Algebraic methods for counting Euclidean embeddings of rigid graphs

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Abstract. The study of (minimally) rigid graphs is motivated by numerous applications, mostly in robotics and bioinformatics. A major open problem concerns the number of embeddings of such graphs, up to rigid motions, in Euclidean space. We capture embeddability by polynomial systems with suitable structure, so that their mixed volume, which bounds the number of common roots, to yield interesting upper bounds on the number of embeddings. We focus on $\mathbb{R}^2$ and $\mathbb{R}^3$, where Laman graphs and 1-skeleta of convex simplicial polyhedra, respectively, admit inductive Henneberg constructions. We establish the first general lower bound in $\mathbb{R}^3$ of about 2.52^n, where n denotes the number of vertices. Moreover, our implementation yields upper bounds for $n \leq 10$ in $\mathbb{R}^2$ and $\mathbb{R}^3$, which reduce the existing gaps, and tight bounds up to $n = 7$ in $\mathbb{R}^3$.

Keywords: rigid graph, Euclidean embedding, Henneberg construction, polynomial system, root bound, cyclohexane caterpillar

1 Introduction

Rigid graphs (or frameworks) constitute an old but still active area of research due to certain deep mathematical questions, as well as numerous applications, e.g. mechanism theory [9, 10], and structural bioinformatics [5].

Given graph $G = (V, E)$ and a collection of edge lengths $l_{ij} \in \mathbb{R}^+$, for $(i, j) \in E$, a *Euclidean embedding* in $\mathbb{R}^d$ is a mapping of V to a set of points in $\mathbb{R}^d$, such that l_{ij} equals the Euclidean distance between the images of the i-th and j-th vertices, for $(i, j) \in E$. Euclidean embeddings impose no requirements on whether the edges cross each other or not. A graph is *generically rigid* in $\mathbb{R}^d$ iff, for generic edge lengths, admits a finite number of embeddings in $\mathbb{R}^d$, modulo rigid motions. A graph is *minimally rigid* iff it is no longer rigid once any edge is removed. In the sequel, generically minimally rigid graphs are referred to as rigid.

A graph is called *Laman* iff $|E| = 2|V| - 3$ and, additionally, all of its induced subgraphs on $k < |V|$ vertices have $\leq 2k - 3$ edges. The Laman graphs are precisely the rigid graphs in $\mathbb{R}^2$; they also admit inductive constructions. In $\mathbb{R}^3$ there is no analogous combinatorial characterization of rigid graphs, but the 1-skeleta, or edge graphs, of (convex) simplicial polyhedra are rigid in $\mathbb{R}^3$, and admit inductive constructions.

We deal with the problem of computing the maximum number of distinct planar and spatial Euclidean embeddings of rigid graphs, up to rigid motions, as a function of the number of vertices. To study upper bounds, we define a square polynomial system, expressing the edge length constraints, whose real solutions correspond precisely to the different embeddings. Here is a system expressing embeddability in $\mathbb{R}^3$, where (x_i, y_i, z_i) are the coordinates of the i-th vertex, and 3 vertices are fixed to discard translations and rotations:

$$\begin{cases} x_i = a_i,\ y_i = b_i,\ z_i = c_i, & i = 1,2,3, \quad a_i, b_i, c_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = l_{ij}^2, & (i,j) \in E - \{(1,2),(1,3),(2,3)\} \end{cases} \tag{1}$$

All nontrivial equations are quadratic; there are $2n-4$ for Laman graphs, and $3n-9$ for 1-skeleta of simplicial polyhedra, where n is the number of vertices. The classical Bézout bound on the number of roots equals the product of the polynomials' degrees, and yields 4^{n-2} and 8^{n-3}, respectively.

For the planar and spatial case, the best upper bounds are $\binom{2n-4}{n-2} \approx 4^{n-2}/\sqrt{\pi(n-2)}$ and $\frac{2^{n-3}}{n-2}\binom{2n-6}{n-3} \approx 8^{n-3}/\big((n-2)\sqrt{\pi(n-3)}\big)$, respectively [2, 3]. In applications, it is crucial to know the number of embeddings for small n. The main result in this direction was to show that the Desargues (or 3-prism) graph ($n = 6$) admits 24 embeddings in $\mathbb{R}^2$ [3]. This led the same authors to lower bounds in $\mathbb{R}^2$: $24^{\lfloor (n-2)/4 \rfloor} \simeq 2.21^n$ obtained by a caterpillar graph constructed by concatenating copies of the Desargues graph, and $2 \cdot 12^{\lfloor (n-3)/3 \rfloor} \simeq 2.29^n/6$ obtained by a Desargues fan [4].

Bernstein's bound for a polynomial system exploits the sparseness of the equations to bound the number of common roots. It is bounded by Bézout's bound and typically much tighter. We have implemented specialized software that constructs all rigid graphs up to isomorphism, for small n, and computes the respective Bernstein's bounds. Our main contribution is twofold, besides some straightforward upper bounds in Lemmas 2 and 7. First, we derive the first general lower bound in $\mathbb{R}^3$:

$$16^{\lfloor (n-3)/3 \rfloor} \simeq 2.52^n,\ n \geq 9,$$

by designing a cyclohexane caterpillar. Second, we obtain improved upper and lower bounds for $n \leq 10$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Tables 1 and 2). Moreover, we establish tight bounds for $n \leq 7$ in $\mathbb{R}^3$ by appropriately formulating the polynomial system. We apply Bernstein's Second theorem to show that the naive polynomial system cannot yield tight mixed volumes.

The rest of the paper is structured as follows: Section 2 discusses the case $d = 2$, Section 3 presents our algebraic tools and our implementation, Section 4 deals with $\mathbb{R}^3$, and we conclude with open questions.

Some results appeared in [6] in preliminary form.

[4] We have corrected the exponent of the original statement.

2 Planar embeddings of Laman graphs

Laman graphs admit inductive constructions, starting with a triangle, and followed by a sequence of Henneberg-1 (or H_1) and Henneneg-2 steps (or H_2). Each such step adds a new vertex: H_1 connects it to two existing vertices, H_2 connects it to 3 existing vertices having at least one edge among them, which is removed.We represent each Laman graph by $\triangle s_4 \ldots, s_n$, where $s_i \in \{1, 2\}$; this is its Henneberg sequence. A Laman graph is called H_1 iff it can be constructed using only H_1; otherwise it is called H_2. Since two generic circles intersect in two real points, H_1 exactly doubles the maximum number of embeddings. It follows that a H_1 graph has 2^{n-2} embeddings.

One can easily verify that every $\triangle 2$ graph is isomorphic to a $\triangle 1$ graph and that every $\triangle 12$ graph is isomorphic to a $\triangle 11$ graph. Consequently, all Laman graphs with $n = 4, 5$ are H_1 and they have 4 and 8 embeddings, respectively. For $n = 6$, there are 3 possibilities: the graph is either H_1, $K_{3,3}$, or the Desargues graph. Since the $K_{3,3}$ graph has at most 16 embeddings [9, 10] and the Desargues graph has 24 embeddings [3], the latter is the uppper bound. Using our software (Section 3), we construct all Laman graphs with $n = 7, \ldots, 10$, and compute their respective mixed volumes, thus obtaining the following upper bounds.

Lemma 1. *The maximum number of Euclidean embeddings for Laman graphs with* $n = 7, \ldots, 10$ *is 64, 128, 512 and 2048, respectively.*

Table 1 summarizes our results for $n \leq 10$. The lower bound for $n = 9$ follows from the Desargues fan. All other lower bounds follow from the fact that H_1 doubles the number of embeddings.

We now establish an upper bound, which improves upon the existing ones when our graph contains many degree-2 vertices.

Lemma 2. *Let* G *be a Laman graph with* $k \geq 4$ *degree-2 vertices. Then, the number of planar embeddings of* G *is bounded above by* $2^{k-4}4^{n-k}$.

Proof. The removal of a degree-2 vertex cannot destroy any other degree-2 vertex (because the remaining graph is also Laman), although it may create new ones. Since the remaining graph has $n-k$ vertices, the Bézout bound of its polynomial system is equal to 4^{n-k} and thus the number of embeddings is bounded above by $2^{k-4}4^{n-k}$.

3 An algebraic interlude

This section introduces mixed volumes and discusses our computer-assisted proofs.

Given a polynomial f in n variables, its support is the set of exponents in $\mathbb{N}^n$ corresponding to nonzero terms (or monomials). The Newton polytope of f is the convex hull of its support and lies in $\mathbb{R}^n$. Consider polytopes $P_i \subset \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}, \lambda_i \geq 0$, for $i = 1, \ldots, n$. Consider the Minkowski sum $\lambda_1 P_1 + \cdots + \lambda_n P_n \in \mathbb{R}^n$: its (Euclidean) volume is a homogeneous polynomial of degree n in the λ_i. The coefficient of $\lambda_1 \cdots \lambda_n$ is the *mixed volume* of $P_1, \ldots, P_n$. If $P_1 = \cdots = P_n$, then the mixed volume is $n!$ times the volume of P_1. We focus on $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Theorem 3. [1] *Let $f_1 = \cdots = f_n = 0$ be a polynomial system in n variables with real coefficients, where the f_i have fixed supports. The number of isolated common solutions in $(\mathbb{C}^*)^n$ is bounded above by the* mixed volume *of (the Newton polytopes of) the f_i. This bound is tight for generic coefficients of the f_i's.*

Bernstein's Second theorem below was used in $\mathbb{R}^2$ [8]; we apply it to $\mathbb{R}^3$. Given $v \in \mathbb{R}^n - \{0\}$ and polynomial f_i, $\partial_v f_i$ is the polynomial obtained by keeping only the terms whose exponents minimize inner product with v; its Newton polytope is the face of the Newton polytope of f_i supported by v.

Theorem 4. [1] *If for all $v \in \mathbb{R}^n - \{0\}$ the face system $\partial_v f_1 = \ldots = \partial_v f_n = 0$ has no solutions in $(\mathbb{C}^*)^n$, then the mixed volume of the f_i exactly equals the number of solutions in $(\mathbb{C}^*)^n$, and all solutions are isolated. Otherwise, the mixed volume is a strict upper bound on the number of isolated solutions.*

In order to bound the number of embeddings of rigid graphs, we have developed specialized software that constructs all Laman graphs and all 1-skeleta of simplicial polyhedra with $n \leq 10$. Our computational platform is SAGE [5]. We construct all graphs using the Henneberg steps, which we implemented in Python, using SAGE's interpreter. We classify all graphs up to isomorhism using SAGE's interface with N.I.C.E., an open-source isomorphism check software, keeping for each graph the Henneberg sequence with largest number H_1.

For each graph we construct a system whose real solutions express all possible embeddings, using formulation (2). We bound the number of its (complex) solutions by mixed volume. Notice that, by genericity, solutions have no zero coordinates. For every Laman graph, to discard translations and rotations, we assume that one edge is of unit length, aligned with an axis, with one of its vertices at the origin. In $\mathbb{R}^3$, a third vertex is also fixed in a coordinate plane. Depending on the choice of the fixed edge, we obtain different systems hence different mixed volumes, and we use their minimum.

We used an Intel Core2, at 2.4GHz, with 2GB of RAM. We tested more that $20,000$ graphs, computed the mixed volume of more than $40,000$ systems, taking a total time of about 2 days. Tables 1 and 2 summarize our results.

4 Spatial embeddings of 1-skeleta of simplicial polyhedra

This section extends the previous results to 1-skeleta of (convex) simplicial polyhedra, which are rigid in $\mathbb{R}^3$ [7]. For such a graph (V, E), we have $|E| = 3|V| - 6$ and all of the induced subgraphs on $k < |V|$ vertices have $\leq 3k - 6$ edges.

Consider any $k+2$ vertices forming a cycle with $\geq k-1$ diagonals, $k \geq 1$. The extended Henneberg-k step (or H_k), $k = 1, 2, 3$, corresponds to adding a vertex, connecting it to the $k+2$ vertices, and removing $k-1$ diagonals among them. A graph is the 1-skeleton of a simplicial polyhedron in $\mathbb{R}^3$ iff it has a construction starting with the 3-simplex, followed by any sequence of H_1, H_2, H_3 [4].

[5] `http://www.sagemath.org/`

Since 3 spheres intersect generically in two points, H_1 exactly doubles the maximum number of embeddings. In order to discard translations and rotations, we fix a (triangular) facet of the polytope; we choose wlog the first 3 vertices and obtain system (1) of dimension $3n$. Let $v = (0,0,0,0,0,0,0,0,0,-1,\dots,-1) \in \mathbb{R}^{3n}$, the face system is:

$$\begin{cases} x_i = a_i, y_i = b_i, z_i = c_i, & i = 1,2,3, \quad a_i, b_i, c_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = 0, & (i,j) \in E,\ i,j \notin \{1,2,3\}, \\ x_i^2 + y_i^2 + z_i^2 = 0, & (i,j) \in E:\ i \notin \{1,2,3\},\ j \in \{1,2,3\}. \end{cases}$$

This system has $(a_1, b_1, c_1, \dots, a_3, b_3, c_3, 1, 1, \gamma\sqrt{2}, \dots, 1, 1, \gamma\sqrt{2}) \in (\mathbb{C}^*)^{3n}$ as a solution, where $\gamma = \pm\sqrt{-1}$. According to Theorem 4, the mixed volume is not a tight bound on the number of solutions in $(\mathbb{C}^*)^{3n}$. This was also observed, for $\mathbb{R}^2$, in [8]. To remove spurious solutions let $w_i = x_i^2 + y_i^2 + z_i^2$, for $i = 1, \dots, n$. This yields an equivalent system, with lower mixed volume, which will be used in our computations:

$$\begin{cases} x_i = a_i, y_i = b_i, z_i = c_i, & i = 1,2,3, \\ w_i = x_i^2 + y_i^2 + z_i^2, & i = 1, \dots, n, \\ w_i + w_j - 2x_ix_j - 2y_iy_j - 2z_iz_j = l_{ij}^2, & (i,j) \in E - \{(1,2),(1,3),(2,3)\}. \end{cases} \tag{2}$$

For $n = 4$, the only simplicial polytope is the 3-simplex, which has 2 embeddings. For $n = 5$, there is a unique 1-skeleton of a simplicial polyhedron [4], and it is obtained from the 3-simplex by H_1, hence it has exactly 4 embeddings.

Lemma 5. *The 1-skeleton of a simplicial polyhedron for $n = 6$ has at most 16 embeddings and this is tight.*

Proof. There are two non-isomorphic graphs G_1, G_2 [4]. The mixed volumes are 8 and 16. G_2 is the graph of the cyclohexane, which admits 16 different embeddings [5]. To see this, the cyclohexane is a 6-cycle with known lengths between vertices at distance 1 (adjacent) and 2. Alternatively, G_2 corresponds to a Stewart platform parallel robot with 16 configurations, where triangles define the platform and base, and 6 lenghts link the triangles in a jigsaw shape.

Theorem 6. *There exist edge lengths for which the cyclohexane caterpillar construction has $16^{\lfloor (n-3)/3 \rfloor} \simeq 2.52^n$ embeddings, for $n \geq 9$.*

Proof. We glue copies of cyclohexanes sharing a common triangle, each adding 3 vertices. The final graph is the 1-skeleton of a simplicial polytope, and we apply Lemma 5.

Table 2 summarizes our results for $n \leq 10$. The upper bounds for $n = 7, \dots, 10$ are computed by our software. The lower bound for $n = 9$ is from Theorem 6. All other lower bounds are obtained by considering a H_1 construction. Lastly, we state without proof a result similar to Lemma 2.

Lemma 7. *Let G be the 1-skeleton of a simplicial polyhedron with $k \geq 9$ degree-3 vertices. The number of embeddings of G is bounded above by $2^{k-9}8^{n-k}$.*

n =	3	4	5	6	7	8	9	10
lower	2	4	8	24	48	96	288	576
upper	2	4	8	24	64	128	512	2048

Table 1. Bounds for Laman graphs.

n =	4	5	6	7	8	9	10
lower	2	4	16	32	64	256	512
upper	2	4	16	32	160	640	2560

Table 2. Bounds for 1-skeleta of simplicial polyhedra.

5 Further work

The most important and oldest problem in rigidity theory is the combinatorial characterization of rigid graphs in $\mathbb{R}^3$. Since we deal with Henneberg constructions, it is important to determine the effect of each step on the number of embeddings: H_1 doubles their number; we conjecture that H_2 multiplies it by ≤ 4 and H_3 by ≤ 8, but these may not always be tight. Our conjecture has been verified for small n.

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