The Sum can be Weaker Than Each Part

Generic Attacks against the Sum of Two Hash Functions

Gaëtan Leurent[1] Lei Wang[2]

[1] Inria, France

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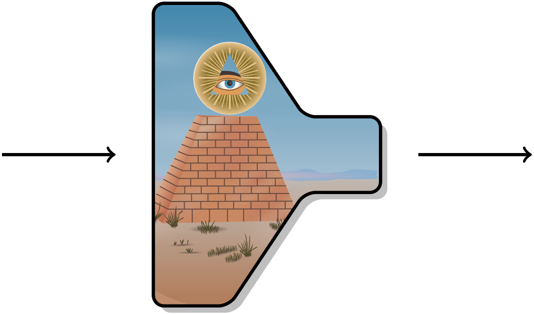
An Ideal Hash Function: the Random Oracle



- Public Random Oracle
- The output can be used as a fingerprint of the document

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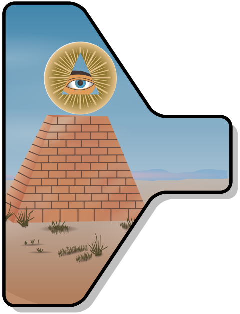


`0x1d66ca77ab361c6f`

- Public Random Oracle
- The output can be used as a fingerprint of the document

Concrete security goals

Preimage attack

Given F and $\overline{H}$, find M s.t. $F(M) = \overline{H}$.
Ideal security: 2^n.

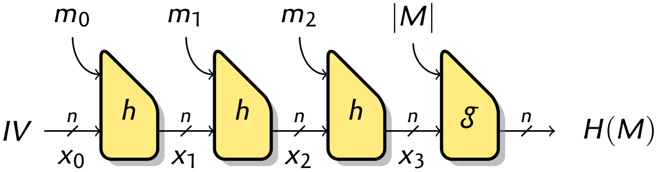
Second-preimage attack

Given F and M_1, find $M_2 \neq M_1$ s.t. $F(M_1) = F(M_2)$.
Ideal security: 2^n.

Collision attack

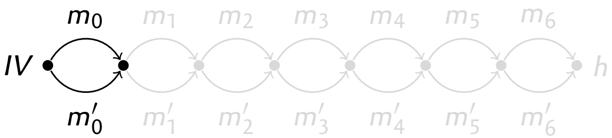
Given F, find $M_1 \neq M_2$ s.t. $F(M_1) = F(M_2)$.
Ideal security: $2^{n/2}$.

Iterated hash function (Merkle-Damgård)



- n-bit state, compression function
- Security with ideal compression function:
  - Collisions: $2^{n/2}$ (optimal)
  - Preimages: 2^n (optimal)
  - Second-preimage: 2^{n-t}
  - Non-ideal after $2^{n/2}$: multi-collisions, herding, long 2nd-preimage

Joux's multicollision attack



1. Find a collision pair m_0 / m'_0 starting from IV
2. Find a collision pair m_1 / m'_1 starting from $x_1 = h(IV, m_0)$
3. Repeat k times
4. This yields 2^k messages with the same hash:

$$\begin{matrix} m_0 m_1 m_2 \ldots & m'_0 m_1 m_2 \ldots & m_0 m'_1 m_2 \ldots & m'_0 m'_1 m_2 \ldots \\ m_0 m_1 m'_2 \ldots & m'_0 m_1 m'_2 \ldots & m_0 m'_1 m'_2 \ldots & m'_0 m'_1 m'_2 \ldots \end{matrix}$$

- Complexity $k \cdot 2^{n/2}$ vs. $\approx 2^{\frac{2^k - 1}{2^k} n}$ for a random function

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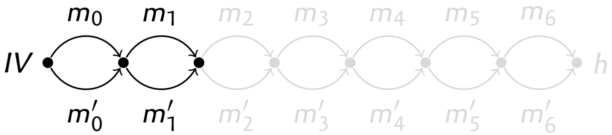

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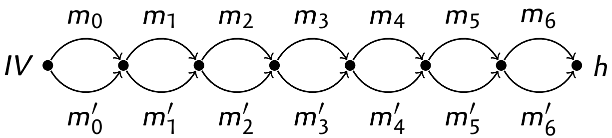

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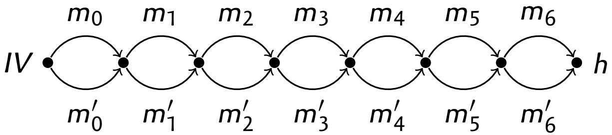

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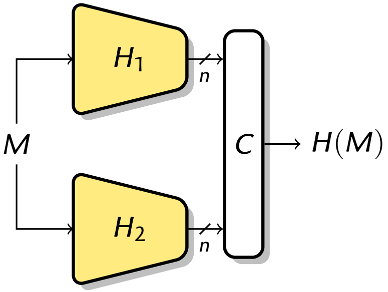
Combining two hash functions



"In order to make the PRF as secure as possible, it uses two hash algorithms in a way which should guarantee its security if either algorithm remains secure."

– RFC 2246 (TLS 1.0)

Classical combiners:

- Concatenation: $H_1(M) \| H2(M)$
- Xor: $H_1(M) \oplus H2(M)$

"The whole is greater than the sum of its parts"

– Aristotle

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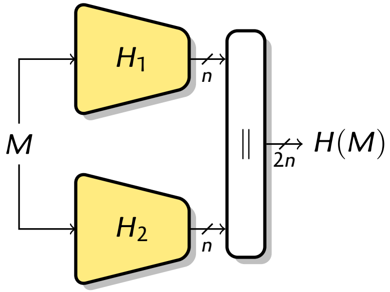

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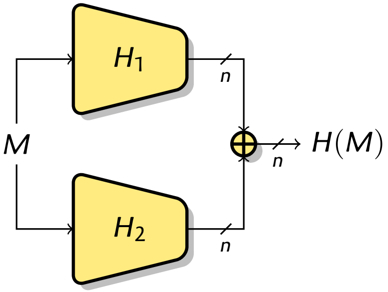

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Known results: Concatenation combiner

- $H(M) = H_1(M) \parallel H_2(M)$
- $2 \times n$-bit internal state, $2n$-bit output

- *Robust combiner* for collisions
  - A collision in H implies a collision in H_1 and H_2

- $2 \times n$-bit internal state can increase security?

- NO: Multicollision attack [Joux '04]
  - Collisions in $2^{n/2}$
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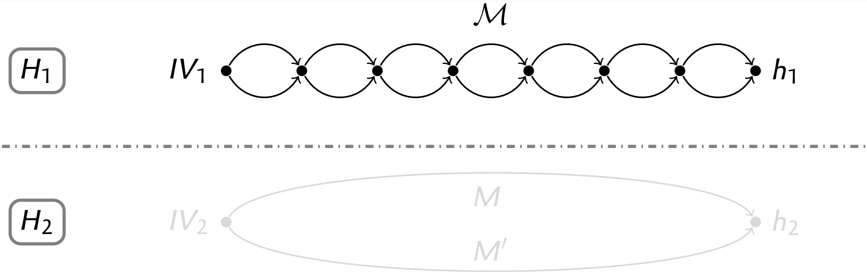

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$$\forall M \in \mathcal{M}, H_1(M) = h_1$$

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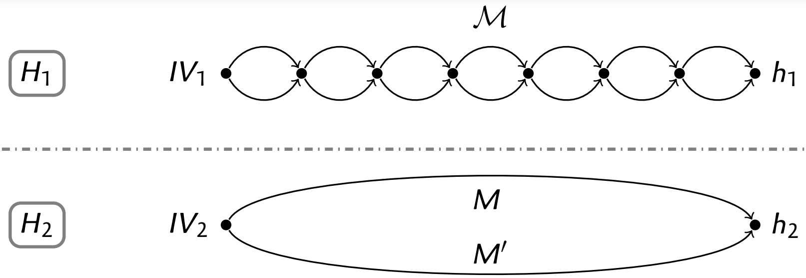

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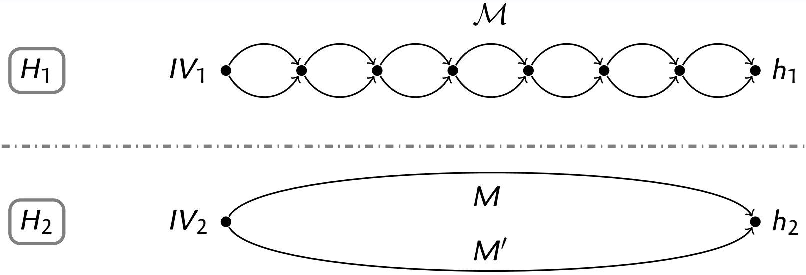

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Generic attacks against combiniers

Concatenation combiner

- $H(M) = H_1(M) \| H_2(M)$
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  - Non-ideal after $2^{n/2}$

XOR combiner

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Suprising result

If H_1 and H_2 are good MD hash functions, $H_1 \oplus H_2$ is weak!

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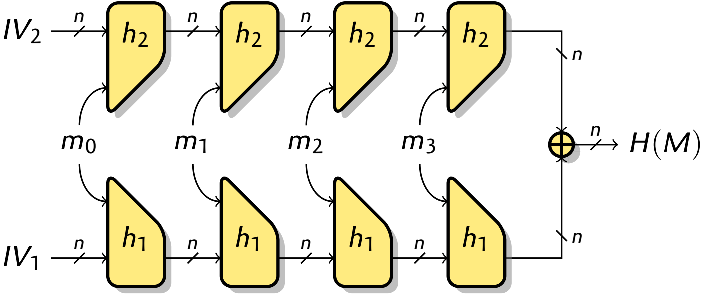
Our target: $H(M) = H_1(M) \oplus H_2(M)$



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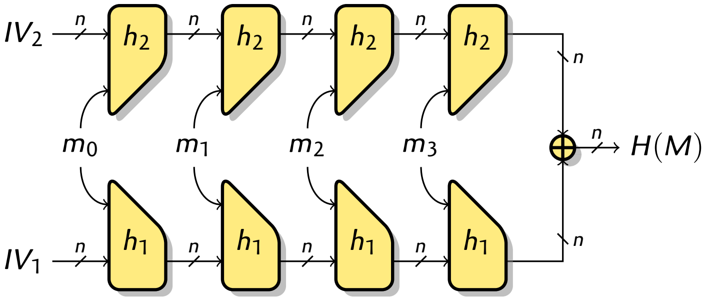

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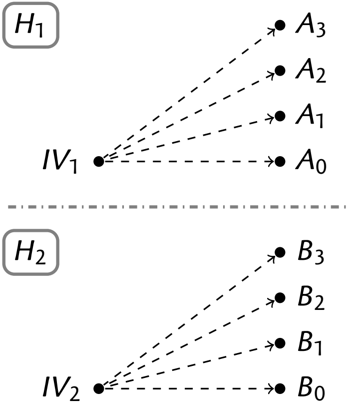
Overview



- Build a structure $\{\mathbf{M}_{ij}\}$ to control H_1 and H_2:
$$(IV_1, IV_2) \overset{\mathbf{M}_{jk}}{\rightsquigarrow} (A_j, B_k)$$
- Horizontal lines: common message M
$$(a^i_j, b^i_k) \xrightarrow{M_i} (a^{i+1}_j, b^{i+1}_k)$$
- Orange lines: alternative messages
$$(a^i_{j_0}, b^i_{k_0}) \xrightarrow{M'_i} (a^{i+1}_{j_0}, b^{i+1}_{k_1})$$
- Message in the structure use a few alternative chunks:
$$\mathbf{M}_{ij} = M_0, M_1, \ldots, M'_{ij}, \ldots$$

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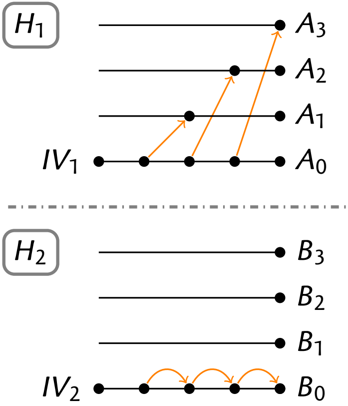

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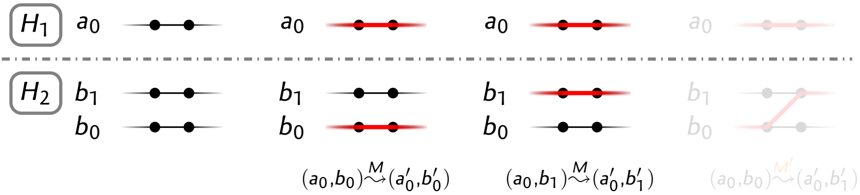
Switch stucture



- Simple case: one H_1-chain, and two H_2-chains
- Input: a_0, b_0, b_1
- Output: M, M', s.t.

$$a'_0 = h_1^*(a_0, M) = h_1^*(a_0, M')$$
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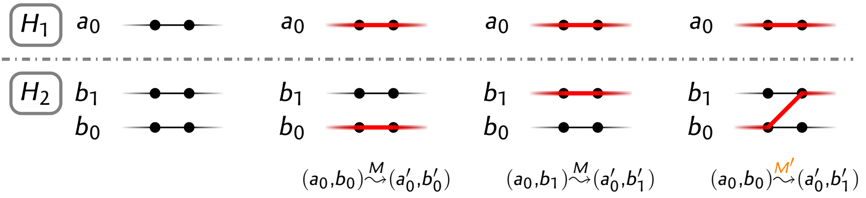

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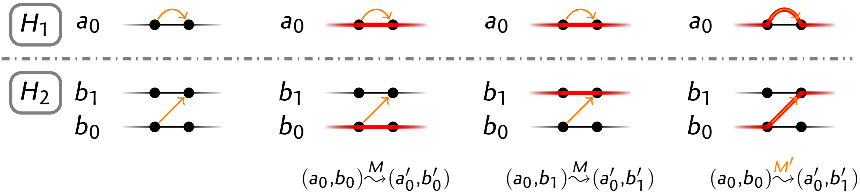

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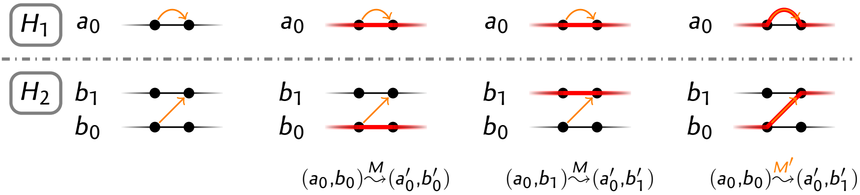

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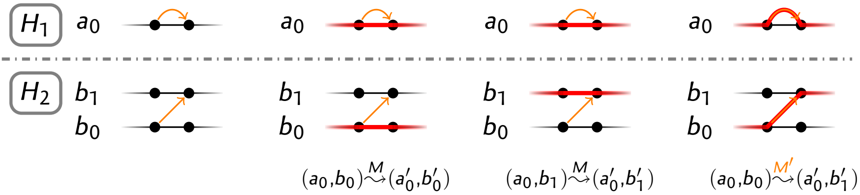

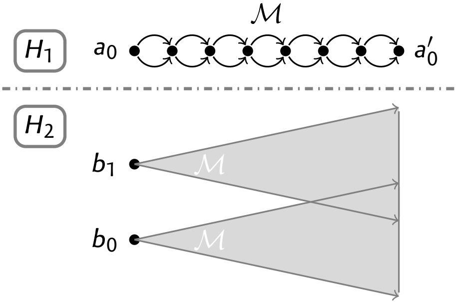

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- Complexity $\approx n \cdot 2^{n/2}$

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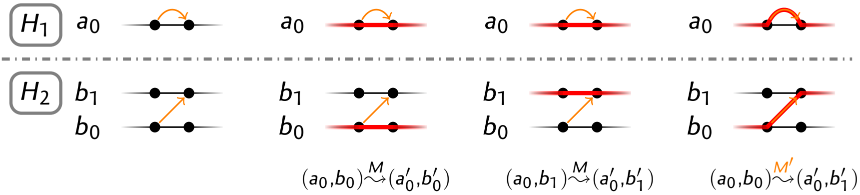

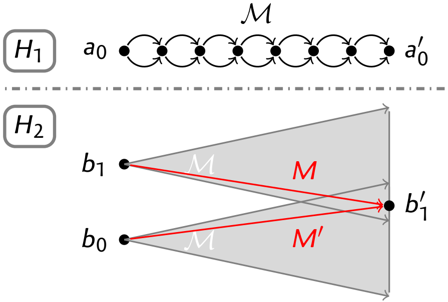

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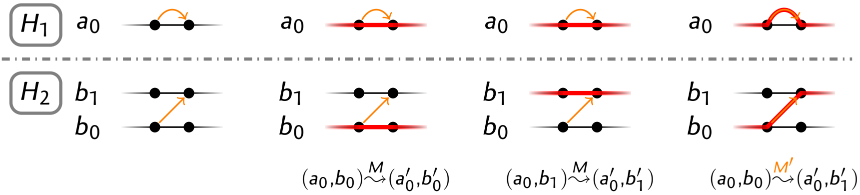

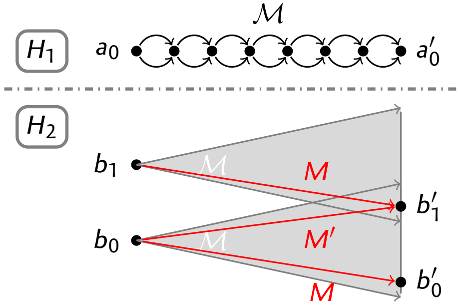

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2. Select $M, M' \in \mathcal{M}$ s.t.
   $h_2^*(b_1, M) = h_2^*(b_0, M') \triangleq b'_1$
3. Set $a'_0 \triangleq h_1^*(a_0, M)$,

   $b'_0 \triangleq h_2^*(b^i_k, M)$

- Complexity $\approx n \cdot 2^{n/2}$

Switch stucture

- We call this structure a switch. It can be used with more chains:
  - Update inactive chains with common message M
  - Jump from (a_j, b_k) to $(a_{j'}, b_k)$ or to $(a_{,} b_{k'})$
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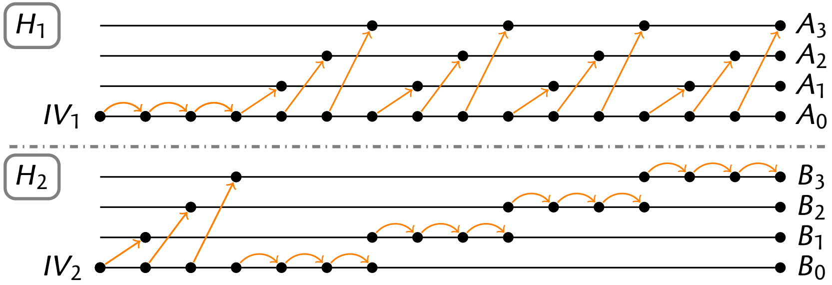

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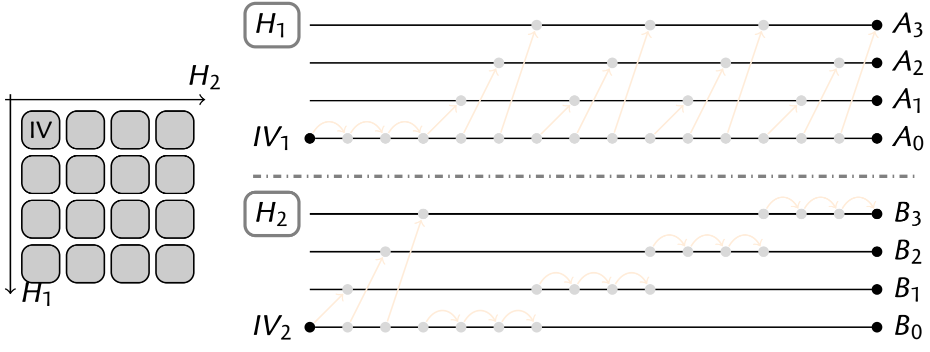

- Control 2^t H_1-chains and 2^t H_2-chains using 2^{2t} switches
- Message length: $n/2 \cdot 2^{2t}$, memory: $n \cdot 2^{2t}$
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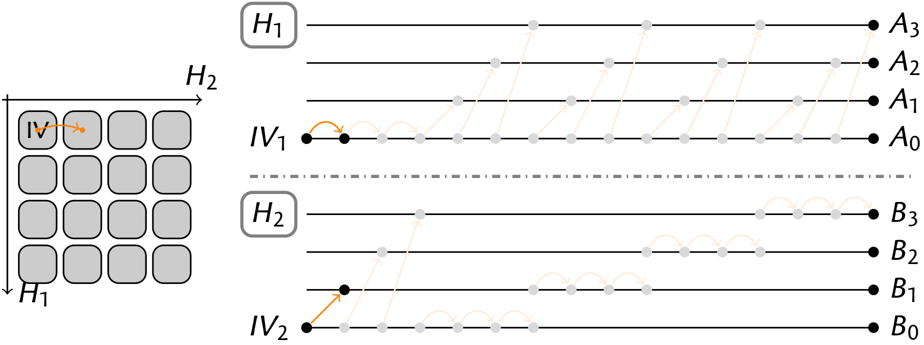

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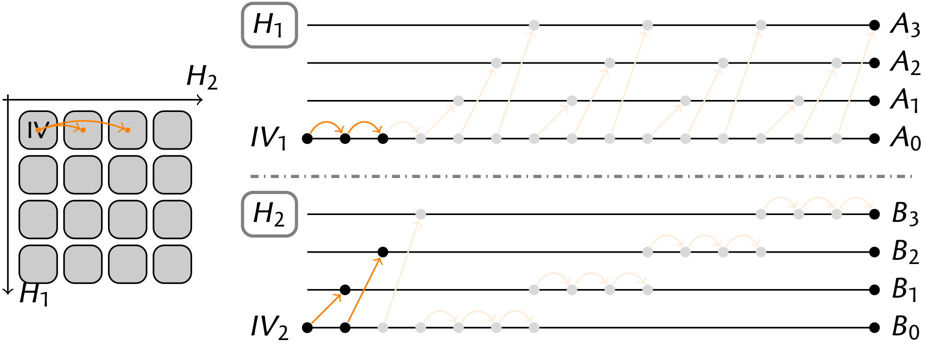

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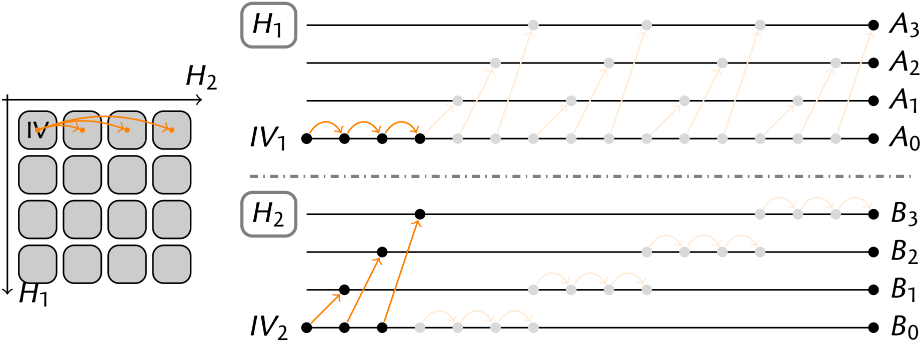

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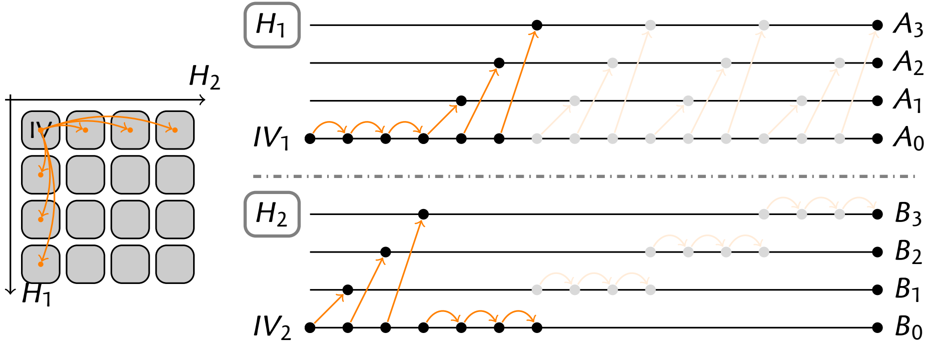

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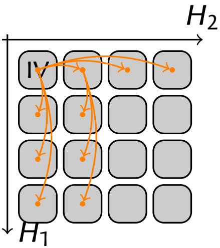

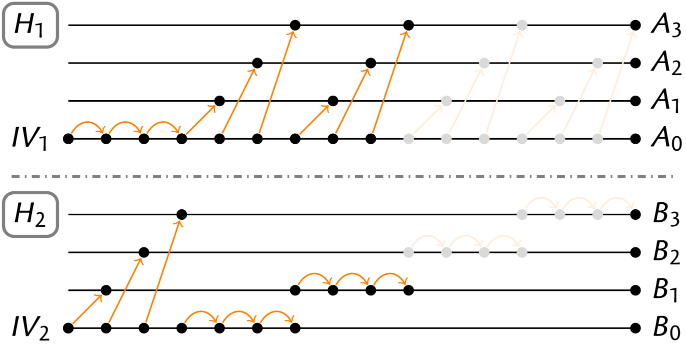

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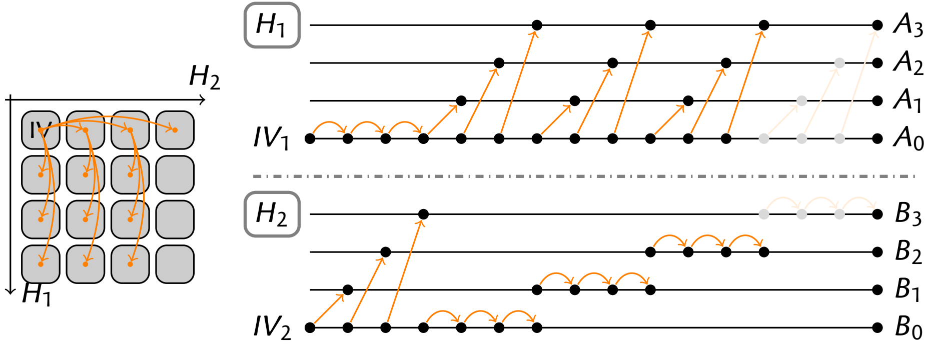

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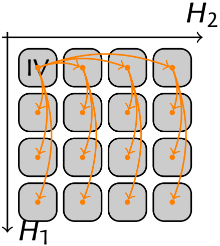

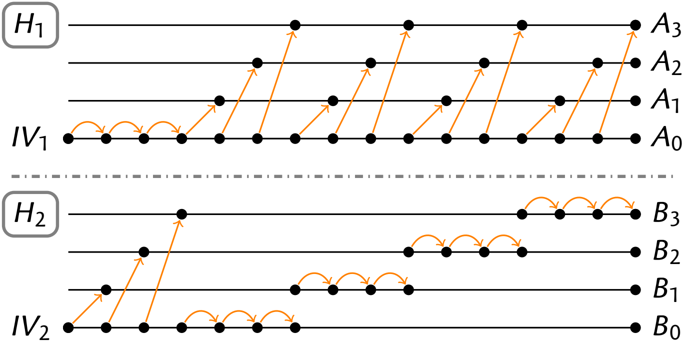

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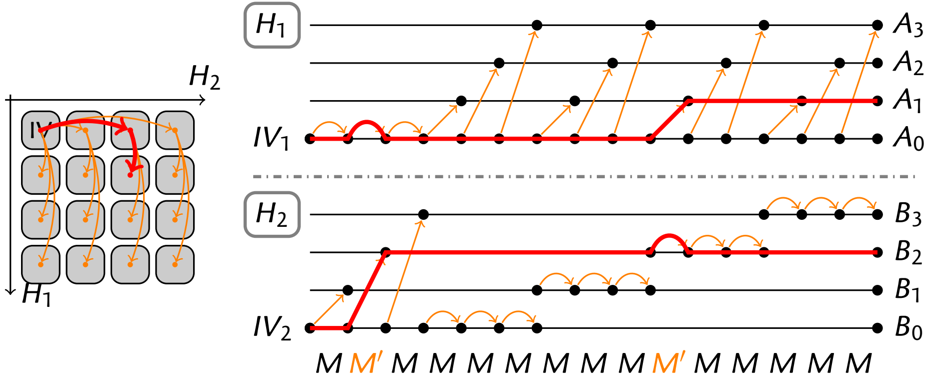

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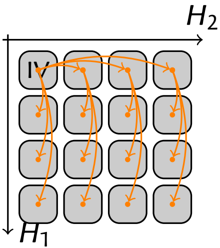

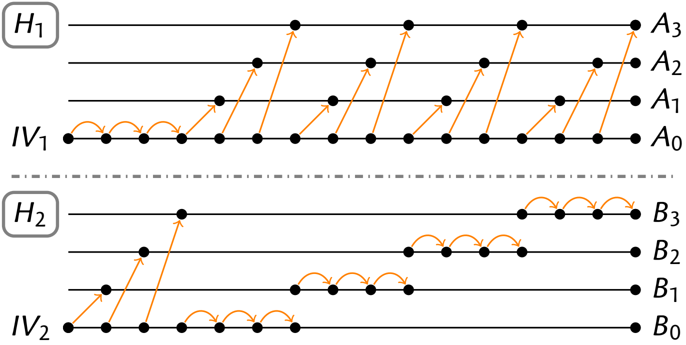

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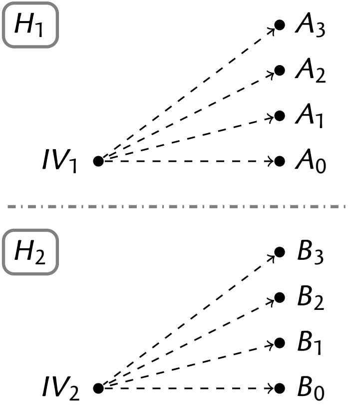
Preimage Attack



1. Build a 2^t-interchange structure $\{\mathbf{M}_{ij}\}$:

$$(IV_1, IV_2) \overset{\mathbf{M}_{jk}}{\rightsquigarrow} (A_j, B_k)$$

   - Complexity: $\tilde{\mathcal{O}}(2^{2t} \cdot 2^{n/2})$

2. Preimage search for $\overline{H}$:
   - For random blocks m, match $\{h_1(A_j, m)\}$ and $\{h_2(B_k, m) \oplus \overline{H}\}$
   - If there is a match (i, j):
     Get $\mathbf{M}_{jk}$, preimage is $\mathbf{M}_{jk} \parallel m$
   - Complexity: $\tilde{\mathcal{O}}(2^{n-t})$

3. Optimal complexity: $\mathcal{O}(n \cdot 2^{5n/6})$
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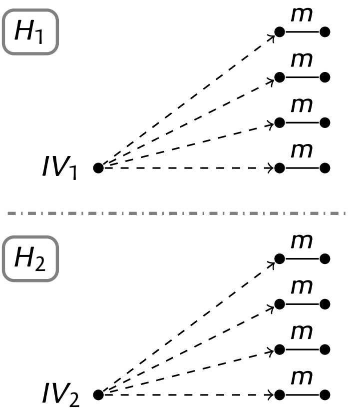

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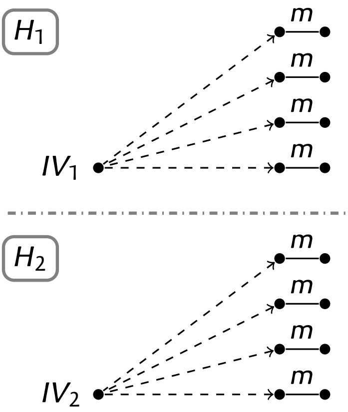

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Extensions

- Works for the HAIFA mode
  - Finalization function, block counter at each round
- Works with internal checksum (GOST)
  - Using pairs of blocks with constant sum
- Works with $H_1(M) \boxplus H_2(M)$
  - Or any easy to invert operation
- For wide-pipe ($\ell > n$ bits of internal state), complexity $\ell/2 \cdot 2^{2n/3+\ell/6}$
  - E.g. 2^{199} for SHA-224 $\oplus$ BLAKE-224
- Variants with shorter messages:
  - Time complexity 2^{n-m} with length 2^{2m}; memory 2^{2m} ($m < n/6$)
- Can be extended to the sum of three or more (k) hash functions
  - Complexity $\mathcal{O}(n^{k-1} \cdot 2^{5n/6})$

Conclusion

- New technique to control two hash functions independently
  - Stronger than previous techniques (multi-collisions, diamonds)
- Preimage attack for $H_1(M) \oplus H_2(M)$ with complexity: $n/2 \cdot 2^{5n/6}$

- The sum of two good narrow-pipe hash functions is a bad hash function
  - Whirlpool $\oplus$ Streebog: complexity 2^{435} ($n = 512$)
  - SHA-512 $\oplus$ Whirlpool: complexity 2^{461} ($n = 512$)
  - SHA-224 $\oplus$ BLAKE-224: complexity 2^{199} ($n = 224$)