New uses in Symmetric Cryptography:
An equation between Practical needs and Mathematical concepts

Clémence Bouvier [1,2]

including joint works with Augustin Bariant[2], Pierre Briaud[1,2], Anne Canteaut[2], Pyrros Chaidos[3], Gaëtan Leurent[2], Léo Perrin[2] and Vesselin Velichkov[4,5]

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SORBONNE UNIVERSITÉ

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Some definitions

Definition

Cryptology: science of secret messages.
Eth. from the Greek kryptós *(hidden) and* lógos *(word).*

$$\text{Cryptology} = \text{Cryptography} + \text{Cryptanalysis}$$

Definition

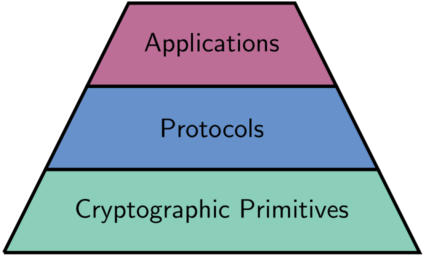
Cryptography: methods used to transform a plaintext in an unintelligible one.

Definition

Cryptanalysis: methods used to recover the plaintext from the ciphertext.

Cryptographic primitives

A primitive is the building block of security



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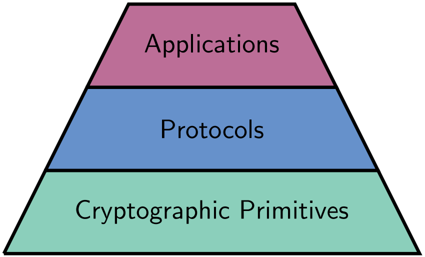


Applications in everyday life!

Example:

- ⋆ Encrypting email communications: PGP
- ⋆ Securing a website: HTTPS
- ⋆ Internet of Things (IoT)
- ⋆ ...

Lifecycle of a primitive

Conception

- ⋆ Specification of the algorithm
- ⋆ Justification of design choices
- ⋆ First security analysis

Publication

Analysis

- ⋆ Trying to break algorithms

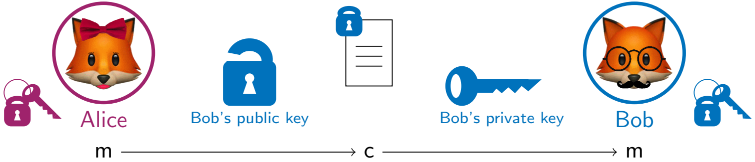
Standardization

Deployment

- ⋆ Implementation of algorithms

Asymmetric VS Symmetric

⋆ Asymmetric: RSA, Diffie-Hellman, . . .



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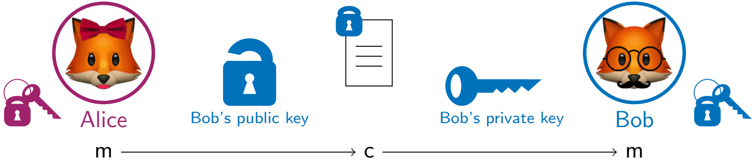
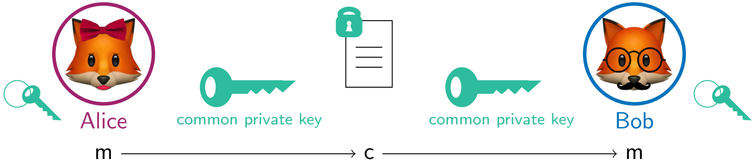


⋆ Symmetric: AES, DES, Triple-DES, ...



Symmetric cryptography

We assume that a key is already shared.

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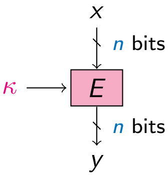
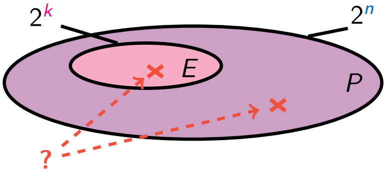


Block cipher

Symmetric cryptography



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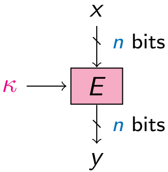

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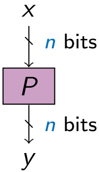

Random permutation

- ⋆ input: n-bit block x
- ⋆ parameter: k-bit key κ
- ⋆ output: n-bit block $y = E_\kappa(x)$
- ⋆ symmetry: E and E^{-1} use the same κ

$\Rightarrow$ Block cipher: family of 2^k permutations of n bits.

Content

New uses in Symmetric Cryptography:
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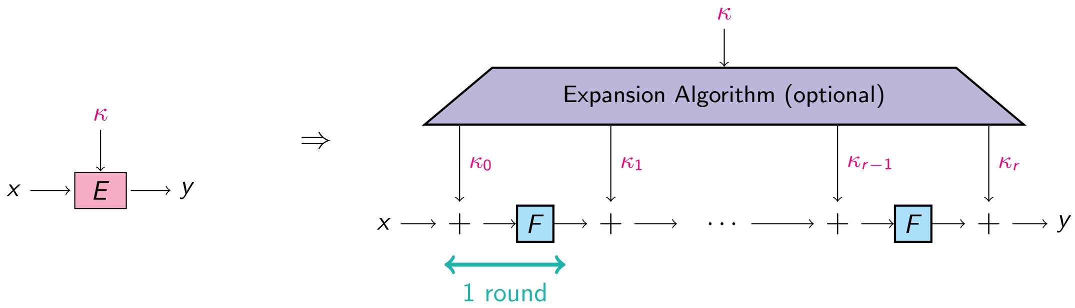
Iterated constructions

$\Rightarrow$ How to build a block cipher?

By iterating a round function.



Performance constraints! The primitive must be fast.

A need of new primitives

Problem: Analyzing the security of new symmetric primitives

Protocols requiring new primitives:

- ⋆ Multiparty Computation (MPC)
- ⋆ Homomorphic Encryption (FHE)
- ⋆ Systems of Zero-Knowledge (ZK) proofs

  Example: SNARKs, STARKs, Bulletproofs

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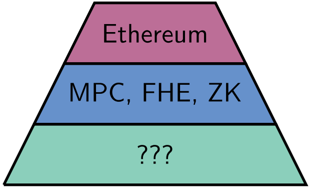

$\Rightarrow$ What differs from the "usual" case?

Comparison with "usual" case

A new environment

"Usual" case

- Field size:
  $\mathbb{F}_{2^n}$, with $n \simeq 4, 8$ (AES: $n = 8$).
- Operations:
  logical gates/CPU instructions

Arithmetization-friendly

- Field size:
  $\mathbb{F}_q$, with $q \in \{2^n, p\}, p \simeq 2^n$, $n \geq 64$.
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- Operations:
  $$y \leftarrow E(x)$$
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The block cipher MiMC

- Minimize the number of multiplications in $\mathbb{F}_{2^n}$.
- Construction of MiMC_3 [Albrecht et al., Eurocrypt16]:
  - n-bit blocks (n odd ≈ 129): $x \in \mathbb{F}_{2^n}$
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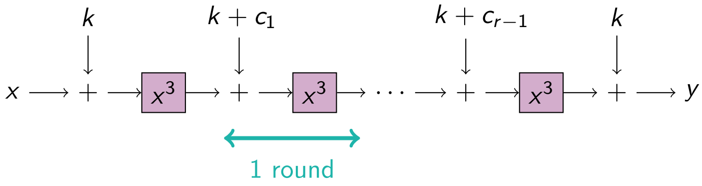

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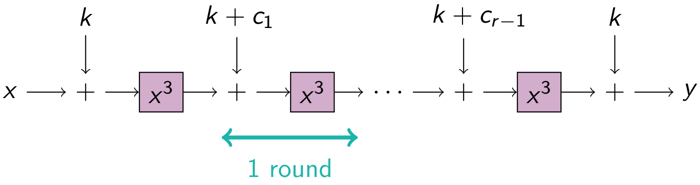
$$R := \lceil n \log_3 2 \rceil \ .$$

n	129	255	769	1025
R	82	161	486	647

Number of rounds for MiMC.



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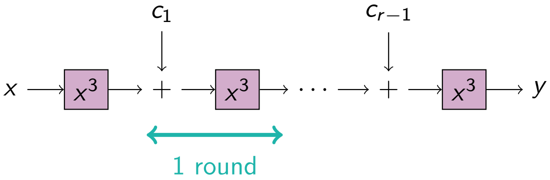

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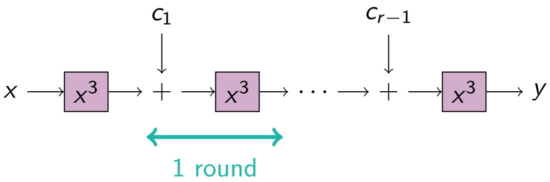

☞ *Bouvier, Canteaut, Perrin*
On the Algebraic Degree of Iterated Power Functions

Algebraic degree

Let $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$, there is **a unique multivariate polynomial** in $\mathbb{F}_2[x_1, \ldots x_n] / \left((x_i^2 + x_i)_{1 \le i \le n}\right)$:

$$f(x_1, ..., x_n) = \sum_{u \in \mathbb{F}_2^n} a_u x^u, \text{ where } a_u \in \mathbb{F}_2, \; x^u = \prod_{i=1}^{n} x_i^{u_i} \; .$$

This is the **Algebraic Normal Form (ANF)** of f.

Definition

Algebraic Degree of $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$:

$$\deg^a(f) = \max \left\{ \mathrm{wt}(u) : u \in \mathbb{F}_2^n, a_u \neq 0 \right\} \; ,$$

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$$\deg^a(F) = \max\{\deg^a(f_i),\ 1 \le i \le m\} .$$

where $F(x) = (f_1(x), \dots f_m(x))$.

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If $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is a permutation, then

$$\boxed{\deg^a(F) \leq n - 1}$$

Higher-order differential attack

Exploiting a low algebraic degree

For any affine subspace $\mathcal{V} \subset \mathbb{F}_2^n$ with $\dim \mathcal{V} \geq \deg^a(F) + 1$, we have a 0-sum distinguisher:

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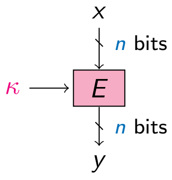

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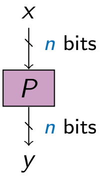

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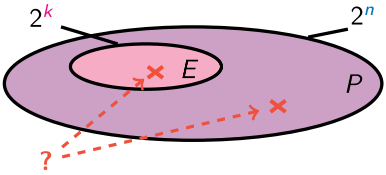

First Plateau

Round i of MiMC_3: $x \mapsto (x + c_{i-1})^3$.

For r rounds:

- ⋆ Upper bound [Eichlseder et al., Asiacrypt20]: $\lceil r \log_2 3 \rceil$.
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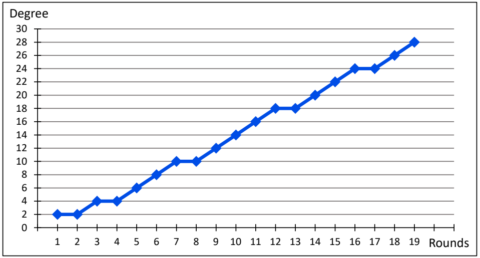

Algebraic degree observed for $n = 31$.

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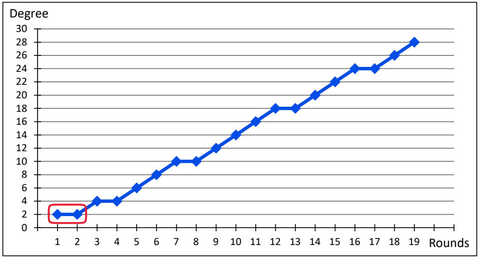

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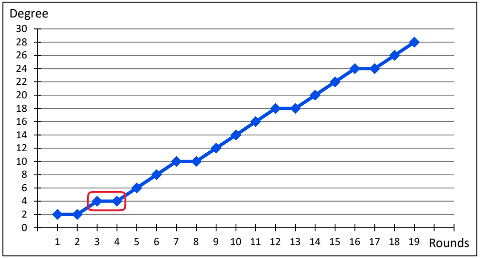

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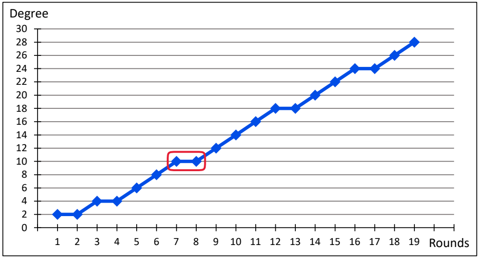

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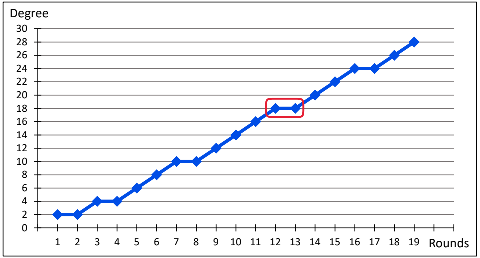
$$9 = [1001]_2 \; 6 = [110]_2 \; 3 = [11]_2$$



Algebraic degree observed for $n = 31$.

First Plateau

Round i of MiMC$_3$: $x \mapsto (x + c_{i-1})^3$.

For r rounds:

- ⋆ Upper bound [Eichlseder et al., Asiacrypt20]: $\lceil r \log_2 3 \rceil$.
- ⋆ Aim: determine $B_3^r := \max_c \deg^a \mathrm{MIMC}_{3,c}[r]$.

Definition

There is a **plateau** whenever $B_3^r = B_3^{r-1}$.

⋆ Round 1: $B_3^1 = 2$

$$\mathcal{P}_1(x) = x^3, \quad (c_0 = 0)$$

$$3 = [11]_2$$

⋆ Round 2: $B_3^2 = 2$

$$\mathcal{P}_2(x) = x^9 + c_1 x^6 + c_1^2 x^3 + c_1^3$$

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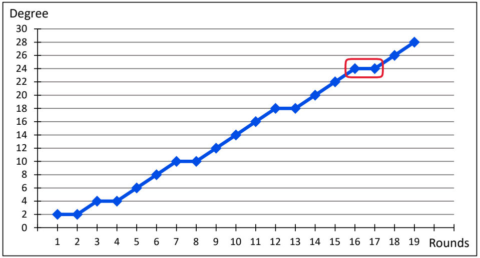

Algebraic degree observed for $n = 31$.

An upper bound

Proposition

Set of exponents that might appear in the polynomial:

$$\mathcal{E}_r = \{3j \bmod (2^n - 1) \text{ where } j \preceq i,\ i \in \mathcal{E}_{r-1}\}$$

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Example:

$$\mathcal{P}_1(x) = x^3 \quad \Rightarrow \quad \mathcal{E}_1 = \{3\} \ .$$

$$3 = [11]_2 \quad \xrightarrow{\succeq} \quad \begin{cases} [00]_2 = 0 & \xrightarrow{\times 3} \quad 0 \\ [01]_2 = 1 & \xrightarrow{\times 3} \quad 3 \\ [10]_2 = 2 & \xrightarrow{\times 3} \quad 6 \\ [11]_2 = 3 & \xrightarrow{\times 3} \quad 9 \end{cases}$$

$$\mathcal{E}_2 = \{0, 3, 6, 9\} \ ,$$

$$\mathcal{P}_2(x) = x^9 + c_1 x^6 + c_1^2 x^3 + c_1^3 \ .$$

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No exponent $\equiv 5, 7 \bmod 8 \Rightarrow$ No exponent $2^{2k} - 1$

$$\mathcal{E}_r \subseteq \{ \begin{array}{cccccccc} 0 & 3 & 6 & 9 & 12 & \cancel{15} & 18 & \cancel{21} \\ 24 & 27 & 30 & 33 & 36 & \cancel{39} & 42 & \cancel{45} \\ 48 & 51 & 54 & 57 & 60 & \cancel{63} & 66 & \cancel{69} \\ \ldots & 3^r\} \end{array}$$

Example: $63 = 2^{2\times 3} - 1 \notin \mathcal{E}_4 = \{0, 3, \ldots, 81\}$ $\Rightarrow B_3^4 < 6 = wt(63)$

$\forall e \in \mathcal{E}_4 \backslash \{63\}, wt(e) \leq 4$ $\Rightarrow B_3^4 \leq 4$

Bounding the degree

Theorem

After r rounds of MiMC, the algebraic degree is

$$B_3^r \leq 2 \times \lceil \lfloor \log_2(3^r) \rfloor / 2 - 1 \rceil$$

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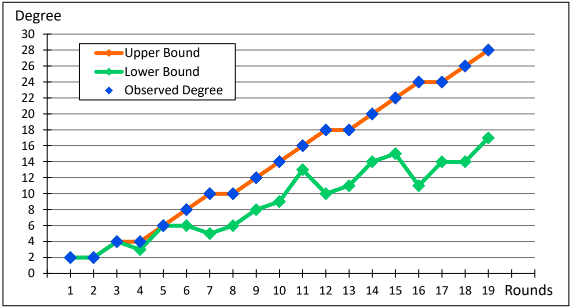
And a lower bound if $3^r < 2^n - 1$:

$$B_3^r \geq wt(3^r)$$



Exact degree

Maximum-weight exponents:

Let $k_r = \lfloor \log_2 3^r \rfloor$.

$\forall r \in \{4, \dots, 16265\} \backslash \mathcal{F}$ with $\mathcal{F} = \{465, 571, \dots\}$:

- if $k_r = 1 \bmod 2$,

$$\omega_r = 2^{k_r} - 5 \in \mathcal{E}_r,$$

- if $k_r = 0 \bmod 2$,

$$\omega_r = 2^{k_r} - 7 \in \mathcal{E}_r.$$

Example:

$$123 = 2^7 - 5 = 2^{k_5} - 5 \qquad \in \mathcal{E}_5,$$
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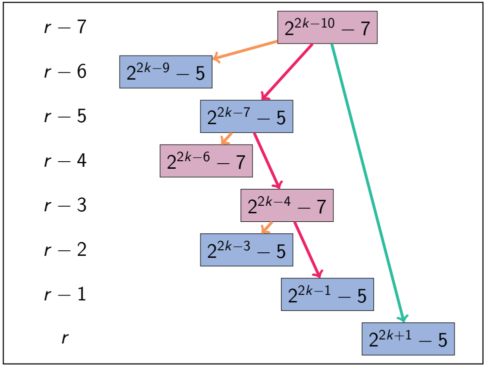

Constructing exponents.

$$\exists \ell \text{ s.t. } \quad \omega_{r-\ell} \in \mathcal{E}_{r-\ell} \Rightarrow \omega_r \in \mathcal{E}_r$$

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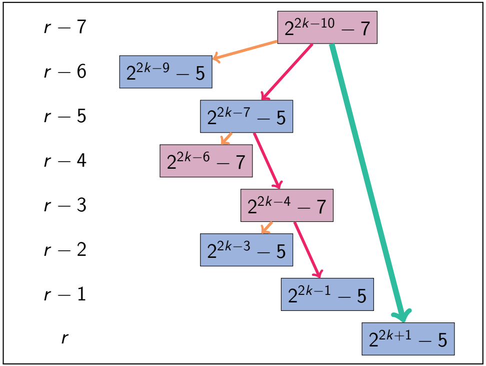

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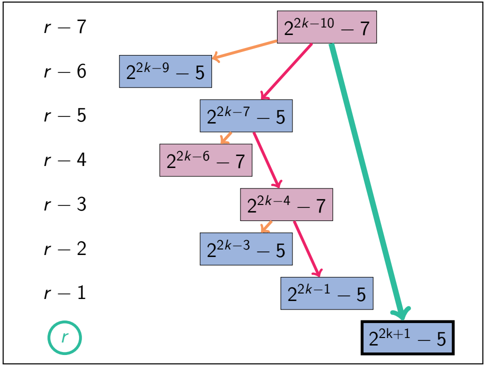

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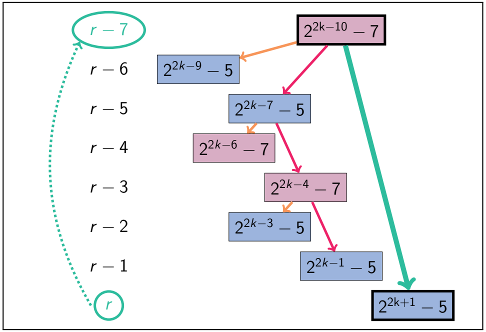

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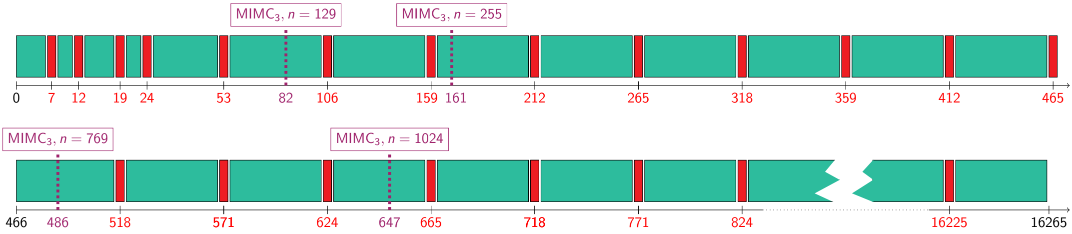
Covered rounds

Idea of the proof:

⋆ inductive proof: existence of "good" ℓ

Rounds for which we are able to exhibit a maximum-weight exponent.



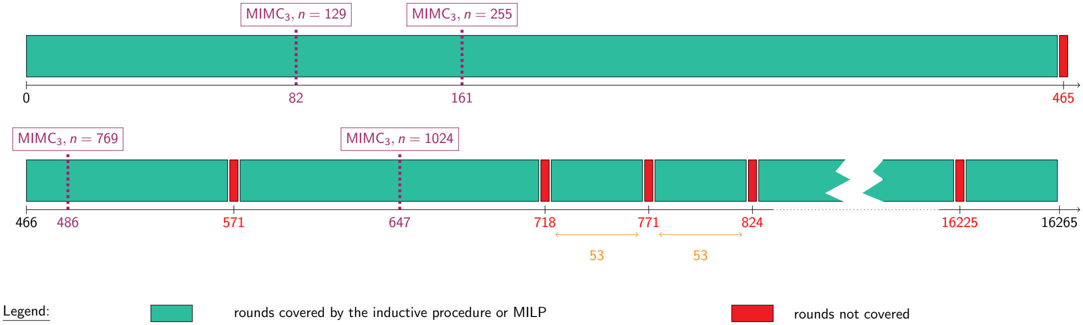
Legend: rounds covered by the inductive procedure; rounds not covered

Covered rounds

Idea of the proof:

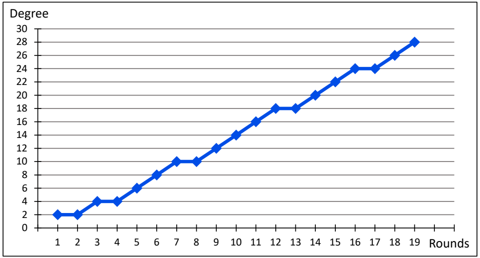
- ⋆ inductive proof: existence of "good" ℓ
- ⋆ MILP solver (`PySCIPOpt`)

Rounds for which we are able to exhibit a maximum-weight exponent.



Plateau

$\Rightarrow$ plateau when $k_r = \lfloor \log_2 3^r \rfloor = 1 \bmod 2$ and $k_{r+1} = \lfloor \log_2 3^{r+1} \rfloor = 0 \bmod 2$



Algebraic degree observed for $n = 31$.

If we have a plateau

$$B_3^r = B_3^{r+1} \,,$$

Then the next one is

$$B_3^{r+4} = B_3^{r+5} \quad \text{or} \quad B_3^{r+5} = B_3^{r+6} \,.$$

Music in MIMC$_3$

- Patterns in sequence $(k_r)_{r>0}$:

$\Rightarrow$ denominators of semiconvergents of $\log_2(3) \simeq 1.5849625$

$$\mathfrak{D} = \{\boxed{1}, \boxed{2}, 3, 5, \boxed{7}, \boxed{12}, 17, 29, 41, \boxed{53}, 94, 147, 200, 253, 306, \boxed{359}, \ldots\} ,$$

$$\log_2(3) \simeq \frac{a}{b} \quad \Leftrightarrow \quad 2^a \simeq 3^b$$

- **Music theory:**
  - perfect octave 2:1
  - perfect fifth 3:2

$$2^{19} \simeq 3^{12} \quad \Leftrightarrow \quad 2^7 \simeq \left(\frac{3}{2}\right)^{12} \quad \Leftrightarrow \quad \text{7 octaves} \sim \text{12 fifths}$$

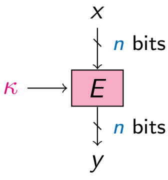
Higher-order differential attack

Exploiting a low algebraic degree

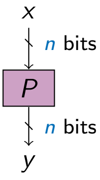
For any affine subspace $\mathcal{V} \subset \mathbb{F}_2^n$ with $\dim \mathcal{V} \geq \deg^a(F) + 1$, we have a 0-sum distinguisher:

$$\bigoplus_{x \in \mathcal{V}} F(x) = 0.$$

Random permutation: degree $= n - 1$



Block cipher



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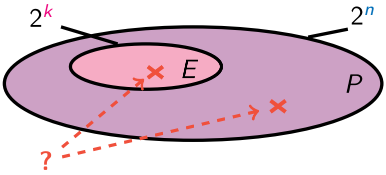
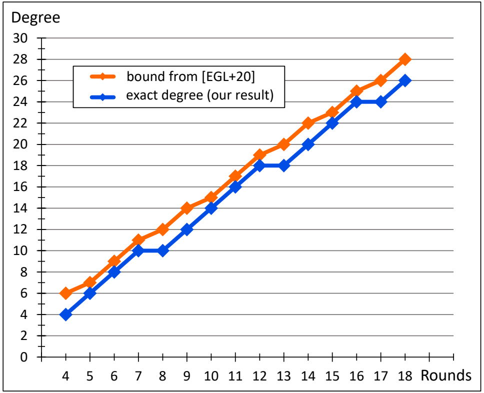


Comparison to previous work

First Bound: $\lceil r \log_2 3 \rceil \quad \Rightarrow \quad$ Exact degree: $2 \times \lceil \lfloor r \log_2 3 \rfloor / 2 - 1 \rceil$.



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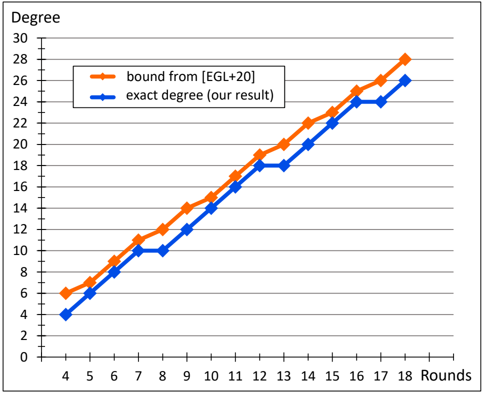


For $n = 129$, MIMC$_3$ $= 82$ rounds

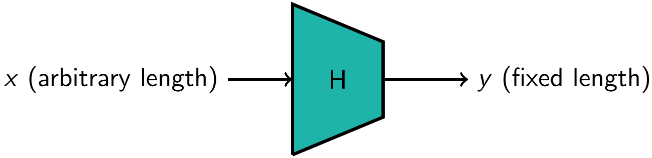
Rounds	Time	Data	Source
80/82	2^{128}XOR	2^{128}	[EGL+20]
81/82	2^{128}XOR	2^{128}	New
80/82	2^{125}XOR	2^{125}	New

Secret-key distinguishers ($n = 129$)

Hash Functions

Definition

Hash function: $H : \mathbb{F}_q^\ell \to \mathbb{F}_q^h, x \mapsto y = H(x)$ where ℓ is arbitrary and h is fixed.



- ⋆ Preimage resistance: Given y it must be *infeasible* to find x s.t.

$$H(x) = y \ .$$

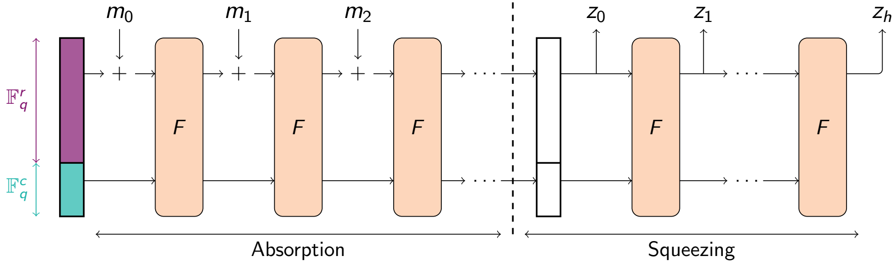
- ⋆ Collision resistance: It must be *infeasible* to find $x \neq x'$ s.t.

$$H(x) = H(x') \ .$$

Sponge construction

Parameters:

- ⋆ rate $r > 0$
- ⋆ capacity $c > 0$
- ⋆ permutation of $\mathbb{F}_q^r \times \mathbb{F}_q^c$



Hash function in sponge framework.

Some values of p

Parameter p given by Standardized Elliptic Curves.

Example:

- ⋆ Curve BLS12-381 $\log_2 p = 381$

$$p = 4002409555221667393417789825735904156556882819939007885332 \\ 058136124031650490837864442687629129015664037894272559787$$

- ⋆ Curve BLS12-377 $\log_2 p = 377$

$$p = 2586644260129690940106527336948935335363935127549146605398 \\ 842626667204683483408227749688881395733601244403214581 77$$

Substitution-Permutation Network (SPN)

- S-Box layer → Confusion

  Example:

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} x_0^d & x_1^d & \dots & x_{m-1}^d \end{pmatrix}.$$

- Linear layer → Diffusion

  Example:

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} x_0 & x_1 & \dots & x_{m-1} \end{pmatrix} \times M.$$

- Constants addition

  Example:

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} x_0 & x_1 & \dots & x_{m-1} \end{pmatrix} + \begin{pmatrix} c_0 & c_1 & \dots & c_{m-1} \end{pmatrix}.$$

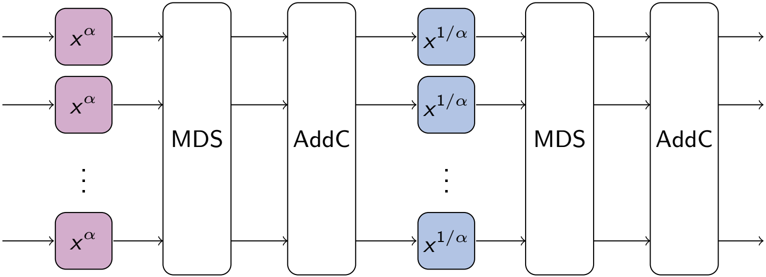
Rescue

[Aly et al., ToSC20]

- S-Box layer
- Linear layer: MDS
- Round constants addition: AddC

$S : x \mapsto x^\alpha$, and $S^{-1} : x \mapsto x^{1/\alpha}$ $(\alpha = 3)$

$R \approx 10$



The 2 steps of round i of Rescue.

Rescue

[Aly et al., ToSC20]

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Curve BLS12-381:

$$p = 4002409555221667393417789825735904156556882819939007885332 \\ 058136124031650490837864442687629129015664037894272559787$$

$$\alpha = 5$$

$$\alpha^{-1} = 3201927644177333914734231860588723325245506255951206308265 \\ 646508899225320392670291554150103303212531230315418047829$$

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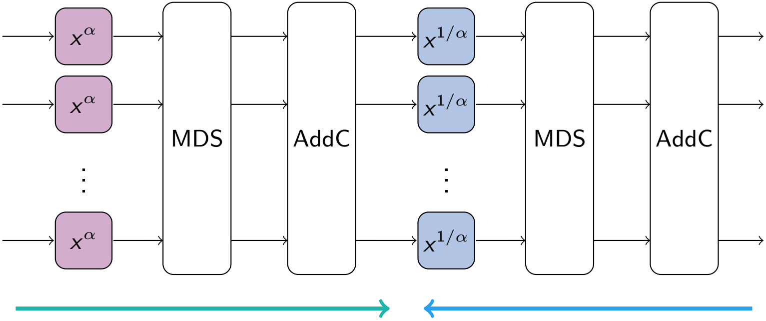


The 2 steps of round i of Rescue.

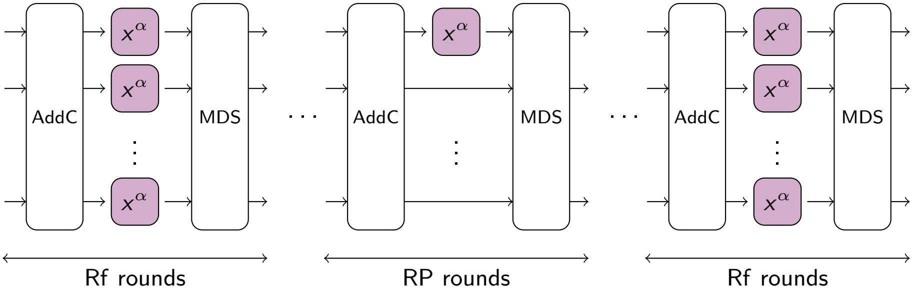
Poseidon

[Grassi et al., USENIX21]

- S-Box layer
- Linear layer: MDS
- Round constants addition: AddC

$S : x \mapsto x^\alpha, \ (\alpha = 3)$

$R = \text{RF} + \text{RP} \approx 50$



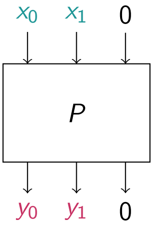
Overview of Poseidon.

Ethereum Challenges

A Cryptanalysis Challenge for ZK-friendly Hash Functions!
In November 2021, by the Ethereum Foundation.

Definition

Constrained Input Constrained Output (CICO) problem:
Find $X, Y \in \mathbb{F}_q^{t-u}$ s.t. $P(X, 0^u) = (Y, 0^u)$.



CICO problem when $t = 3$, $u = 1$.

Tricks for SPN

⋆ Solving Univariate systems:

Find the roots of a polynomial $P \in \mathbb{F}_p[X]$.

Tricks for SPN

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- Solving Multivariate systems:

  From polynomial equations on variables $X_i \in \mathbb{F}_p$:

$$\begin{cases} P_1(X_1, \ldots X_n) = 0 \\ P_2(X_1, \ldots X_n) = 0 \\ \qquad\qquad \vdots \\ P_n(X_1, \ldots X_n) = 0, \end{cases}$$

compute a Gröbner basis...

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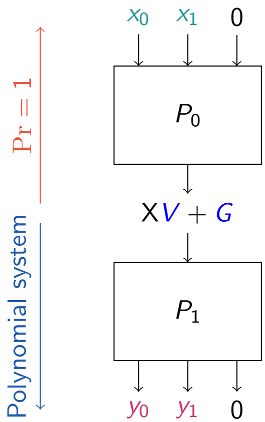

A 2-staged trick.

Consequence for the Challenge

Category	Parameters	Security Level (bits)	Bounty
~~Easy~~	~~r=6~~	~~9~~	~~$2,000~~
~~Easy~~	~~r=10~~	~~15~~	~~$4,000~~
~~Medium~~	~~r=14~~	~~22~~	~~$6,000~~
~~Hard~~	~~r=18~~	~~28~~	$12,000
~~Hard~~	~~r=22~~	~~34~~	$26,000

(a) *Feistel-MiMC*

Category	Parameters	Security Level (bits)	Bounty
~~Easy~~	~~N=4, m=3~~	~~25~~	~~$2,000~~
Easy	N=6, m=2	25	$4,000
Medium	N=7, m=2	29	$6,000
Hard	N=5, m=3	30	$12,000
Hard	N=8, m=2	33	$26,000

(b) *Rescue*

Category	Parameters	Security Level (bits)	Bounty
~~Easy~~	~~RP=3~~	~~8~~	~~$2,000~~
~~Easy~~	~~RP=8~~	~~16~~	~~$4,000~~
~~Medium~~	~~RP=13~~	~~24~~	~~$6,000~~
Hard	RP=19	32	$12,000
Hard	RP=24	40	$26,000

(c) *Poseidon*

☞ *Bariant, Bouvier, Leurent, Perrin*
Practical Algebraic Attacks against some Arithmetization-oriented Hash Functions

Goals and Principles

Anemoi

Bouvier, Briaud, Chaidos, Perrin, Velichkov

A family of hash functions exploiting the link between arithmetization-friendliness and CCZ-equivalence.

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Bouvier, Briaud, Chaidos, Perrin, Velichkov

A family of hash functions exploiting the link between arithmetization-friendliness and CCZ-equivalence.

Design goals:

- ⋆ Compatibility with Various Proof Systems.
- ⋆ Low number of multiplications
- ⋆ Fast and secure

CCZ-equivalence

Definition

$F : \mathbb{F}_q \rightarrow \mathbb{F}_q$ and $G : \mathbb{F}_q \rightarrow \mathbb{F}_q$ are **CCZ-equivalent**

$$\Gamma_F = \left\{ (x, F(x)) \mid x \in \mathbb{F}_q \right\} = \mathcal{A}(\Gamma_G) = \left\{ \mathcal{A}(x, F(x)) \mid x \in \mathbb{F}_q \right\},$$

where $\mathcal{A}$ is an affine permutation.

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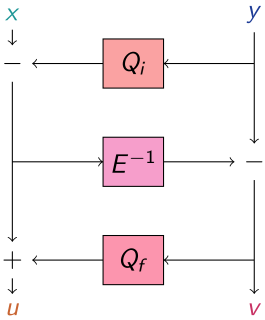
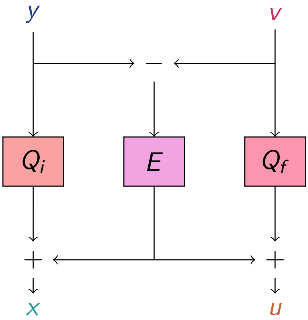
where $\mathcal{A}$ is an affine permutation.

High-degree permutation



Open `Flystel` $\mathcal{H}$.

Low-degree function



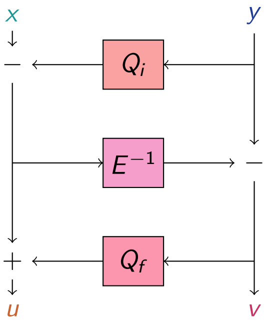
Closed `Flystel` $\mathcal{V}$.

CCZ-equivalence

$$\Gamma_{\mathcal{H}} = \mathcal{A}(\Gamma_{\mathcal{V}})$$

$$\{(x, y), (u, v)\} = \mathcal{A}(\{(y, v), (x, u)\})$$

High-degree permutation



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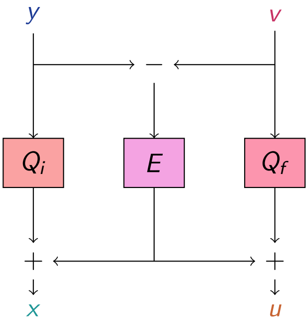
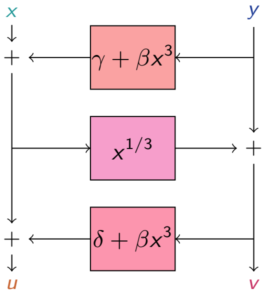


Closed Flystel $\mathcal{V}$.

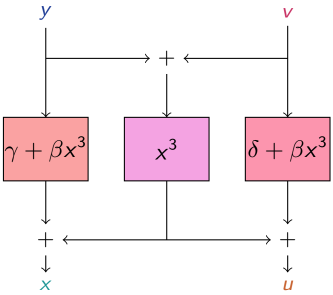
Flystel in $\mathbb{F}_{2^n}$

$$\mathcal{H} : \begin{cases} \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} & \to \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} \\ (x,y) \mapsto & \left(x + \beta y^3 + \gamma + \beta \left(y + (x + \beta y^3 + \gamma)^{1/3}\right)^3 + \delta\ , \right. \\ & \left. \quad y + (x + \beta y^3 - \gamma)^{1/3}\right)\ . \end{cases}$$

$$\mathcal{V} : \begin{cases} \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} & \to \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} \\ (x,y) & \mapsto \left((y+v)^{1/3} + \beta y^3 + \gamma\ , \right. \\ & \left. \quad (y+v)^{1/3} + \beta v^3 + \delta\right)\ , \end{cases}$$



Open Flystel$_2$.

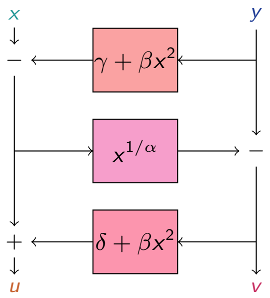


Closed Flystel$_2$.

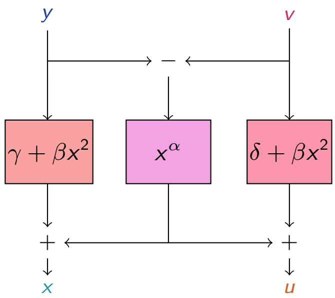
Flystel in $\mathbb{F}_p$

$$\mathcal{H} : \begin{cases} \mathbb{F}_p \times \mathbb{F}_p & \to \mathbb{F}_p \times \mathbb{F}_p \\ (x, y) & \mapsto \left(x - \beta y^2 - \gamma + \beta \left(y - (x - \beta y^2 - \gamma)^{1/\alpha}\right)^2 + \delta \, , \right. \\ & \left. \quad y - (x - \beta y^2 - \gamma)^{1/\alpha}\right) . \end{cases}$$

$$\mathcal{V} : \begin{cases} \mathbb{F}_p \times \mathbb{F}_p & \to \mathbb{F}_p \times \mathbb{F}_p \\ (y, v) & \mapsto \left((y - v)^{1/\alpha} + \beta y^2 + \gamma \, , \right. \\ & \left. \quad (v - y)^{1/\alpha} + \beta v^2 + \delta\right) . \end{cases}$$



Open Flystel$_p$.

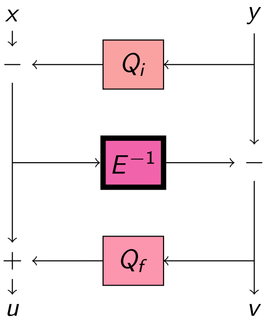


Closed Flystel$_p$.

Advantage of CCZ-equivalence

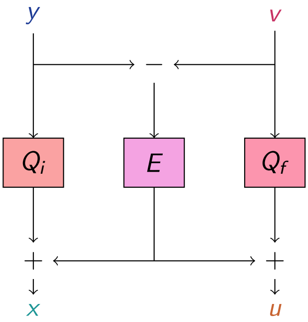
⋆ High Degree Evaluation.

High-degree permutation



Open Flystel $\mathcal{H}$.

Low-degree function



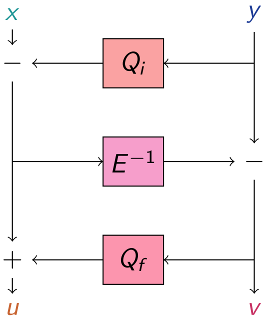
Closed Flystel $\mathcal{V}$.

Advantage of CCZ-equivalence

- ⋆ High Degree Evaluation.
- ⋆ Low Cost Verification.

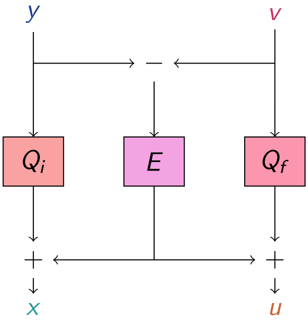
$$(u, v) == \mathcal{H}(x, y) \Leftrightarrow (x, u) == \mathcal{V}(y, v)$$

High-degree permutation



Open Flystel $\mathcal{H}$.

Low-degree function



Closed Flystel $\mathcal{V}$.

The SPN Structure

Let

$$X = \begin{pmatrix} x_0 & x_1 & \dots & x_{\ell-1} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_0 & y_1 & \dots & y_{\ell-1} \end{pmatrix} \text{ with } x_i, y_i \in \mathbb{F}_q \ .$$

The internal state of `Anemoi` can be represented as:

$$\begin{pmatrix} X \\ Y \end{pmatrix} .$$

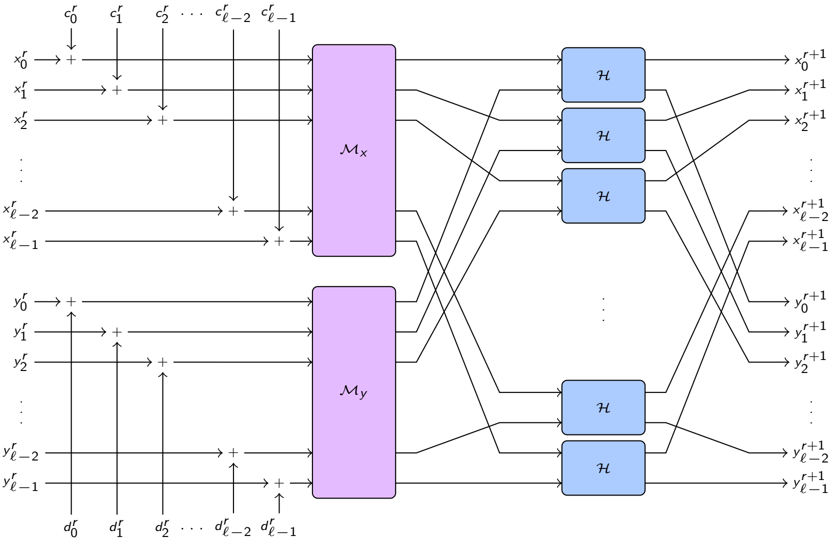
Addition of constants and the linear layer as:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} C \\ D \end{pmatrix}, \quad \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} X\mathcal{M}_x \\ Y\mathcal{M}_y \end{pmatrix} .$$

And the S-Box layer as:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} {}^t\mathcal{H}(x_0, y_0) & {}^t\mathcal{H}(x_1, y_1) & \dots & {}^t\mathcal{H}(x_{\ell-1}, y_{\ell-1}) \end{pmatrix} .$$

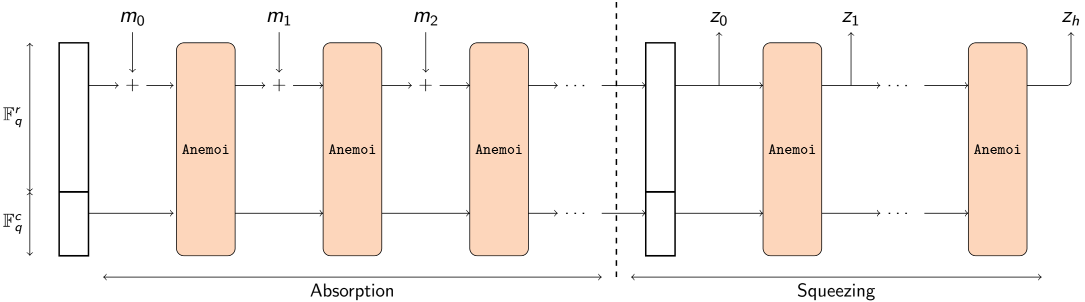
The SPN Structure



Overview of `Anemoi`*.*

New Mode

- Hash function:
  - input: arbitrary length
  - ouput: fixed length

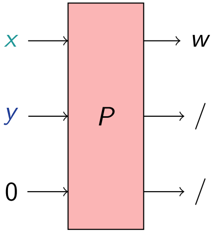


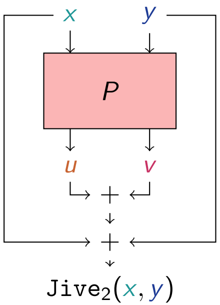
New Mode

- Hash function:
  - input: arbitrary length
  - ouput: fixed length
- Compression function:
  - input: fixed length
  - output: length 1

Dedicated mode $\Rightarrow$ 2 words in 1

$$(x, y) \mapsto x + y + u + v \ .$$





Comparison to previous work

s	$\log_2 q$	m	Rescue	Poseidon	Anemoi
128	192	8	384	363	**200**
	256	6	288	315	**150**
	384	4	216	264	**120**
256	192	8	432	450	**280**
	256	6	432	495	**225**
	384	4	432	444	**200**

(a) *for R1CS.*

s	$\log_2 q$	m	Rescue	Poseidon	Anemoi
128	192	8	1280	4003	**560**
	256	6	768	2265	**360**
	384	4	432	1032	**240**
256	192	8	1440	5714	**784**
	256	6	1152	4245	**540**
	384	4	864	1932	**784**

(b) *for Plonk.*

Number of constraints for Rescue, Poseidon and Anemoi when $\alpha = 5$.

Conclusions

- Algebraic degree of $MIMC_3$
  - a tight upper bound, up to 16265 rounds: $2 \times \lceil \lfloor \log_2(3^r) \rfloor / 2 - 1 \rceil$.
  - minimal complexity for higher-order differential attack

☞ More details on eprint.iacr.org/2022/366
and to appear in *Designs, Codes and Cryptography*

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- Practical attacks against arithmetization-oriented hash functions

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Conclusions

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- Anemoi
  - a new family of ZK-friendly hash functions
  - new observations of fundamental interest

Open Problem

Cryptanalysis and designing of arithmetization-oriented primitives remain to be explored!

And the opinion of mathematicians would be of great help to us!

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Observation

$$\forall\ 1 \leq t \leq 21,\ \forall\ x \in \mathbb{Z}/3^t\mathbb{Z},\ \exists\ \varepsilon_2, \ldots, \varepsilon_{2t+2} \in \{0,1\},\ \text{s.t. } x = \sum_{j=2}^{2t+2} \varepsilon_j 4^j \bmod 3^t\ .$$

Is this true for any t? Should we consider more ε_j for larger t?

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Is this true for any t? Should we consider more ε_j for larger t?

Thanks for your attention

Sporadic Cases

Bound on ℓ

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Let: $k_r = \lfloor r \log_2 3 \rfloor$, $b_r = k_r \bmod 2$ and

$$\mathcal{L}_r = \{\ell,\ 1 \leq \ell < r,\ \text{s.t. } k_{r-\ell} = k_r - k_\ell\}\ .$$

Proposition

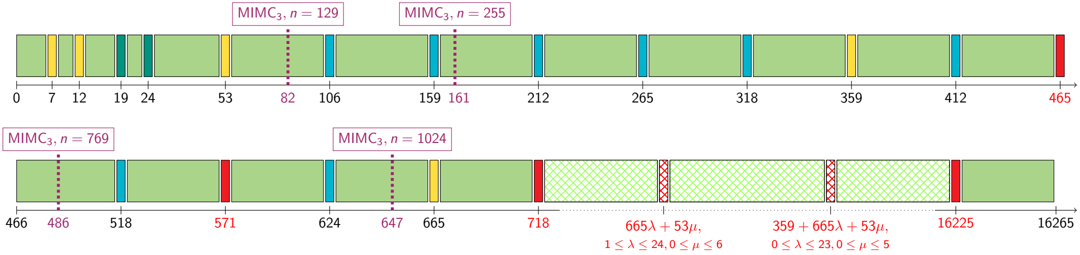
Let $r \geq 4$, and $\ell \in \mathcal{L}_r$ s.t.:

- $\ell = 1, 2,$
- $2 < \ell \leq 22$ s.t. $k_r \geq k_\ell + 3\ell + b_r + 1$, and ℓ is even, or ℓ is odd, with $b_{r-\ell} = \overline{b_r}$;
- $2 < \ell \leq 22$ is odd s.t. $k_r \geq k_\ell + 3\ell + \overline{b_r} + 5$

Then $\omega_{r-\ell} \in \mathcal{E}_{r-\ell}$ implies that $\omega_r \in \mathcal{E}_r$.

Covered Rounds

Rounds for which we are able to exhibit a maximum-weight exponent.



Legend: Rounds for which we are able to construct an exponent.

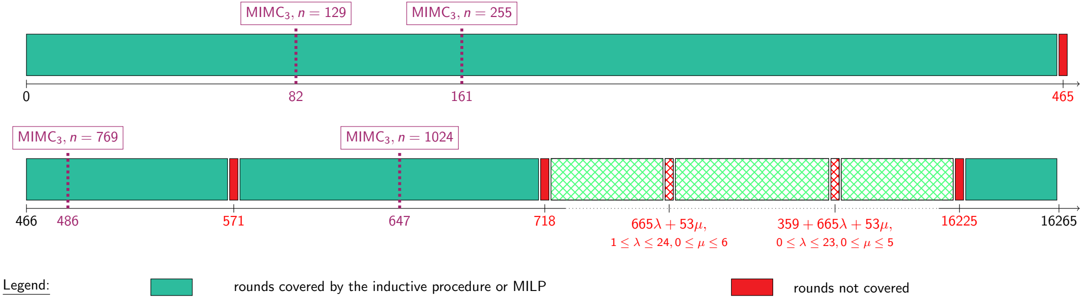
- semiconvergents of $\log_2(3)$: MILP
- "good" ℓ
- no "good" ℓ: MILP
- no "good" ℓ ($\ell \geq 53$): MILP

Rounds likely to be covered by solving the conjecture.

- no "good" ℓ: no result with MILP

Covered Rounds

Rounds for which we are able to exhibit a maximum-weight exponent.



MILP Solver

Let

$$\text{Mult}_3 : \begin{cases} \mathbb{N}^{\mathbb{N}} & \to \mathbb{N}^{\mathbb{N}} \\ \{j_0, \ldots, j_{\ell-1}\} & \mapsto \{(3j_0) \bmod (2^n - 1), \ldots, (3j_{\ell-1}) \bmod (2^n - 1)\} \end{cases} ,$$

and

$$\text{Cover} : \begin{cases} \mathbb{N}^{\mathbb{N}} & \to \mathbb{N}^{\mathbb{N}} \\ \{j_0, \ldots, j_{\ell-1}\} & \mapsto \{k \preceq j_i, i \in \{0, \ldots, \ell - 1\}\} \end{cases} .$$

So that:

$$\mathcal{E}_r = \text{Mult}_3\big(\text{Cover}(\mathcal{E}_{r-1})\big) .$$

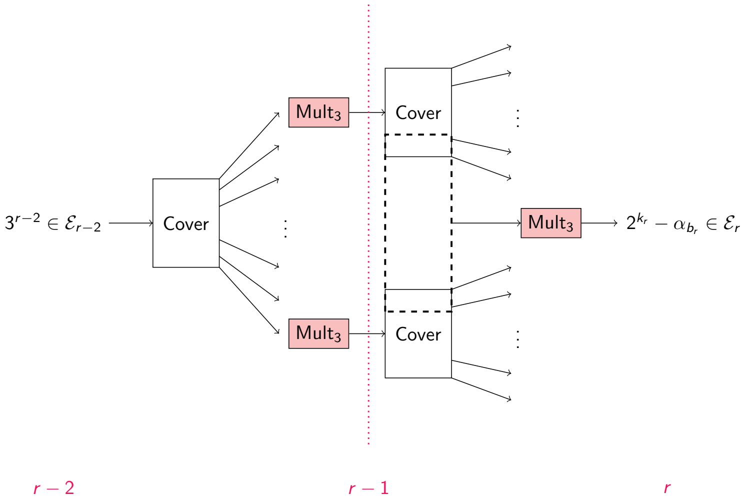
$\Rightarrow$ MILP problem solved using PySCIPOpt

$$\boxed{\text{existence of a solution} \quad \Leftrightarrow \quad \omega_r \in (\text{Mult}_3 \circ \text{Cover})^{\ell}(\{3^{r-\ell}\})}$$

With $\ell = 1$:

$$3^{r-1} \in \mathcal{E}_{r-1} \longrightarrow \boxed{\text{Cover}} \longrightarrow \boxed{\text{Mult}_3} \longrightarrow 2^{k_r} - \alpha_{b_r} \in \mathcal{E}_r$$

MILP Solver (2 rounds)



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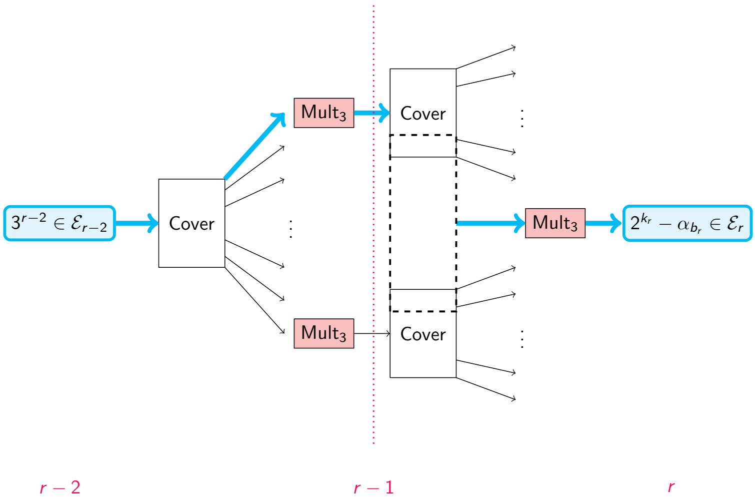

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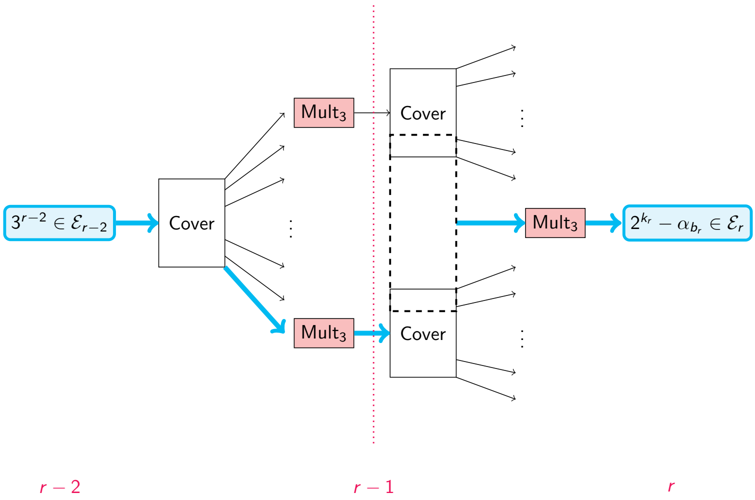
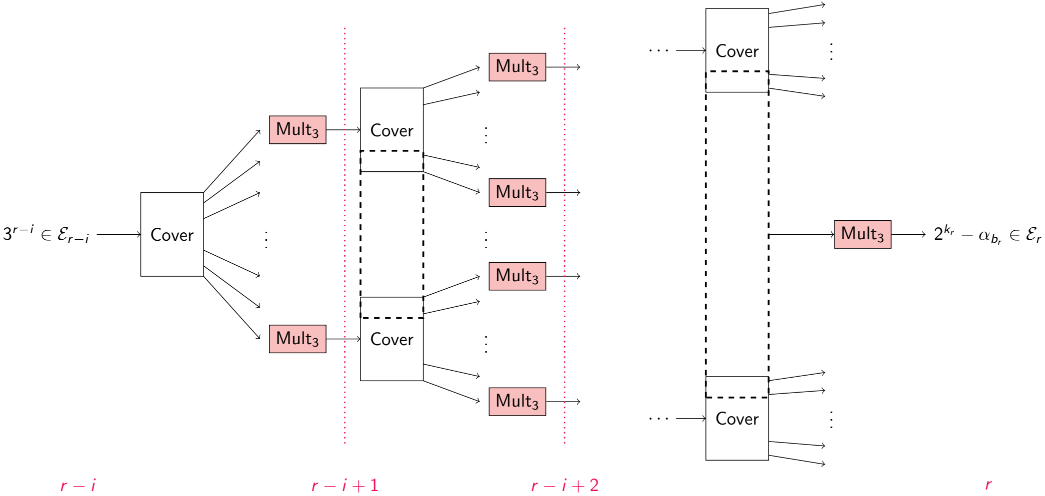


MILP Solver (i rounds)



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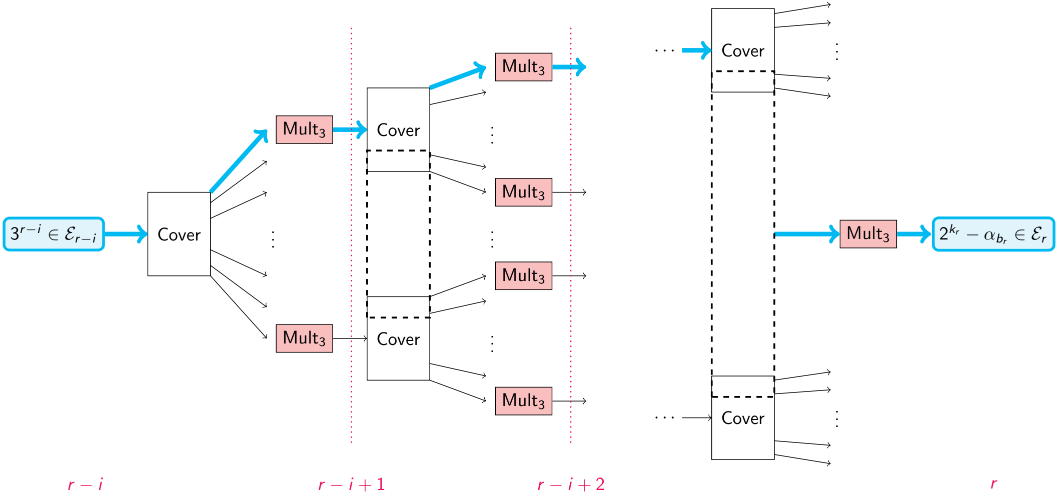

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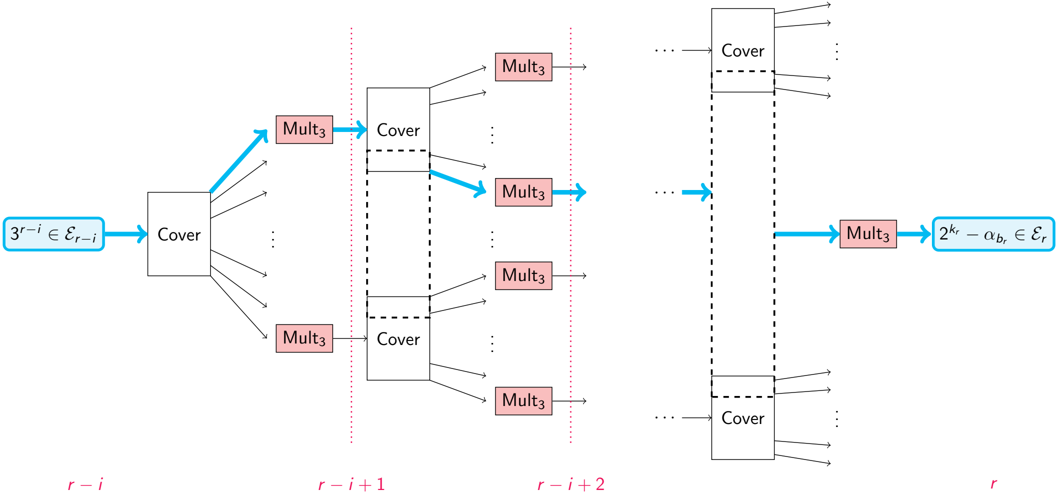

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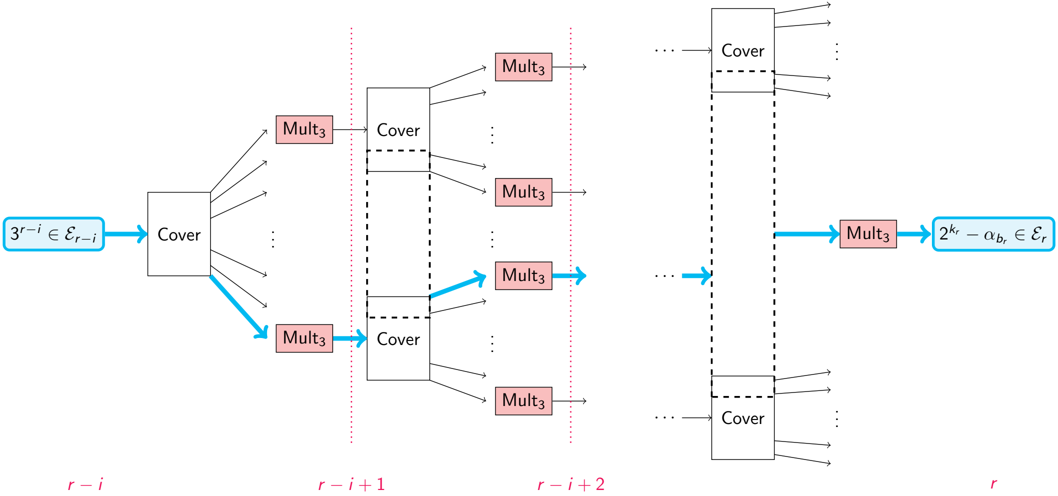

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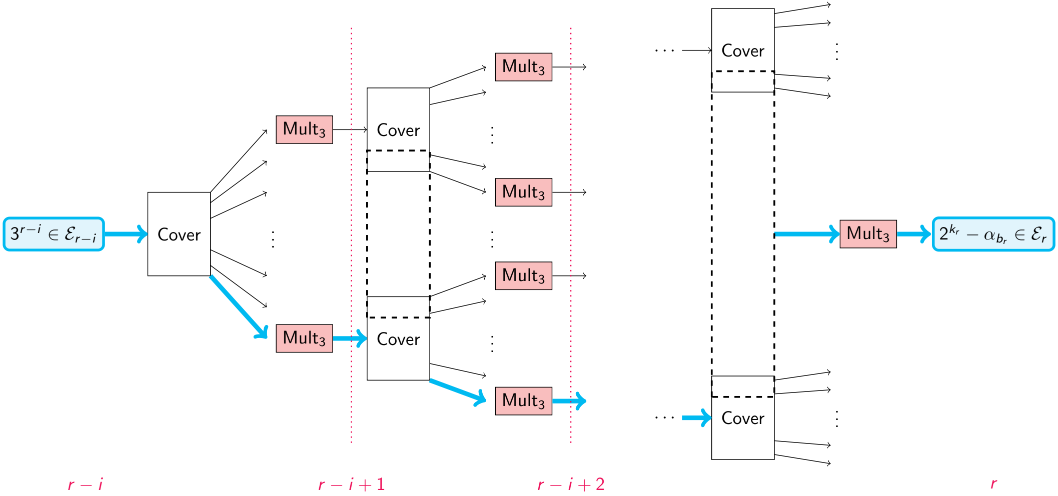
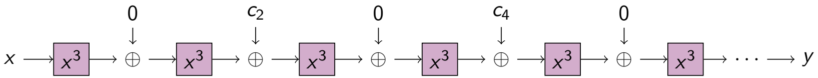


$MiMC_9$ and form of coefficients

- $MIMC_3[2r]$



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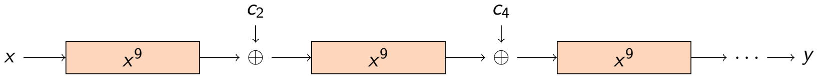

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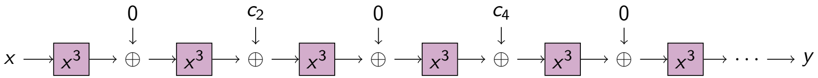

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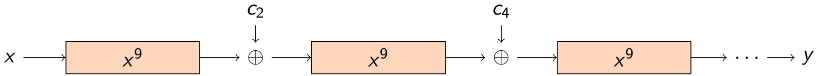

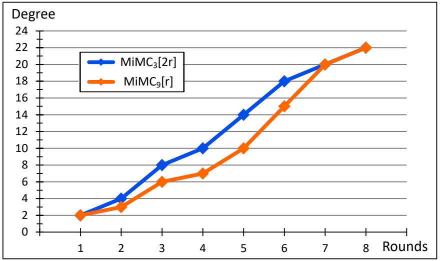

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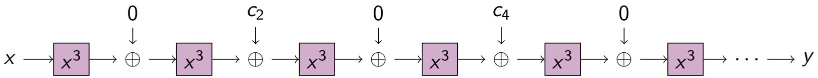

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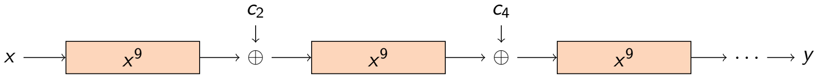

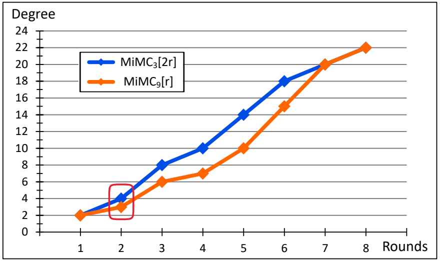

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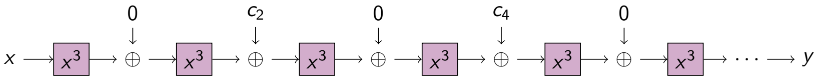
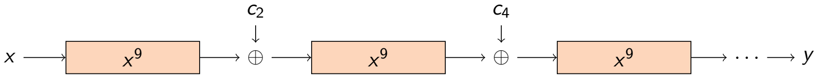
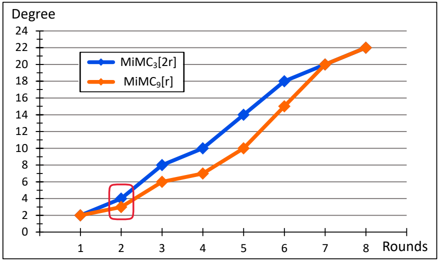


- ⋆ $MIMC_9[r]$





Example: coefficients of maximum weight exponent monomials at round 4

$$27 : c_1^{18} + c_3^2 \qquad 57 : c_1^8$$
$$30 : c_1^{17} \qquad 75 : c_1^2$$
$$51 : c_1^{10} \qquad 78 : c_1$$
$$54 : c_1^9 + c_3$$

Other Quadratic functions

Proposition

Let $\mathcal{E}_r$ be the set of exponents in the univariate form of $\text{MIMC}_9[r]$. Then:

$$\forall\, i \in \mathcal{E}_r,\ i \bmod 8 \in \{0, 1\}\ .$$

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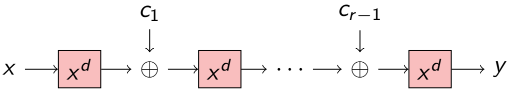
Gold Functions: x^3, x^9, $\ldots$



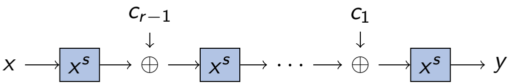
Proposition

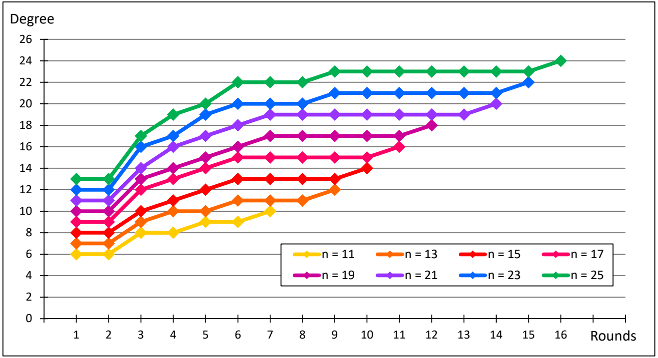
Let $\mathcal{E}_r$ be the set of exponents in the univariate form of MIMC$_d[r]$, where $d = 2^j + 1$. Then:

$$\forall\, i \in \mathcal{E}_r,\ i \bmod 2^j \in \{0, 1\}\ .$$

Algebraic degree of $MiMC_3^{-1}$

Inverse: $F : x \mapsto x^s, s = (2^{n+1} - 1)/3 = [101..01]_2$





Some ideas studied

Plateau between rounds 1 and 2, for $s = (2^{n+1} - 1)/3 = [101..01]_2$:

- ⋆ Round 1: $B_s^1 = wt(s) = (n+1)/2$
- ⋆ Round 2: $B_s^2 = \max\{wt(is), \text{ for } i \preceq s\} = (n+1)/2$

Proposition

For $i \preceq s$ such that $wt(i) \geq 2$:

$$wt(is) \in \begin{cases} [wt(i) - 1, (n-1)/2] & \text{if } \; wt(i) \equiv 2 \bmod 3 \\ [wt(i), (n-1)/2] & \text{if } \; wt(i) \equiv 0 \bmod 3 \\ [wt(i), (n+1)/2] & \text{if } \; wt(i) \equiv 1 \bmod 3 \end{cases}$$

Some ideas studied

Plateau between rounds 1 and 2, for $s = (2^{n+1} - 1)/3 = [101..01]_2$:

- ⋆ Round 1: $B_s^1 = wt(s) = (n+1)/2$
- ⋆ Round 2: $B_s^2 = \max\{wt(is), \text{ for } i \preceq s\} = (n+1)/2$

Proposition

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$$wt(is) \in \begin{cases} [wt(i) - 1, (n-1)/2] & \text{if } wt(i) \equiv 2 \bmod 3 \\ [wt(i), (n-1)/2] & \text{if } wt(i) \equiv 0 \bmod 3 \\ [wt(i), (n+1)/2] & \text{if } wt(i) \equiv 1 \bmod 3 \end{cases}$$

Next rounds: another plateau at $n-2$?

$$\boxed{r_{n-2} \geq \left\lceil \frac{1}{\log_2 3} \left(2 \left\lceil \frac{n-1}{4} \right\rceil + 1 \right) \right\rceil}$$