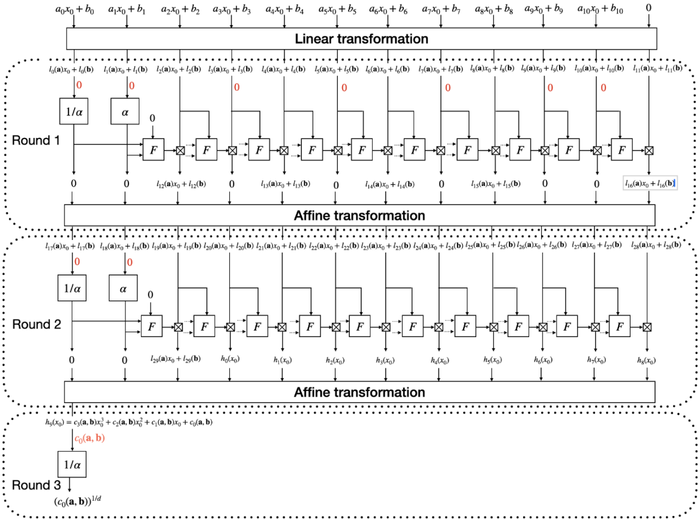
THANK YOU FOR YOUR ATTENTION!

COLLISION FROM THE CICO PROBLEM

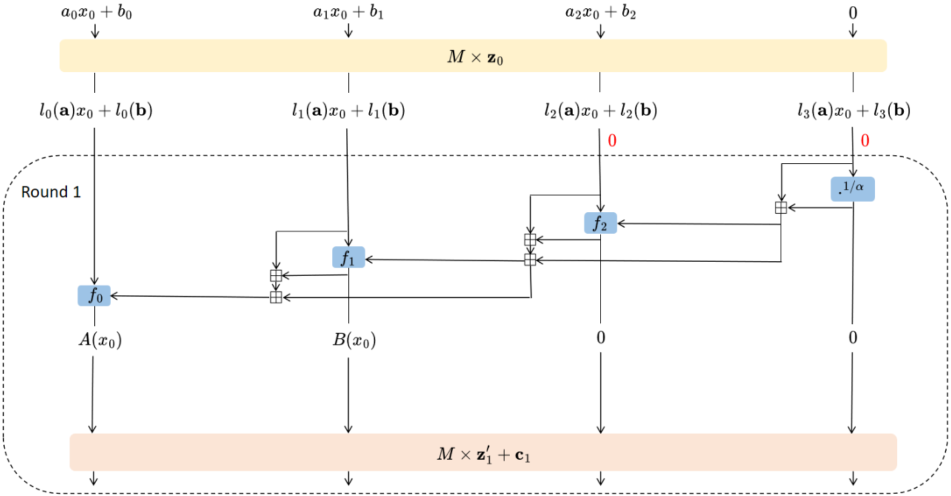
- Suppose you know x such that $P(x \mid\mid 0^c) = (y \mid\mid 0^c)$.



GRIFFIN TRICK



ARION TRICK



$a_0x_0 + b_0$ $a_1x_0 + b_1$ $a_2x_0 + b_2$ 0

$M \times \mathbf{z}_0$

$l_0(\mathbf{a})x_0 + l_0(\mathbf{b})$ $l_1(\mathbf{a})x_0 + l_1(\mathbf{b})$ $l_2(\mathbf{a})x_0 + l_2(\mathbf{b})$ $l_3(\mathbf{a})x_0 + l_3(\mathbf{b})$

0 0

Round 1

$\cdot^{1/\alpha}$

f_2

f_1

f_0

$A(x_0)$ $B(x_0)$ 0 0

$M \times \mathbf{z}_1' + \mathbf{c}_1$