New uses in Symmetric Cryptography:
From Cryptanalysis to Designing

Clémence Bouvier [1,2]

including joint works with Augustin Bariant[2], Pierre Briaud[1,2], Anne Canteaut[2], Pyrros Chaidos[3], Gaëtan Leurent[2], Léo Perrin[2] and Vesselin Velichkov[4,5]

[1]Sorbonne Université, [2]Inria Paris,
[3]National & Kapodistrian University of Athens, [4]University of Edinburgh, [5]Clearmatics, London

May 20th, 2022

Some motivations

A Cryptanalysis Challenge for ZK-friendly Hash Functions!
In November 2021, by the Ethereum Fundation.

Category	Parameters	Security Level (bits)	Bounty
~~Easy~~	~~r=6~~	~~9~~	~~$2,000~~
~~Easy~~	~~r=10~~	~~15~~	~~$4,000~~
~~Medium~~	~~r=14~~	~~22~~	~~$6,000~~
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(a) *Feistel-MiMC*

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Easy	N=6, m=2	25	$4,000
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(c) *Poseidon*

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Practical Algebraic Attacks against some Arithmetization-oriented Hash Functions

Total Bounty Budget: $200 000

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More and more primitives that need to be better understood!

Content

New uses in Symmetric Cryptography:
From Cryptanalysis to Designing.

1. Emerging uses in symmetric cryptography
   - A need of new primitives
   - Comparison with “usual” case
2. On the algebraic degree of $MiMC_3$
   - Preliminaries
   - Exact degree
   - Integral attacks
3. Practical Attacks
4. Anemoi
   - CCZ-equivalence
   - New Mode

Background

Symmetric cryptography: we assume that a key is already shared.

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- ⋆ Block cipher

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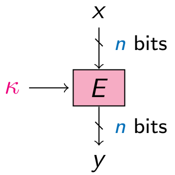
- ⋆ symmetry: E and E^{-1} use the same κ



Block cipher

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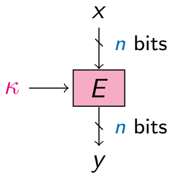

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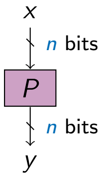

Random permutation

A need of new primitives

Problem: Analyzing the security of new symmetric primitives

Protocols requiring new primitives:

- ⋆ Multiparty Computation (MPC)
- ⋆ Homomorphic Encryption (FHE)
- ⋆ Systems of Zero-Knowledge (ZK) proofs
  Example: SNARKs, STARKs, Bulletproofs

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Primitives designed to minimize the number of multiplications in finite fields.

$\Rightarrow$ What differs from the "usual" case?

Comparison with "usual" case

A new environment

"Usual" case

- Field size:
  $\mathbb{F}_{2^n}$, with $n \simeq 4, 8$ (AES: $n = 8$).
- Operations:
  logical gates/CPU instructions

Arithmetization-friendly

- Field size:
  $\mathbb{F}_q$, with $q \in \{2^n, p\}, p \simeq 2^n$, $n \geq 64$.
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"Usual" case

- Operations:
  $$y \leftarrow R(x)$$
- Efficiency:
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- Minimize the number of multiplications in $\mathbb{F}_{2^n}$.
- Construction of MiMC_3 [Albrecht et al., EC16]:
  - n-bit blocks (n odd ≈ 129)
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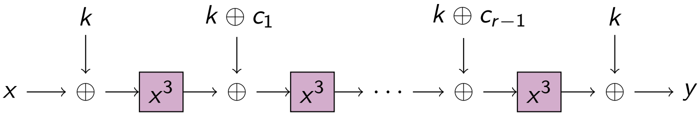
  - decryption : replacing x^3 by x^s where $s = (2^{n+1} - 1)/3$



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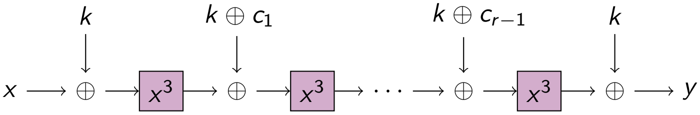
n	129	255	769	1025
R	82	161	486	647

Number of rounds for MiMC instances.



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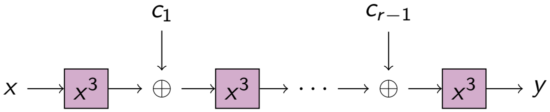

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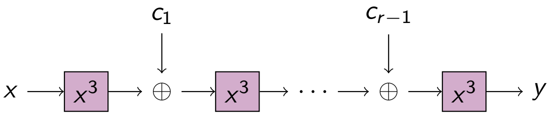

☞ *Bouvier, Canteaut, Perrin*
On the Algebraic Degree of Iterated Power Functions

Algebraic degree

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$, there is **a unique multivariate polynomial** in $\mathbb{F}_2[x_1, \ldots x_n] / \left((x_i^2 + x_i)_{1 \le i \le n}\right)$:

$$f(x_1, ..., x_n) = \sum_{u \in \mathbb{F}_2^n} a_u x^u, \text{ where } a_u \in \mathbb{F}_2,\ x^u = \prod_{i=1}^{n} x_i^{u_i} .$$

This is the **Algebraic Normal Form (ANF)** of f.

Definition

Algebraic Degree of $f : \mathbb{F}_2^n \to \mathbb{F}_2$:

$$\deg^a(f) = \max \left\{ \mathrm{wt}(u) : u \in \mathbb{F}_2^n, a_u \neq 0 \right\} ,$$

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If $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, then

$$\deg^a(F) = \max\{\deg^a(f_i), \; 1 \leq i \leq m\} \; .$$

where $F(x) = (f_1(x), \dots f_m(x))$.

Algebraic degree

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$. Then using the isomorphism $\mathbb{F}_2^n \simeq \mathbb{F}_{2^n}$,
there is **a unique univariate polynomial representation** on $\mathbb{F}_{2^n}$ of degree at most $2^n - 1$:

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If $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is a permutation, then

$$\boxed{\deg^a(F) \le n - 1}$$

Higher-order differential attack

Exploiting a low algebraic degree

For any affine subspace $\mathcal{V} \subset \mathbb{F}_2^n$ with $\dim \mathcal{V} \geq \deg^a(F) + 1$, we have a 0-sum distinguisher:

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Random permutation: degree $= n - 1$

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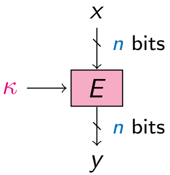

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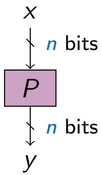

Random permutation

First Plateau

Round i of MiMC_3: $x \mapsto (x + c_{i-1})^3$.

For r rounds:

- ⋆ Upper bound [Eichlseder et al., AC20]: $\lceil r \log_2 3 \rceil$.
- ⋆ Aim: determine $B_3^r := \max_c \deg^a \mathrm{MIMC}_{3,c}[r]$.

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$$9 = [1001]_2 \; 6 = [110]_2 \; 3 = [11]_2$$

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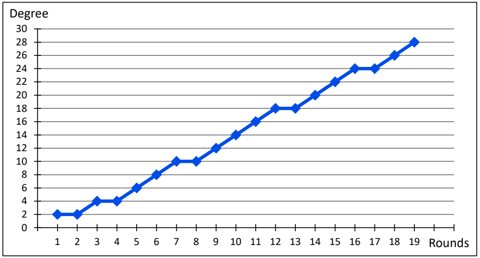

Algebraic degree observed for $n = 31$.

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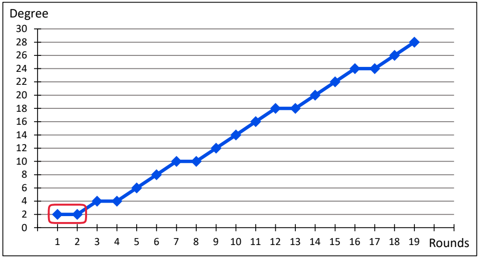

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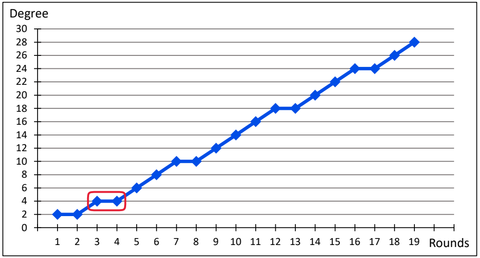

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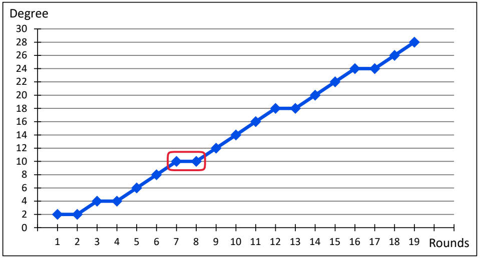

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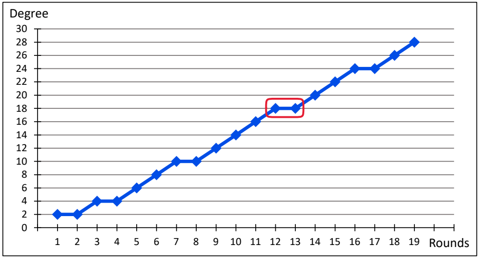
$$\mathcal{P}_2(x) = x^9 + c_1 x^6 + c_1^2 x^3 + c_1^3$$

$$9 = [1001]_2 \; 6 = [110]_2 \; 3 = [11]_2$$

Definition

There is a **plateau** whenever $B_3^r = B_3^{r-1}$.



Algebraic degree observed for $n = 31$.

First Plateau

Round i of MiMC$_3$: $x \mapsto (x + c_{i-1})^3$.

For r rounds:

- ⋆ Upper bound [Eichlseder et al., AC20]: $\lceil r \log_2 3 \rceil$.
- ⋆ Aim: determine $B_3^r := \max_c \deg^a \text{MIMC}_{3,c}[r]$.

Definition

There is a **plateau** whenever $B_3^r = B_3^{r-1}$.

⋆ Round 1: $B_3^1 = 2$

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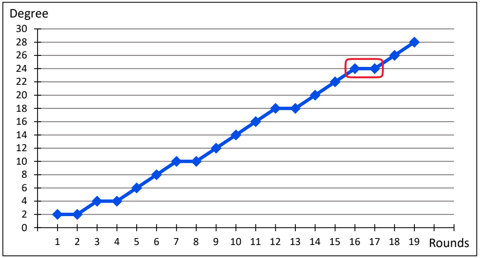

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An upper bound

Proposition

Set of exponents that might appear in the polynomial:

$$\mathcal{E}_r = \{3j \bmod (2^n - 1) \text{ where } j \preceq i,\ i \in \mathcal{E}_{r-1}\}$$

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Example:

$$\mathcal{P}_1(x) = x^3 \quad \Rightarrow \quad \mathcal{E}_1 = \{3\} \ .$$

$$3 = [11]_2 \quad \xrightarrow{\succeq} \quad \begin{cases} [00]_2 = 0 & \xrightarrow{\times 3} \quad 0 \\ [01]_2 = 1 & \xrightarrow{\times 3} \quad 3 \\ [10]_2 = 2 & \xrightarrow{\times 3} \quad 6 \\ [11]_2 = 3 & \xrightarrow{\times 3} \quad 9 \end{cases}$$

$$\mathcal{E}_2 = \{0, 3, 6, 9\} \ ,$$

$$\mathcal{P}_2(x) = x^9 + c_1 x^6 + c_1^2 x^3 + c_1^3 \ .$$

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No exponent $\equiv 5, 7 \bmod 8 \Rightarrow$ No exponent $2^{2k} - 1$

$$\mathcal{E}_r \subseteq \{ \begin{array}{cccccccc} 0 & 3 & 6 & 9 & 12 & \not{15} & 18 & \not{21} \\ 24 & 27 & 30 & 33 & 36 & \not{39} & 42 & \not{45} \\ 48 & 51 & 54 & 57 & 60 & \not{63} & 66 & \not{69} \\ & & & & & & & \\ \dots & 3^r\} & & & & & & \end{array}$$

Example: $63 = 2^{2\times 3} - 1 \notin \mathcal{E}_4 = \{0, 3, \dots, 81\}$ $\Rightarrow B_3^4 < 6 = wt(63)$

$\forall e \in \mathcal{E}_4 \backslash \{63\}, wt(e) \leq 4$ $\Rightarrow B_3^4 \leq 4$

Bounding the degree

Theorem

After r rounds of MiMC, the algebraic degree is

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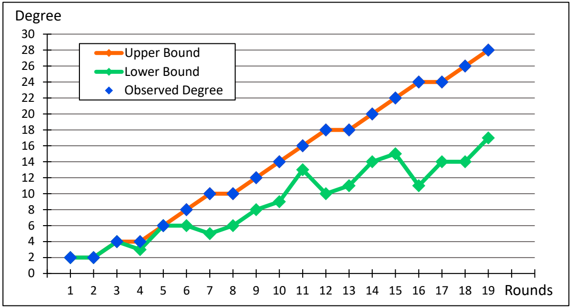
And a lower bound if $3^r < 2^n - 1$:

$$B_3^r \geq wt(3^r)$$



Exact degree

Maximum-weight exponents:

Let $k_r = \lfloor r \log_2 3 \rfloor$.

$\forall r \in \{4, \ldots, 16265\} \setminus \mathcal{F}$ with $\mathcal{F} = \{465, 571, \ldots\}$:

- ⋆ if k_r is odd,

$$\omega_r = 2^{k_r} - 5 \in \mathcal{E}_r,$$

- ⋆ if k_r is even,

$$\omega_r = 2^{k_r} - 7 \in \mathcal{E}_r.$$

Example:

$$123 = 2^7 - 5 = 2^{k_5} - 5 \qquad \in \mathcal{E}_5,$$
$$4089 = 2^{12} - 7 = 2^{k_8} - 7 \qquad \in \mathcal{E}_8.$$

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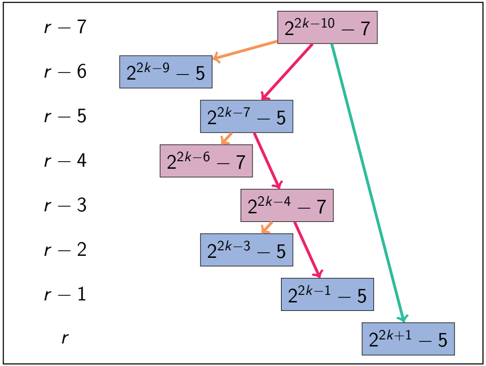

Constructing exponents.

$$\exists \ell \text{ s.t. } \quad \omega_{r-\ell} \in \mathcal{E}_{r-\ell} \Rightarrow \omega_r \in \mathcal{E}_r$$

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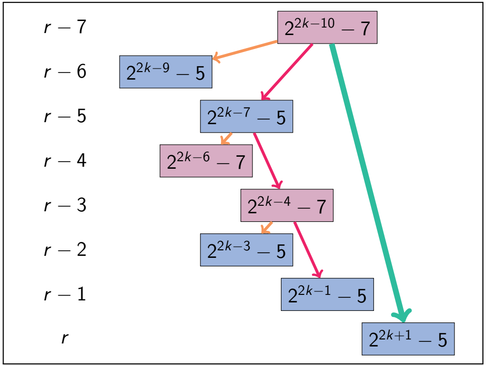

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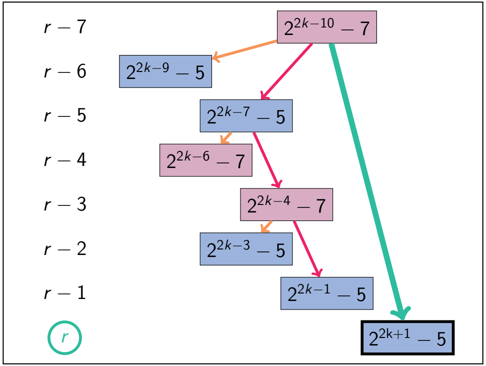

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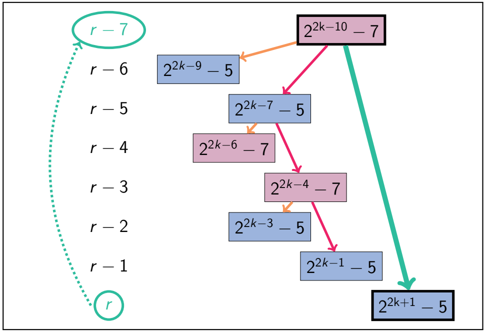

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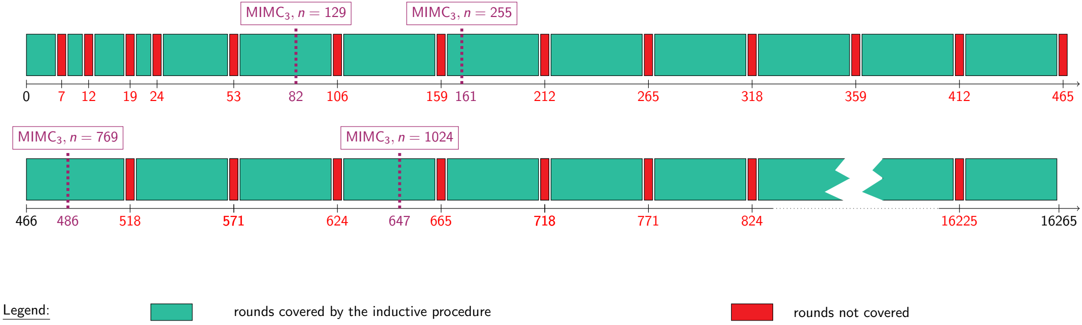
Covered rounds

Idea of the proof:

- ⋆ inductive proof: existence of "good" ℓ

Rounds for which we are able to exhibit a maximum-weight exponent.



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- ⋆ MILP solver (`PySCIPOpt`)

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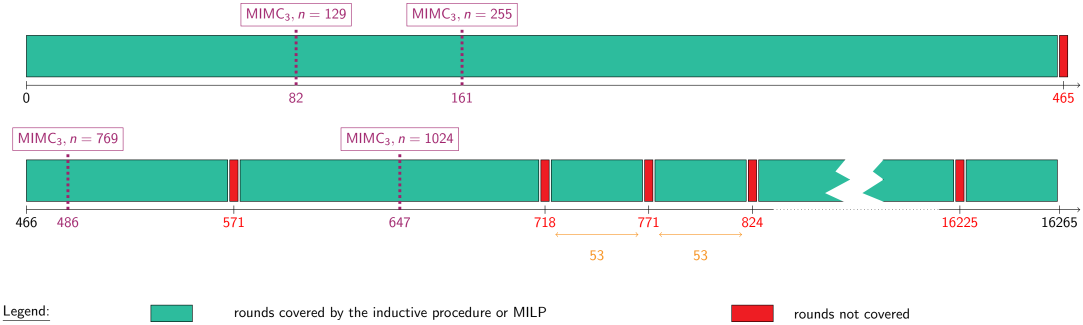
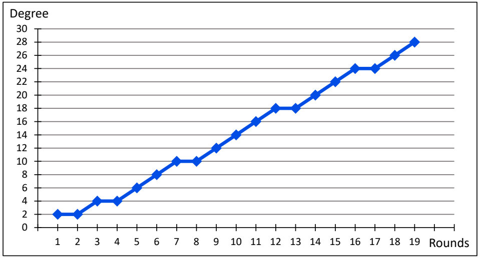


Plateau

$\Rightarrow$ plateau when $k_r = \lfloor r \log_2 3 \rfloor$ is odd and $k_{r+1} = \lfloor (r+1) \log_2 3 \rfloor$ is even



Algebraic degree observed for $n = 31$.

If we have a plateau

$$B_3^r = B_3^{r+1} \ ,$$

Then the next one is

$$B_3^{r+4} = B_3^{r+5} \quad \text{or} \quad B_3^{r+5} = B_3^{r+6} \ .$$

Music in MIMC$_3$

- Patterns in sequence $(k_r)_{r>0}$:

$$\Rightarrow \text{denominators of semiconvergents of } \log_2(3) \simeq 1.5849625$$

$$\mathfrak{D} = \{\boxed{1}, \boxed{2}, 3, 5, \boxed{7}, \boxed{12}, 17, 29, 41, \boxed{53}, 94, 147, 200, 253, 306, \boxed{359}, \ldots\},$$

$$\log_2(3) \simeq \frac{a}{b} \quad \Leftrightarrow \quad 2^a \simeq 3^b$$

- **Music theory:**
  - perfect octave 2:1
  - perfect fifth 3:2

$$2^{19} \simeq 3^{12} \quad \Leftrightarrow \quad 2^7 \simeq \left(\frac{3}{2}\right)^{12} \quad \Leftrightarrow \quad \text{7 octaves} \sim \text{12 fifths}$$

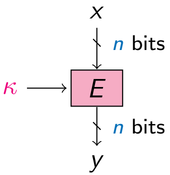
Higher-order differential attack

Exploiting a low algebraic degree

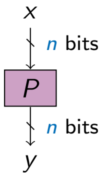
For any affine subspace $\mathcal{V} \subset \mathbb{F}_2^n$ with $\dim \mathcal{V} \geq \deg^a(F) + 1$, we have a 0-sum distinguisher:

$$\bigoplus_{x \in \mathcal{V}} F(x) = 0.$$

Random permutation: degree $= n - 1$



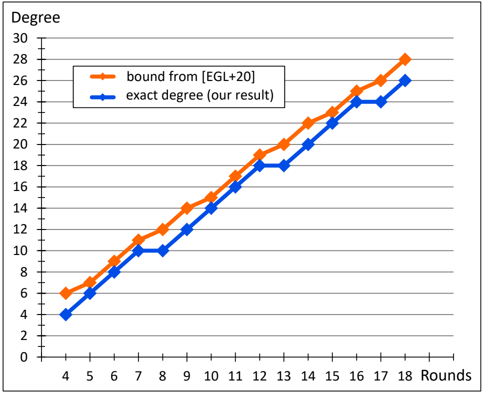
Block cipher



Random permutation

Comparison to previous work

First Bound: $\lceil r \log_2 3 \rceil \quad \Rightarrow \quad$ Exact degree: $2 \times \lceil \lfloor r \log_2 3 \rfloor / 2 - 1 \rceil$.



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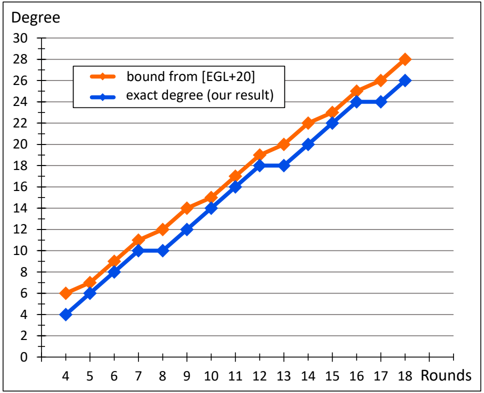


For $n = 129$, MIMC$_3$ $= 82$ rounds

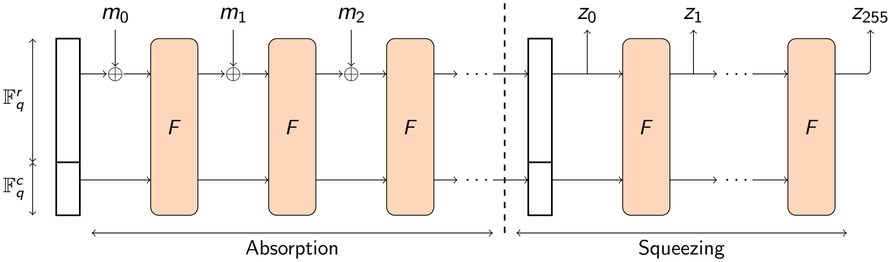
Rounds	Time	Data	Source
80/82	2^{128} XOR	2^{128}	[EGL+20]
81/82	2^{128} XOR	2^{128}	New
80/82	2^{125} XOR	2^{125}	New

Secret-key distinguishers ($n = 129$)

MiMC in a Hash Function

Sponge construction:

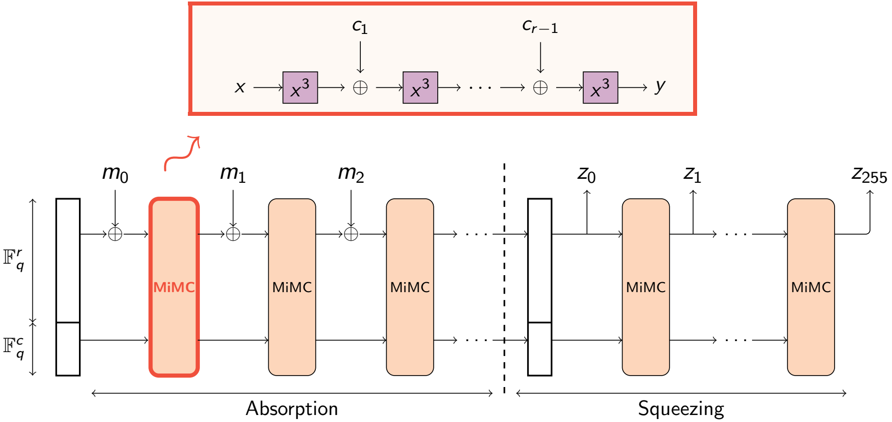
- ⋆ rate $r > 0$
- ⋆ capacity $c > 0$
- ⋆ permutation of $\mathbb{F}_q^r \times \mathbb{F}_q^c$



Hash function in sponge framework.

MiMC in a Hash Function

MiMC-Hash: $MIMC_3$ used as a permutation in a hash function ($\approx$ 90 rounds)



Hash function in sponge framework.

Some values of p

Parameter p given by Elliptic Curves.

Example:

- ⋆ Curve BLS12-381 $\log_2 p = 381$

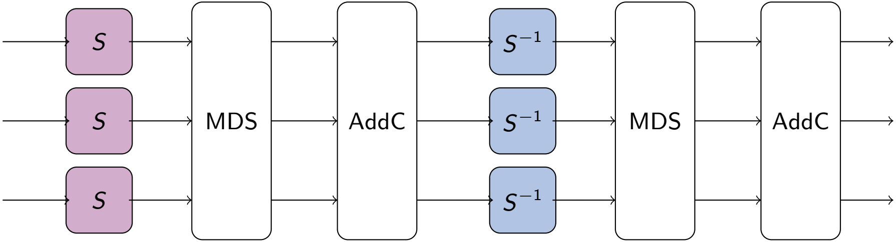
$$p = 4002409555221667393417789825735904156556882819939007885332\\058136124031650490837864442687629129015664037894272559787$$

- ⋆ Curve BLS12-377 $\log_2 p = 377$

$$p = 2586644260129690940106527336948935335363935127549146605398\\842626667204683483408227749688881395733601244403214581 77$$

Rescue

- S-Box layer
- Linear layer
- Constants addition



The 2 steps of round i of Rescue.

Rescue

- S-Box layer
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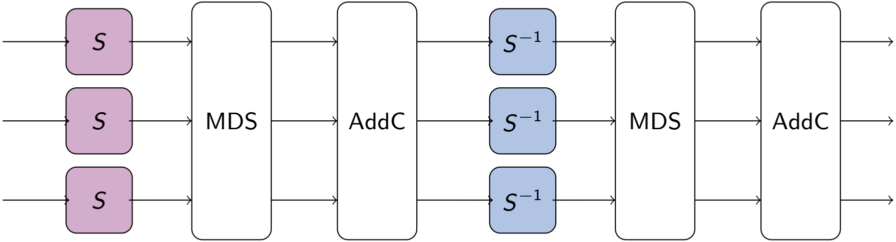
$S : x \mapsto x^{\alpha}$, and $S^{-1} : x \mapsto x^{1/\alpha}$ $(\alpha = 3)$

$R \approx 10$



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$$p = 4002409555221667393417789825735904156556882819939007885332\\058136124031650490837864442687629129015664037894272559787$$

$$\alpha = 5$$

$$\alpha^{-1} = 3201927644177339147342318605887233252455062559512063082656\\4650889922532039267029155415010330321253123031541804782 9$$

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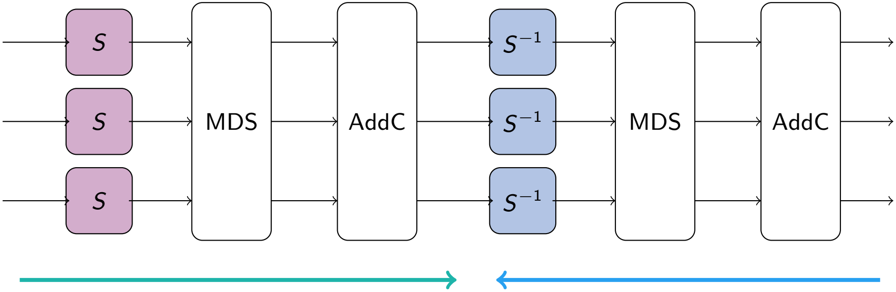

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Poseidon

- S-Box layer
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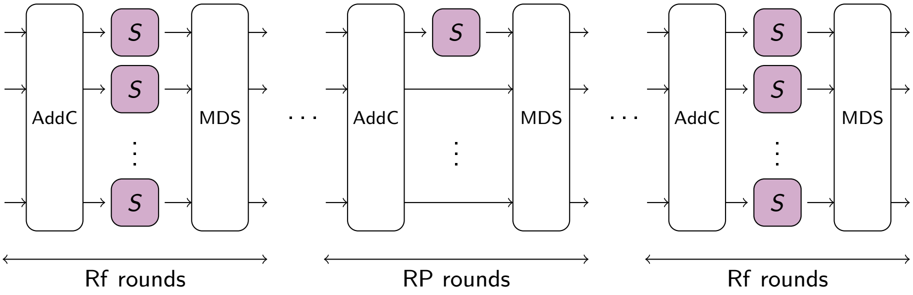

Overview of Poseidon.

Poseidon

- S-Box layer
- Linear layer
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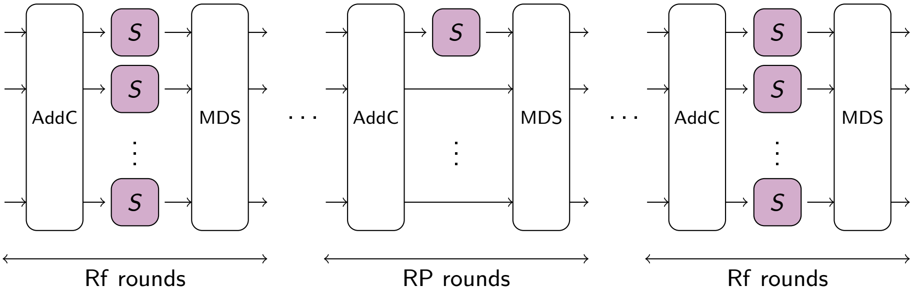
$S : x \mapsto x^{\alpha}$, $(\alpha = 3)$

$R = \text{RF} + \text{RP} \approx 50$



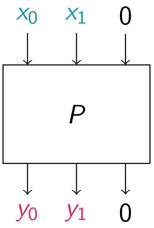
Overview of Poseidon.

CICO Problem

☞ *Bariant, Bouvier, Leurent, Perrin*
Practical Algebraic Attacks against some Arithmetization-oriented Hash Functions

Definition

Constrained Input Constrained Output (CICO) problem:
Find $X, Y \in \mathbb{F}_q^{t-u}$ s.t. $P(X, 0^u) = (Y, 0^u)$.



CICO problem when $t = 3$, $u = 1$.

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- Solving Univariate systems:

  Find the roots of a polynomial $P \in \mathbb{F}_p[X]$.

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$$\begin{cases} P_1(X_1, \ldots X_n) = 0 \\ P_2(X_1, \ldots X_n) = 0 \\ \vdots \\ P_n(X_1, \ldots X_n) = 0, \end{cases}$$

  compute a Gröbner basis...

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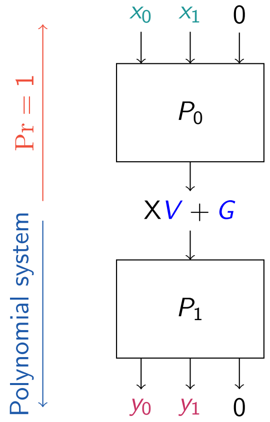

A 2-staged trick.

Goals and Principles

Anemoi

Bouvier, Briaud, Chaidos, Perrin, Velichkov

A family of hash functions exploiting the link between arithmetization-friendliness and CCZ-equivalence.

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- ⋆ Design Consistency. → same structure for all uses

CCZ-equivalence

Definition

$F : \mathbb{F}_q \to \mathbb{F}_q$ and $G : \mathbb{F}_q \to \mathbb{F}_q$ are **CCZ-equivalent**

$$\Gamma_F = \left\{ (x, F(x)) \mid x \in \mathbb{F}_q \right\} = \mathcal{A}(\Gamma_G) = \left\{ \mathcal{A}(x, F(x)) \mid x \in \mathbb{F}_q \right\} ,$$

where $\mathcal{A}$ is an affine permutation.

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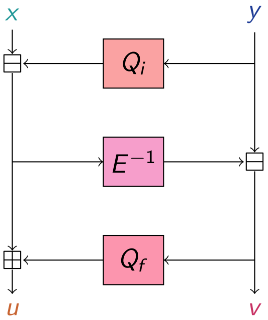
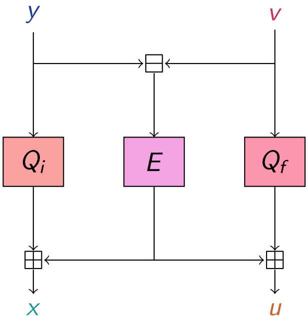


Open Flystel $\mathcal{H}$.

Low-degree function



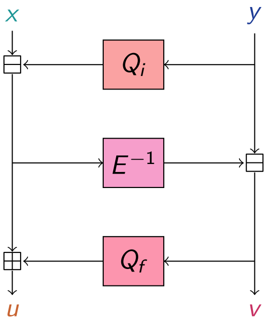
Closed Flystel $\mathcal{V}$.

CCZ-equivalence

$$\Gamma_{\mathcal{H}} = \mathcal{A}(\Gamma_{\mathcal{V}})$$

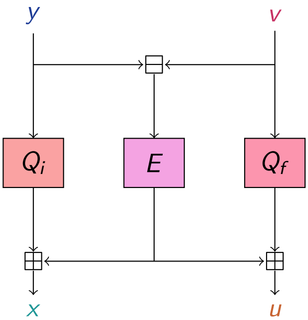
$$\{(x, y), (u, v)\} = \mathcal{A}\left(\{(y, v), (x, u)\}\right)$$

High-degree permutation



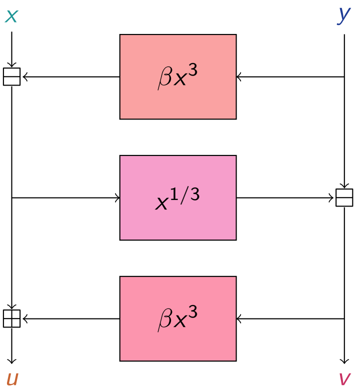
Open Flystel $\mathcal{H}$.

Low-degree function

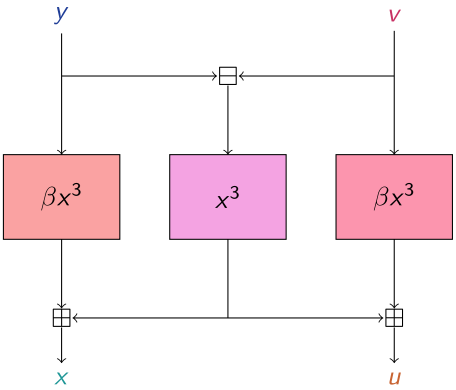


Closed Flystel $\mathcal{V}$.

Flystel in $\mathbb{F}_{2^n}$

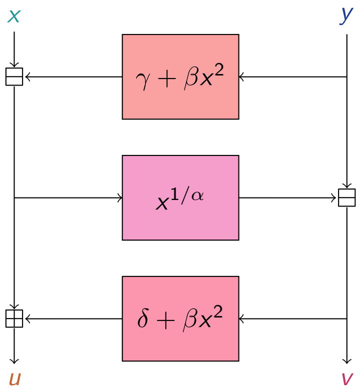


Open Flystel$_2$.

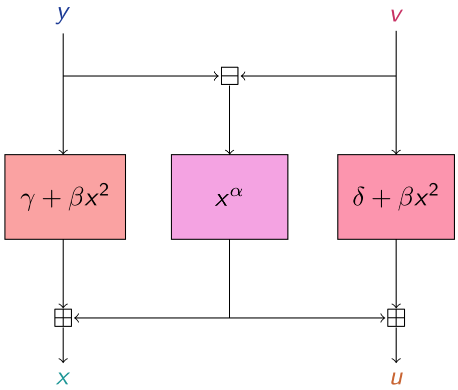


Closed Flystel$_2$.

Flystel in $\mathbb{F}_p$



Open $\texttt{Flystel}_p$.

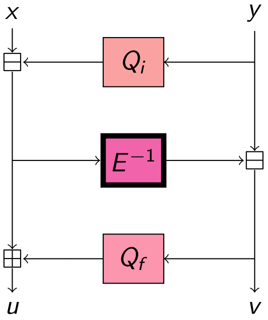


Closed $\texttt{Flystel}_p$.

Advantage of CCZ-equivalence

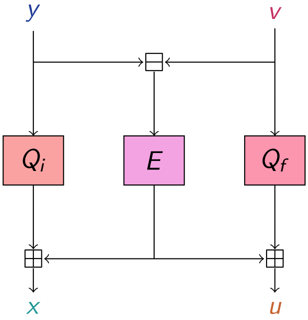
⋆ High Degree Evaluation.

High-degree permutation



Open `Flystel` $\mathcal{H}$.

Low-degree function



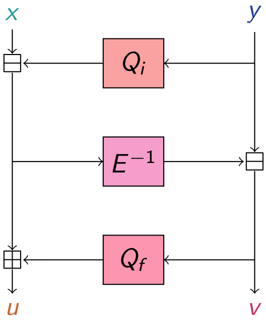
Closed `Flystel` $\mathcal{V}$.

Advantage of CCZ-equivalence

- ⋆ High Degree Evaluation.
- ⋆ Low Cost Verification.

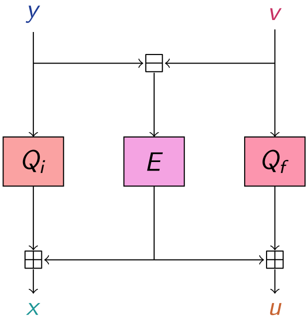
$$(u, v) == \mathcal{H}(x, y) \Leftrightarrow (x, u) == \mathcal{V}(y, v)$$

High-degree permutation



Open Flystel $\mathcal{H}$.

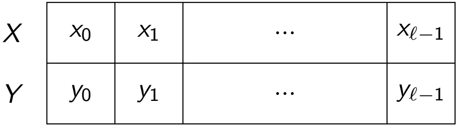
Low-degree function



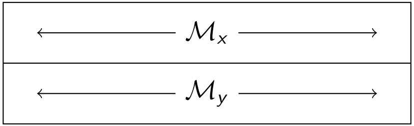
Closed Flystel $\mathcal{V}$.

The SPN Structure

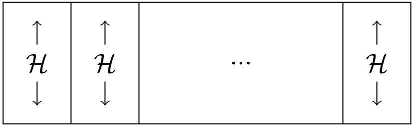
The internal state of `Anemoi` and its basic operations.



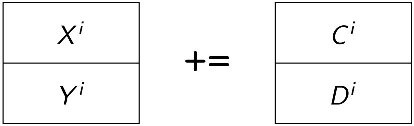
(a) *Internal state*



(b) *The diffusion layer $\mathcal{M}$.*



(c) *The S-box layer $\mathcal{S}$.*



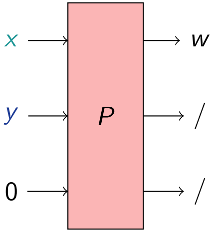
(d) *The constant addition $\mathcal{A}$.*

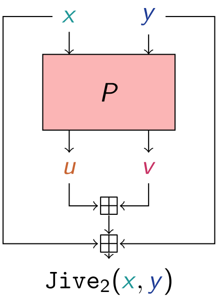
New Mode

- ⋆ Random oracle replacement: `AnemoiRO`
- ⋆ Collision resistant compression function for Merkle-trees: `AnemoiMC`

  Dedicated mode ⇒ 2 words in 1

$$(x, y) \mapsto x + y + u + v \ .$$





Conclusions

- Algebraic degree of MIMC_3
  - a tight upper bound, up to 16265 rounds:

$$2 \times \lceil \lfloor \log_2(3^r) \rfloor / 2 - 1 \rceil .$$

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☞ More details on eprint.iacr.org/2022/366

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  - a new family of ZK-friendly hash functions
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Thanks for your attention

Sporadic Cases

Bound on ℓ

Observation

$$\forall 1 \leq t \leq 21,\ \forall x \in \mathbb{Z}/3^t\mathbb{Z},\ \exists \varepsilon_2, \ldots, \varepsilon_{2t+2} \in \{0,1\},\ \text{s.t. } x = \sum_{j=2}^{2t+2} \varepsilon_j 4^j \bmod 3^t\ .$$

Let: $k_r = \lfloor r \log_2 3 \rfloor$, $b_r = k_r \bmod 2$ and

$$\mathcal{L}_r = \{\ell,\ 1 \leq \ell < r,\ \text{s.t. } k_{r-\ell} = k_r - k_\ell\}\ .$$

Proposition

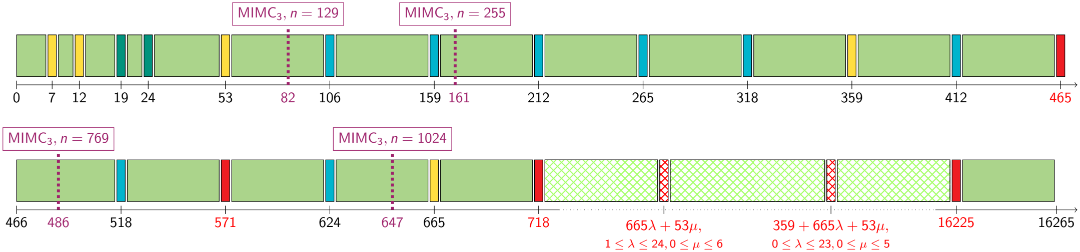
Let $r \geq 4$, and $\ell \in \mathcal{L}_r$ s.t.:

- $\ell = 1, 2$,
- $2 < \ell \leq 22$ s.t. $k_r \geq k_\ell + 3\ell + b_r + 1$, and ℓ is even, or ℓ is odd, with $b_{r-\ell} = \overline{b_r}$;
- $2 < \ell \leq 22$ is odd s.t. $k_r \geq k_\ell + 3\ell + \overline{b_r} + 5$

Then $\omega_{r-\ell} \in \mathcal{E}_{r-\ell}$ implies that $\omega_r \in \mathcal{E}_r$.

Covered Rounds

Rounds for which we are able to exhibit a maximum-weight exponent.



Legend:

Rounds for which we are able to construct an exponent.

- semiconvergents of $\log_2(3)$: MILP
- "good" ℓ
- no "good" ℓ: MILP
- no "good" ℓ ($\ell \geq 53$): MILP

Rounds likely to be covered by solving the conjecture.

- no "good" ℓ: no result with MILP

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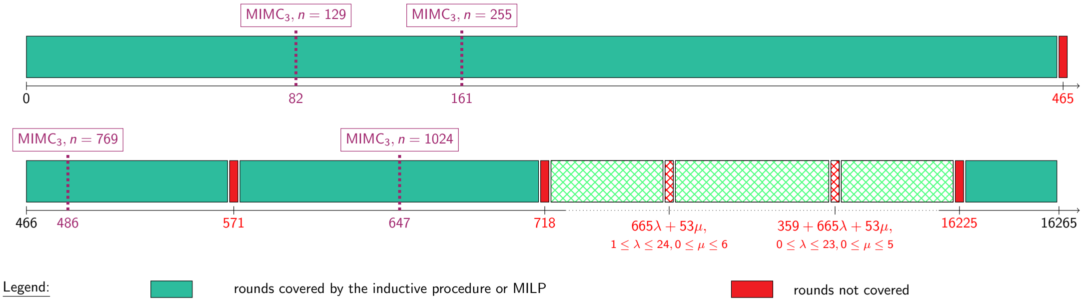


MILP Solver

Let

$$\mathrm{Mult}_3 : \begin{cases} \mathbb{N}^{\mathbb{N}} & \to \mathbb{N}^{\mathbb{N}} \\ \{j_0, \ldots, j_{\ell-1}\} & \mapsto \{(3j_0) \bmod (2^n - 1), \ldots, (3j_{\ell-1}) \bmod (2^n - 1)\} \end{cases},$$

and

$$\mathrm{Cover} : \begin{cases} \mathbb{N}^{\mathbb{N}} & \to \mathbb{N}^{\mathbb{N}} \\ \{j_0, \ldots, j_{\ell-1}\} & \mapsto \{k \preceq j_i, i \in \{0, \ldots, \ell - 1\}\} \end{cases}.$$

So that:

$$\mathcal{E}_r = \mathrm{Mult}_3\big(\mathrm{Cover}(\mathcal{E}_{r-1})\big) .$$

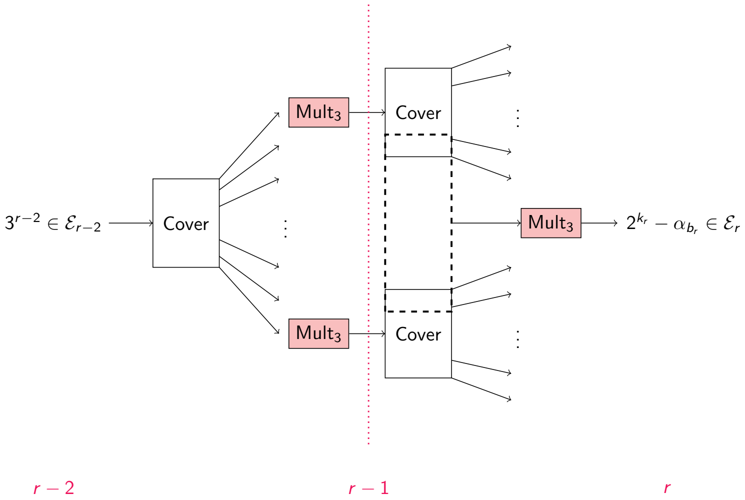
$\Rightarrow$ MILP problem solved using `PySCIPOpt`

$$\boxed{\text{existence of a solution} \quad \Leftrightarrow \quad \omega_r \in (\mathrm{Mult}_3 \circ \mathrm{Cover})^{\ell}(\{3^{r-\ell}\})}$$

<u>With $\ell = 1$</u>:

$$3^{r-1} \in \mathcal{E}_{r-1} \longrightarrow \boxed{\mathrm{Cover}} \longrightarrow \boxed{\mathrm{Mult}_3} \longrightarrow 2^{k_r} - \alpha_{b_r} \in \mathcal{E}_r$$

MILP Solver (2 rounds)



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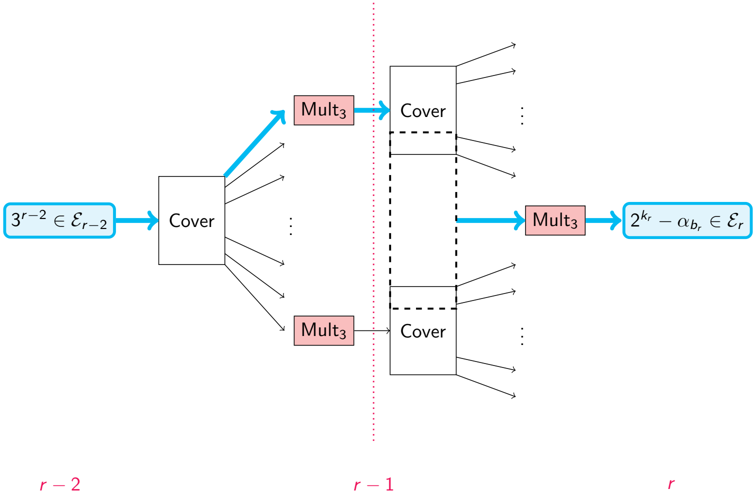

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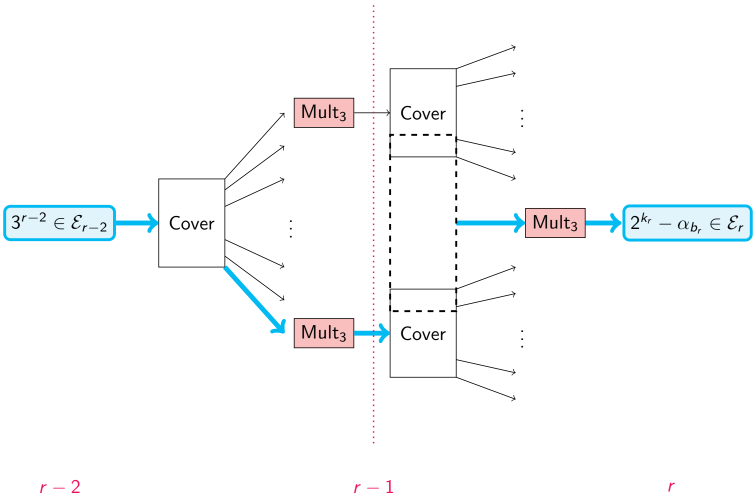
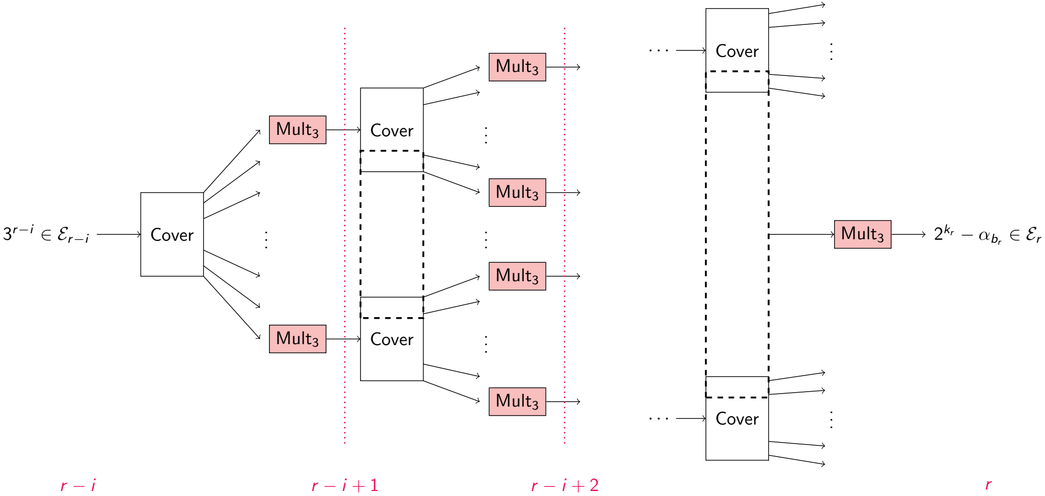


MILP Solver (i rounds)



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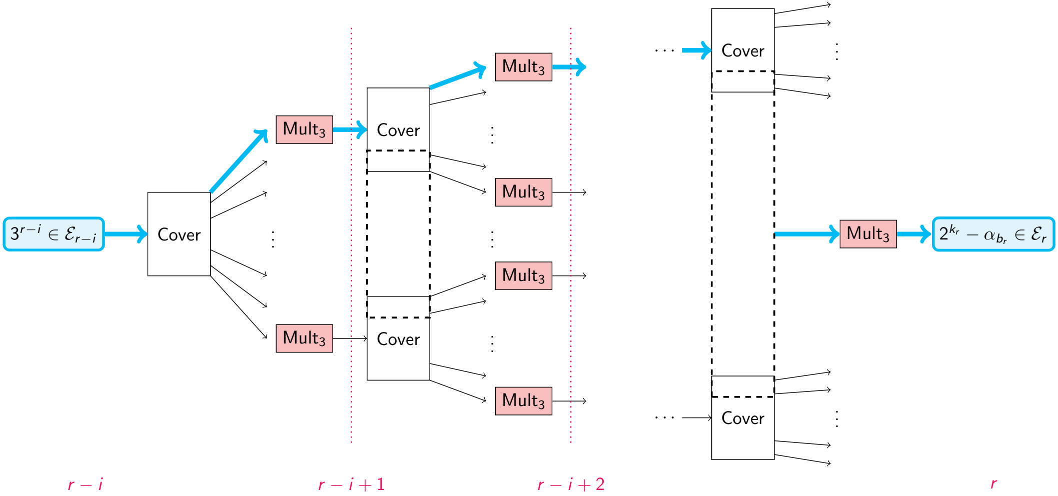

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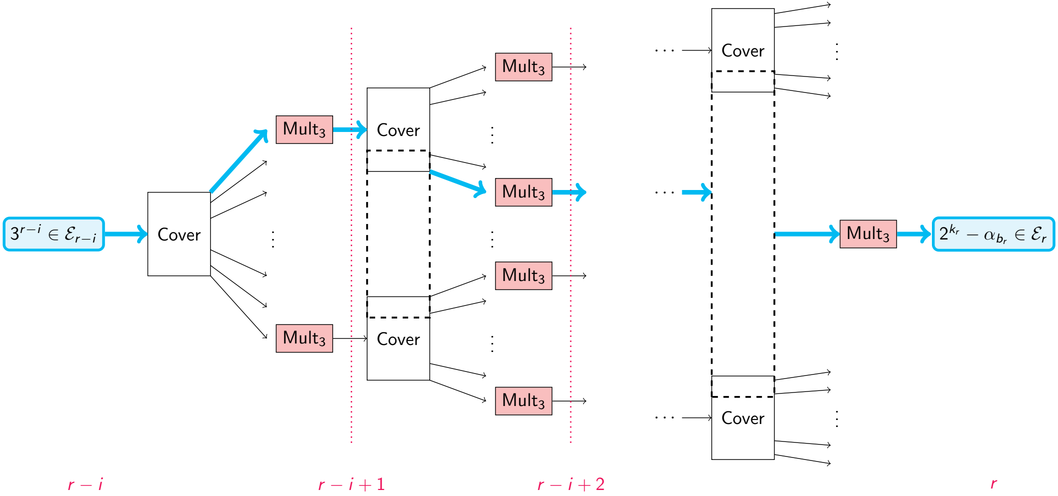

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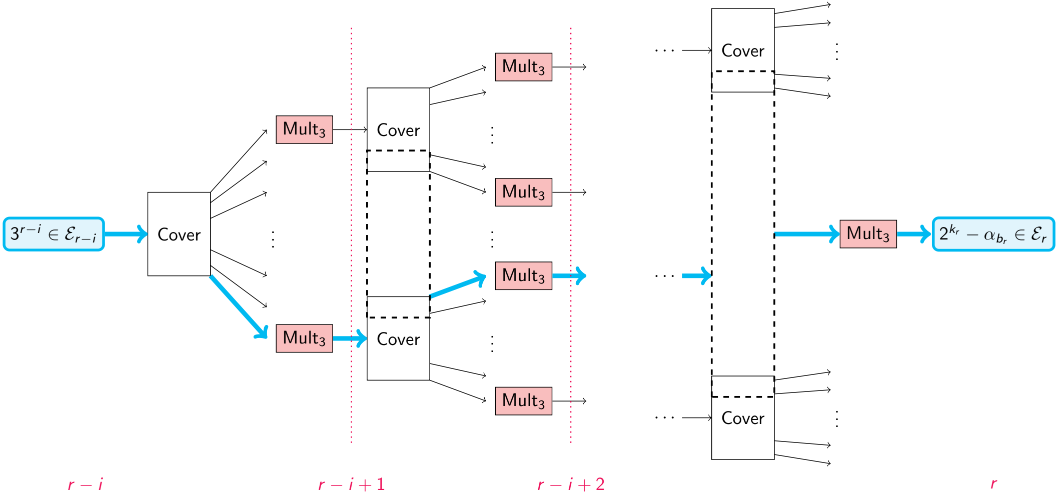

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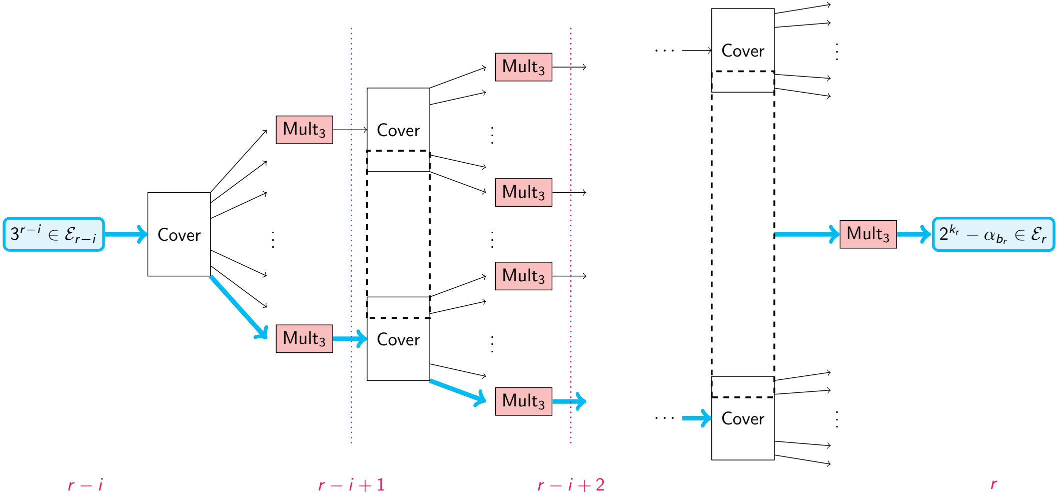
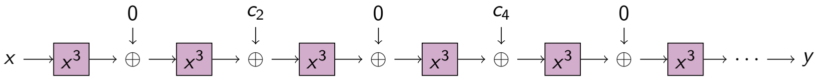


$MiMC_9$ and form of coefficients

- $MIMC_3[2r]$



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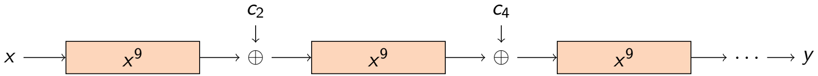

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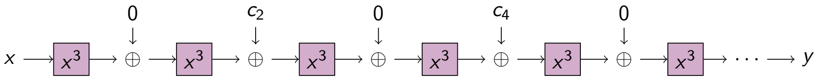

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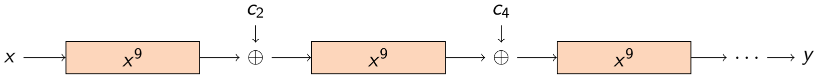

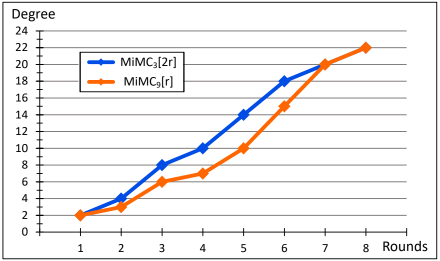

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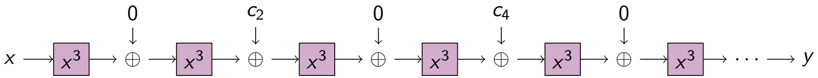

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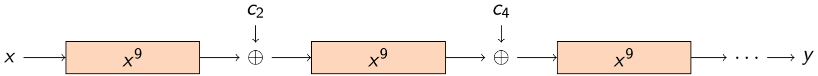

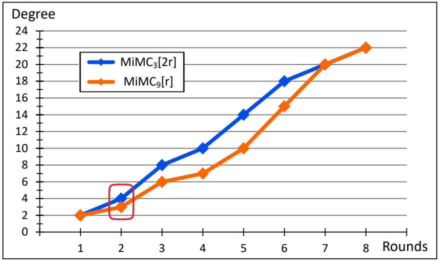

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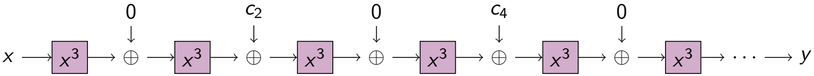

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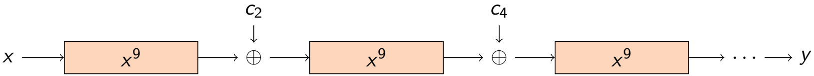

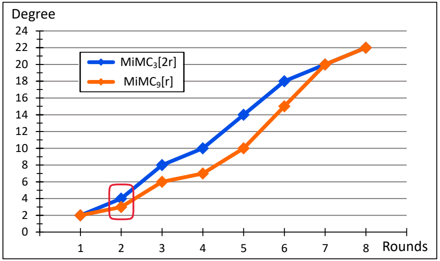

Example: coefficients of maximum weight exponent monomials at round 4

$27 : c_1^{18} + c_3^{2}$

$30 : c_1^{17}$

$51 : c_1^{10}$

$54 : c_1^{9} + c_3$

$57 : c_1^{8}$

$75 : c_1^{2}$

$78 : c_1$

Other Quadratic functions

Proposition

Let $\mathcal{E}_r$ be the set of exponents in the univariate form of MIMC$_9[r]$. Then:

$$\forall\, i \in \mathcal{E}_r,\ i \bmod 8 \in \{0, 1\}\ .$$

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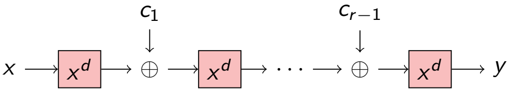
Gold Functions: x^3, x^9, ...



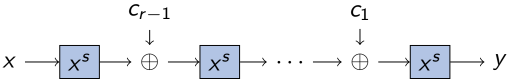
Proposition

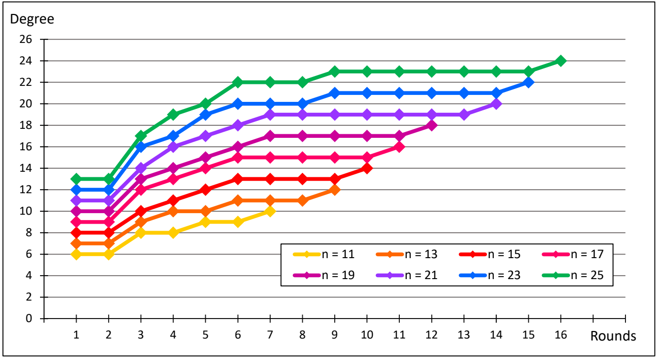
Let $\mathcal{E}_r$ be the set of exponents in the univariate form of MIMC$_d[r]$, where $d = 2^j + 1$. Then:

$$\forall\, i \in \mathcal{E}_r,\ i \bmod 2^j \in \{0, 1\}\ .$$

Algebraic degree of MiMC_3^{-1}

Inverse: $F : x \mapsto x^s, s = (2^{n+1} - 1)/3 = [101..01]_2$





Some ideas studied

Plateau between rounds 1 and 2, for $s = (2^{n+1} - 1)/3 = [101..01]_2$:

- Round 1: $B_s^1 = wt(s) = (n+1)/2$
- Round 2: $B_s^2 = \max\{wt(is), \text{ for } i \preceq s\} = (n+1)/2$

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Next rounds: another plateau at $n-2$?

$$\boxed{r_{n-2} \geq \left\lceil \frac{1}{\log_2 3} \left(2 \left\lceil \frac{n-1}{4} \right\rceil + 1\right) \right\rceil}$$