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# STATIONARY IPA ESTIMATES FOR NON-SMOOTH FUNCTIONS OF THE GI/G/1/$\infty$ WORKLOAD.

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# Stationary IPA Estimates for Non-Smooth Functions of the GI/G/1/$\infty$ Workload.

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**Abstract**

We give stationary estimates for the derivative of the expectation of a non-smooth function of bounded variation $f$ of the workload in a GI/G/1/$\infty$ queue, with respect to a parameter influencing the distribution of the input process. For this, we use an idea of Konstantopoulos and Zazanis [12] based on the Palm inversion formula, however avoiding a limiting argument by performing the level-crossing analysis thereof globally, via Fubini's theorem. This method of proof allows to treat the case where the workload distribution has a mass at discontinuities of $f$ and where the formula of [12] has to be modified. The case where the parameter is the speed of service or/and the time scale factor of the input process is also treated using the same approach.

## 1 Introduction.

Consider a stationary GI/G/1/$\infty$ queue with inter-arrival times $\{\tau_n\}_{n\in\mathbb{Z}}$ and service times $\{\sigma_n(\theta)\}_{n\in\mathbb{Z}}$, where $\theta$ is a real parameter in the interval $\Theta$. Assume that $\mathbb{E}^0\,\sigma(\theta) < \mathbb{E}^0\,\tau_n = \lambda^{-1}$, where $\mathrm{P}^0$ denotes the Palm probability with respect to the arrival process $\{T_n\}_{n\in\mathbb{Z}}$—with the convention $T_0 \leq 0 <$

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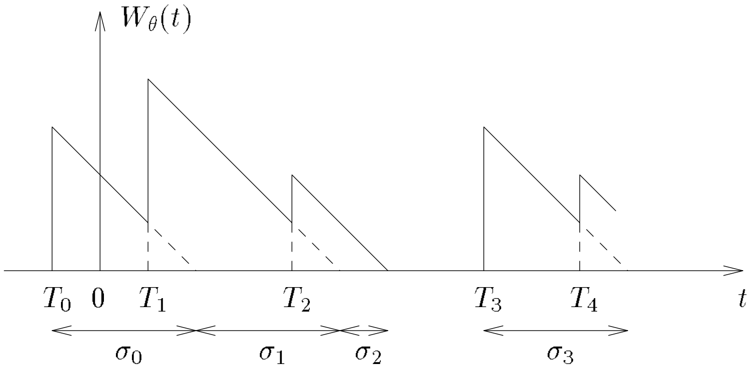



Figure 1: workload of a GI/G/1 queue.

$T_1$. Then the queue is stable and we can define as a stationary process the work $W_\theta(t)$ remaining in the system at time $t$ and given by Lindley's equation—see figure 1:

$$W_\theta(t) = \Big(W_\theta(T_n-) + \sigma_n(\theta) - (t - T_n)\Big)^+, \; t \in [T_n, T_{n+1}), \tag{1}$$

with the notation $x^+ \stackrel{\text{def}}{=} \max(x, 0)$. Given a real function $f$, consider the functional $J(\theta)$ given by

$$J(\theta) \stackrel{\text{def}}{=} \mathbb{E}\, f(W_\theta(0)).$$

We want to estimate, if it exists, the derivative of $J$ with respect to $\theta$. To this end, we use Infinitesimal Perturbation Analysis (IPA), a method first introduced by Ho et al. [11], developed by Suri [15] and validated by Cao [4], Suri and Zazanis [16]. Glasserman [5], Ho and Cao [10] and Konstantopoulos and Zazanis [12] developed further the theory.

Alternative methods have been used to estimate derivatives, namely Likelihood Ratio Method (LRM, see e.g. Reiman and Weiss [14] or Glynn [7]), Smooth Perturbation Analysis (SPA, see Gong and Ho [9], Glasserman and Gong [6]) and Rare Perturbation Analysis (RPA, see Brémaud and Vázquez-Abad [3] and Brémaud [2]).

In this paper, we aim to prove that, under appropriate conditions

$$\frac{\partial}{\partial\theta}\,\mathbb{E}\, f(W_\theta(0)) = \mathbb{E}\,\frac{\partial}{\partial\theta} f(W_\theta(0)) \tag{2}$$

and we give a formula replacing (2) when $f$ is not differentiable but is of bounded variation. This formula was obtained by Konstantopoulos and

Zazanis [12]. However there is one term missing in their formula, due to the difficulty of passing to the limit in their approximation procedure. Our method of proof avoids this passage to the limit and therefore allows for better control of the computations. Moreover, it can be extended in many ways to handle different situations.

The paper is organized as follow: in Section 1, we give a construction of the GI/G/1 queue and derive some basic properties. The main result of the paper is given in Section 3 and the same method is applied to second-order derivatives in Section 4; Sections 5 and 6 show how our method can be extended to other parameters, respectively the speed of the server and the rate of arrival in the system. Finally Section 7 gives conclusive remarks on possible extensions of the results.

## 2 Construction of the GI/G/1 queue.

In a formula like (2), the probability space does not depend on $\theta$. To obtain this independence, we use the inversion representation (see for example Glasserman [5, p. 16]): let $\{\xi_n^\tau\}_{n\in\mathbb{Z}}$ and $\{\xi_n^\sigma\}_{n\in\mathbb{Z}}$ be two sequences of random variables uniformly distributed on $[0,1]$. If $F^\tau(\cdot)$ and $F^\sigma(\cdot,\theta)$ are the respective distribution functions of $\tau$ and $\sigma$, then we can define their inverse functions

$$
\begin{aligned}
G^\tau(\xi) &= \sup(x \geq 0 \ : \ F^\tau(x) \leq \xi), \\
G^\sigma(\xi,\theta) &= \sup(x \geq 0 \ : \ F^\sigma(x,\theta) \leq \xi).
\end{aligned}
$$

Then $\sigma_n(\theta) \stackrel{\text{def}}{=} G^\sigma(\xi_n^\sigma,\theta)$ and $\tau_n \stackrel{\text{def}}{=} G^\tau(\xi_n^\tau)$ have the correct distributions and they will be defined on a probability space independent from $\theta$. We now give the structural conditions on service times that allow IPA:

**A1** *$G^\sigma$ verifies the following conditions:*

*(i)* $\theta \mapsto G^\sigma(\xi,\theta)$ *is differentiable and Lipschitz, that is*

$$|G^\sigma(\xi,\theta_1) - G^\sigma(\xi,\theta_2)| \leq K^\sigma(\xi)|\theta_1-\theta_2|, \ \forall\, \theta_1,\theta_2 \in \Theta;$$

*(ii) there exists* $\theta^* \in \Theta$ *such that*

$$G^\sigma(\xi,\theta) \leq G^\sigma(\xi,\theta^*), \ \forall\, \xi \in [0,1], \ \forall\, \theta \in \Theta.$$

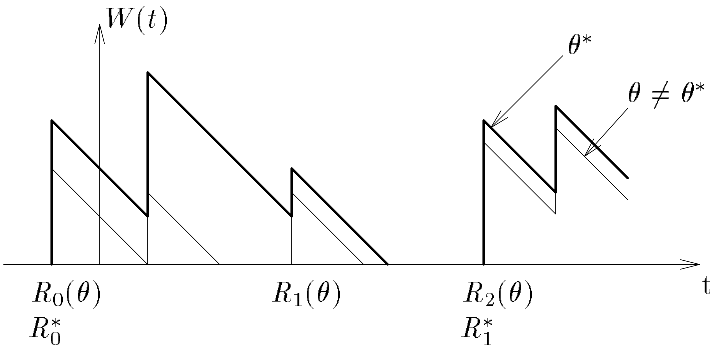



Figure 2: the domination property.

Condition **A1**-$(i)$ ensures that we have enough smoothness with respect to $\theta$ in the distribution of the service times. However, in a number of cases, $\xi_n^\sigma$ will not be directly known, in particular when observing a real experiment; this difficulty can be overcome with the following classical proposition—see e.g. Glasserman [5, p. 16]:

**Proposition** *Suppose that (i) $F^\sigma(\cdot,\theta)$ has a density $\partial_x F^\sigma(\cdot,\theta)$ which is strictly positive on an open interval $I_\theta$ and zero elsewhere; and (ii) $F^\sigma$ is continuously differentiable on $I_\theta \times \Theta$. Then*

$$\sigma'(\theta) = -\frac{\partial_\theta F^\sigma(\sigma(\theta),\theta)}{\partial_x F^\sigma(\sigma(\theta),\theta)}.$$

In the above formula, the prime denotes the derivative with respect to $\theta$. A case of particular interest is when $\theta$ is a scale parameter of the service times, that is when $\sigma(\theta) = \theta\eta$ for some random variable $\eta$. Then we have directly

$$\sigma'(\theta) = \eta = \frac{\sigma(\theta)}{\theta}.$$

In particular, we do not need to know the real distribution of service times unless we actually want to simulate them.

Condition **A1**-$(ii)$ takes care one of IPA's major difficulty, which is that a small perturbation in $\theta$ can make two busy periods of the queue merge and cause a big change in the workload process $W_\theta(t)$. Here we have a $\theta$-independent bound on the size of a busy period of the system, in the following way: take the system with parameter $\theta^*$; since it is stable, there is an infinity of regeneration points, say $\{R_n(\theta^*)\}_{n\in\mathbb{Z}} \stackrel{\text{def}}{=} \{R_n^*\}_{n\in\mathbb{Z}}$, which are the arrival times of incoming customers that find the system empty. We can construct the process $W_\theta(t)$ from the busy period process $\{R_n^*\}_{n\in\mathbb{Z}}$ but with the service times given by $\{\sigma_n(\theta)\}_{n\in\mathbb{Z}}$, so that the following domination property holds—see figure 2:

$$W_\theta(t) \leq W_{\theta^*}(t), \ \forall\, \theta \in \Theta, \ \forall\ t \in \mathbb{R}. \tag{3}$$

With the above construction, we get $\{R_n^*\}_{n\in\mathbb{Z}} \subseteq \{R_n(\theta)\}_{n\in\mathbb{Z}}$, where $\{R_n(\theta)\}_{n\in\mathbb{Z}}$—or simply $\{R_n\}_{n\in\mathbb{Z}}$—denotes the beginning of busy period process for the $\theta$-system. Moreover, we have the boundary property

$$R_-^*(t) \leq R_-(\theta)(t) \leq t < R_+(\theta)(t) \leq R_+^*(t), \tag{4}$$

with the following notation:

$$\begin{aligned} R_-(t) &= \sup(R_n : R_n \leq t), \\ R_+(t) &= \inf(R_n \ : \ R_n > t). \end{aligned}$$

Lindley's equation (1) and Figure 1 show that the workload can be expressed as

$$W_\theta(t) = \sum_{R_-(t)\leq T_n \leq t} \sigma_n(\theta) + R_-(t) - t. \tag{5}$$

Equation (5) gives us a useful expression for $W'_\theta(t)$:

$$W'_\theta(t) = \sum_{R_-(t)\leq T_n \leq t} \sigma'_n(\theta). \tag{6}$$

The last property of $\{R_n\}_{n\in\mathbb{Z}}$ that will be useful is that it is jointly stationary with the arrival process $\{T_n\}_{n\in\mathbb{Z}}$, since both are computed from sequences $\{\xi_n^\tau\}_{n\in\mathbb{Z}}$ and $\{\xi_n^\sigma\}_{n\in\mathbb{Z}}$. This gives us a general purpose lemma that will be used in the next section:

**Lemma 1** *Let $N$ denote the arrival process associated to $\{T_n\}_{n\in\mathbb{Z}}$ and let $\{X_n\}_{n\in\mathbb{Z}}$ be a sequence of stationary marks of $N$. Then*

$$\mathbb{E}^0\Big[\sum_{n\in\mathbb{Z}} X_n \mathbb{1}_{[R_0,R_1)}(T_n)\Big] = \mathbb{E}^0[N([R_0, R_1))\, X_0]. \tag{7}$$

**Proof** Let $\mathrm{P}^R$ be the Palm probability associated with $\{R_n\}_{n\in\mathbb{Z}}$ and $\lambda^R$ be its intensity. If we call $Z$ the random variable inside the expectation of the l.h.s. of equation (7) and if $\theta_t$ is the measurable flow with respect to which the process is stationary, then

$$Z \circ \theta_{T_i} = \sum_{n \in \mathrm{Z}} X_n \mathbb{1}_{[R_-(T_i), R_+(T_i))}(T_n).$$

Since $R_\pm(T_i) = R_\pm(0)$ if $T_i \in [R_-(0), R_+(0))$,

$$\begin{aligned} \sum_{T_i \in [R_-(0), R_+(0))} Z \circ \theta_{T_i} &= \sum_{i \in \mathrm{Z}} \sum_{n \in \mathrm{Z}} X_n \mathbb{1}_{[R_-(0), R_+(0))}(T_n) \mathbb{1}_{[R_-(0), R_+(0))}(T_i) \\ &= N([R_0, R_1)) \sum_{n \in \mathrm{Z}} X_n \, \mathbb{1}_{[R_0, R_1)}(T_n). \end{aligned}$$

If we now apply Neveu's exchange formula—Neveu [13]; see Baccelli and Brémaud [1, p. 11]—between $\mathrm{P}^0$ and $\mathrm{P}^R$, we obtain:

$$\begin{aligned} \lambda \, \mathbb{E}^0 Z &= \lambda^R \, \mathbb{E}^R \sum_{T_i \in [R_-(0), R_+(0))} Z \circ \theta_{T_i} \\ &= \lambda^R \, \mathbb{E}^R \Big[ \sum_{n \in \mathrm{Z}} N([R_0, R_1)) \, X_n \, \mathbb{1}_{[R_0, R_1)}(T_n) \Big] \\ &= \lambda \, \mathbb{E}^0 [N([R_0, R_1)) \, X_0], \end{aligned}$$

which is exactly equality (7). ∎

# 3 An IPA estimator for general non-decreasing functions.

In this section, we show that IPA applies with any non-decreasing *càdlag* function $f$. But since $f$ is not required to be continuous, we cannot apply (2) as such. First, we need to introduce an assumption:

**A2** *The following inequalities hold:*

(*i*) $\mathbb{E}^0[K^\sigma(\xi_0^\sigma)]^4 < \infty$;

(*ii*) $\mathbb{E}^0[N([R_0^*, R_1^*))]^4 < \infty$;

(*iii*) $\mathbb{E}^0[f(W_{\theta^*}(0))]^2 < \infty$.

**Theorem 2** *Let $\mu_f$ be the measure on $\mathbb{R}$ associated with $f$. Assume* **A1** *and* **A2** *hold. Then $J$ admits a right derivative with respect to $\theta$ given by*

$$\begin{aligned} J'_r(\theta) &= \lambda\, \mathbb{E}^0\, W'_\theta(0) \Big[ f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - 1\!\!1_{\{W'_\theta(0)<0\}} \big[ \mu_f(\{W_\theta(0)\}) - \mu_f(\{W_\theta(T_1-)\}) \big] \Big], \end{aligned} \tag{8}$$

*and its left derivative is*

$$\begin{aligned} J'_l(\theta) &= \lambda\, \mathbb{E}^0\, W'_\theta(0) \Big[ f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - 1\!\!1_{\{W'_\theta(0)>0\}} \big[ \mu_f(\{W_\theta(0)\}) - \mu_f(\{W_\theta(T_1-)\}) \big] \Big]. \end{aligned} \tag{9}$$

**Example 1** With $f(w) = 1\!\!1_{\{w \geq x\}}$, Theorem 2 yields

$$\begin{aligned} \frac{\partial_r}{\partial\theta} \mathrm{P}(W_\theta(0) > x) &= \lambda\, \mathbb{E}^0\, W'_\theta(0) \Big[ 1\!\!1_{(W_\theta(T_1-), W_\theta(0)]}(x) \\ &\qquad - 1\!\!1_{\{W'_\theta(0)<0\}} \big[ 1\!\!1_{\{W_\theta(0)=x\}} - 1\!\!1_{\{W_\theta(T_1-)=x\}} \big] \Big] \\ \frac{\partial_l}{\partial\theta} \mathrm{P}(W_\theta(0) > x) &= \lambda\, \mathbb{E}^0\, W'_\theta(0) \Big[ 1\!\!1_{(W_\theta(T_1-), W_\theta(0)]}(x) \\ &\qquad - 1\!\!1_{\{W'_\theta(0)>0\}} \big[ 1\!\!1_{\{W_\theta(0)=x\}} - 1\!\!1_{\{W_\theta(T_1-)=x\}} \big] \Big]. \end{aligned}$$

□

Theorem 2 shows that $J(\theta)$ admits right and left derivatives even when $f$ is not continuous. But in a number of cases, we can get a much more usable formula:

**Corollary 3** *Assume* **A1** *and* **A2** *hold. If $f$ is continuous or if $W_\theta(0)$ and $W_\theta(T_1-)$ admit densities with respect to $\mathrm{P}^0$ then $J(\theta)$ is differentiable and*

$$J'(\theta) = \lambda\, \mathbb{E}^0\, W'_\theta(0)[f(W_\theta(0)) - f(W_\theta(T_1-))]. \tag{10}$$

**Proof** First note that if $f$ is continuous, $w \mapsto \mu_f(\{w\}) \equiv 0$. If $W_\theta(0)$ admits a $\mathrm{P}^0$-density, say $\gamma^0(w)$, we can use the fact that $\mu_f(\{\cdot\}) = 0$ almost everywhere for the Lebesgue measure:

$$
\begin{aligned}
|\,\mathbb{E}^0\,\mathbb{1}_{\{W'_\theta(0)<0\}}\mu_f(\{W_\theta(0)\})| &\le \mathbb{E}^0\,\mu_f(\{W_\theta(0)\}) \\
&= \int_0^\infty \mu_f(\{w\})\gamma^0(w)\,dw = 0.
\end{aligned}
$$

In either case, the result is proved. ■

**Remark** In the case where $f$ admits a derivative $f'$, we can write (10) as

$$
J'(\theta) = \lambda\,\mathbb{E}^0[\int_0^{T_1} W'_\theta(t) f'(W_\theta(t))]dt = \mathbb{E}[W'_\theta(0) f'(W_\theta(0))],
$$

thus obtaining the expected IPA estimate (2). Since our Palm estimate does not require differentiability for $f$, one will want to check whether it is as accurate as the classic IPA estimate: if the system is ergodic, equation (2) gives

$$
\begin{aligned}
J'(\theta) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t W'_\theta(s) f'(W_\theta(s))\,ds \\
&= \lim_{n\to\infty} \frac{1}{T_n}\sum_{k=0}^{n-1}\int_{T_k}^{T_{k+1}} W'_\theta(s) f'(W_\theta(s))\,ds \\
&= \lim_{n\to\infty} \frac{1}{T_n}\sum_{k=0}^{n-1} W'_\theta(T_k)[f(W_\theta(T_k)) - f(W_\theta(T_{k+1}-))],
\end{aligned}
$$

where all the limits are valid $P^0$-a.s. or P-a.s. indifferently. On the other hand, the ergodic theorem reads in case of equation (10)

$$
J'(\theta) = \lim_{n\to\infty} \frac{\lambda}{n}\sum_{k=0}^{n-1} W'_\theta(T_k)[f(W_\theta(T_k)) - f(W_\theta(T_{k+1}-))].
$$

Thus, the estimates based on the same amount of data give very close expressions; in fact, they are even equal when $\lambda$ needs to be estimated. For comparisons between time-average and customer-average estimates, see for instance Glynn and Whitt [8]. □

Before starting the proof of the theorem, let us mention that our derivation is different from Konstantopoulos and Zazanis [12] in two respects: first we do not require an approximation procedure and we treat directly a non decreasing function $f$. This is made possible by the simple crucial observation that

$$f(x) - f(y) = \int_{(x,y]} \mu_f(dz) \quad \text{for all } x \leq y,$$

which allows us to have a better view of the residual terms in the level crossing analysis that follows. The result can be applied to any function of bounded variation if assumption **A2** is verified by both the increasing and decreasing parts of the function. Secondly, we do not need switch back and forth between the Palm probabilities with respect to the arrival process and with respect to the regeneration points as in [12]. However, we retain the fundamental idea of [12] by starting with its expression in term of the Palm probability $\mathrm{P}^0$.

**Proof of Theorem 2** Assume that $f(0) = 0$, so that $f$ is non-negative. The Palm inversion formula—see for instance Baccelli and Brémaud [1, p. 13]—gives

$$\begin{aligned}
\mathbb{E}\, f(W_\theta(0)) &= \lambda\, \mathbb{E}^0 \int_0^{T_1} f(W_\theta(t))\, dt \\
&= \lambda\, \mathbb{E}^0 \int_0^{T_1} \int_{\mathbb{R}_+} \mathbb{1}_{\{W_\theta(t) > x\}}\, \mu_f(dx)\, dt \\
&= \lambda\, \mathbb{E}^0 \int_{\mathbb{R}_+} \int_0^{T_1} \mathbb{1}_{\{W_\theta(t) > x\}}\, dt\, \mu_f(dx)
\end{aligned}$$

and therefore

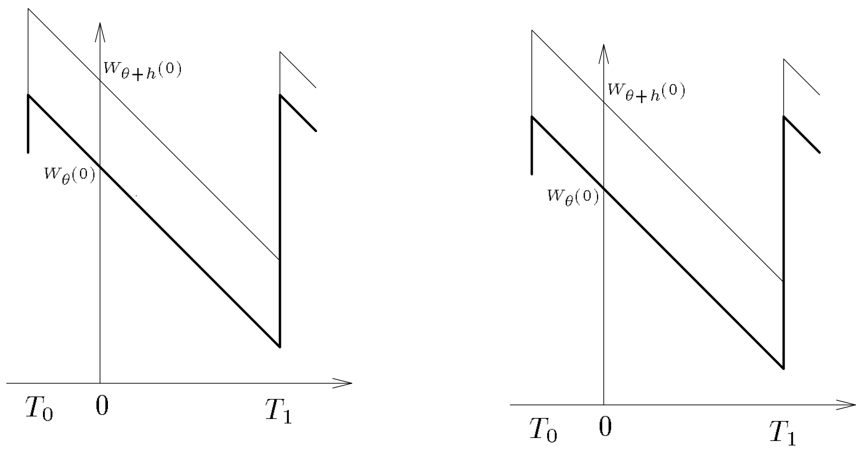
$$\begin{aligned}
&\frac{1}{h}\, \mathbb{E}[f(W_{\theta+h}(0)) - f(W_\theta(0))] \\
&\quad = \frac{\lambda}{h}\, \mathbb{E}^0 \int_{\mathbb{R}_+} \int_0^{T_1} [\mathbb{1}_{\{W_{\theta+h}(t) > x\}} - \mathbb{1}_{\{W_\theta(t) > x\}}]\, dt\, \mu_f(dx).
\end{aligned}$$

In order to simplify the notations, let:

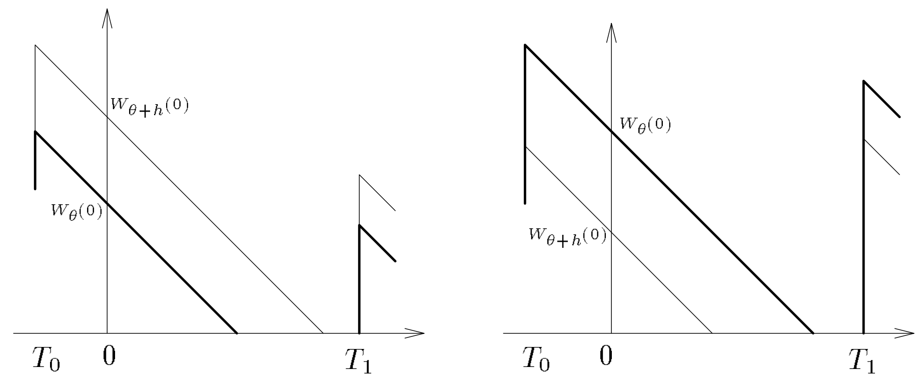
$$\begin{aligned}
\varphi(x,t) &\stackrel{\text{def}}{=} \mathbb{1}_{\{W_{\theta+h}(t) > x\}} - \mathbb{1}_{\{W_\theta(t) > x\}} \\
\Phi(\theta, h) &\stackrel{\text{def}}{=} \int_{\mathbb{R}_+} \int_0^{T_1} \varphi(x,t)\, dt\, \mu_f(dx).
\end{aligned}$$

The first step of our proof is to compute $\lim_{h \to 0} \Phi(\theta, h)/h$. We will have to integrate a function taking its values in $\{-1, 0, 1\}$ with respect to $dt\, \mu_f(dx)$. Define also for any $t \in [0, T_1)$:

$$\Delta W_{\theta,h} \stackrel{\text{def}}{=} W_{\theta+h}(t) - W_\theta(t).$$




Case 1: $W'(0) > 0$ and $W(T_1-) > 0$ Case 2: $W'(0) < 0$ and $W(T_1-) > 0$




Case 1′: $W'(0) > 0$ and $W(T_1-) = 0$ Case 2′: $W'(0) < 0$ and $W(T_1-) = 0$

Figure 3: Four different cases for the computation of $\Phi$.

Assume first that $h > 0$. As shown in Figure 3, we must consider different cases depending on the relative position of $W_\theta(0)$ and $W_{\theta+h}(0)$. We have to add cases 3 and $3'$, where $W'_\theta(0) = 0$, preventing us to guess their relative positions. In fact, all the terms of the formula can be found in the first two cases and we will let the other ones to the reader's attention.

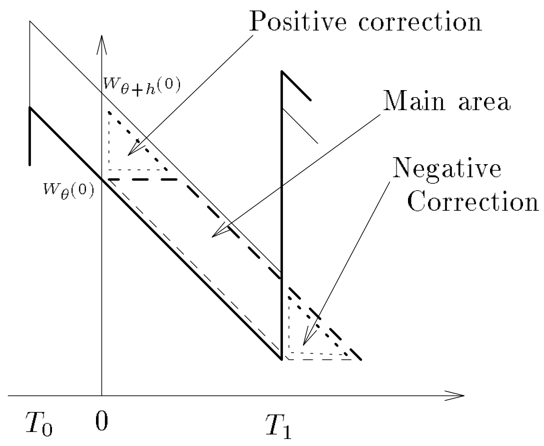



Figure 4: computation of $\Phi$ in case 1.

**Case 1:** for $h$ small enough, $W_{\theta+h}(0) > W_\theta(0)$ and $\varphi = 1$. The way to compute $\Phi(\theta, h)$ can be best understood with the help of Figure 4. $\Phi$ is equal to the area with a dashed border plus the dotted triangle on the left, minus the right one. Here the borders included in the areas are in bold; since all functions are *càdlag*, these borders are the top and right ones.

$$\begin{aligned}\frac{1}{h}\Phi(\theta,h) &= \Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big]\frac{\Delta W_{\theta,h}}{h} \\ &\quad + \frac{1}{h}\int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(0), W_\theta(0)+y])\,dy \\ &\quad - \frac{1}{h}\int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(T_1-), W_\theta(T_1-)+y])\,dy. \qquad (11)\end{aligned}$$

The first term converges to $[f(W_\theta(0)) - f(W_\theta(T_1-))]W'_\theta(0)$. Moreover,

$$\mu_f((W_\theta(0), W_\theta(0)+y]) = \mu_f((W_\theta(0), W_\theta(0)+y)) + \mu_f(\{W_\theta(0)+y\})$$

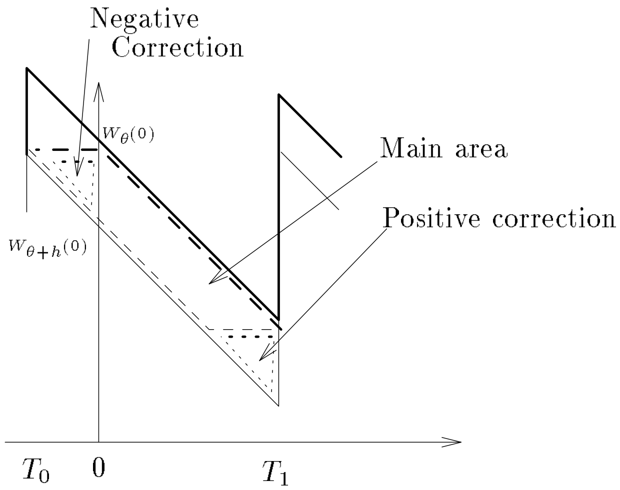



Figure 5: computation of $\Phi$ in case 2.

and since $\mu_f(\{W_\theta(0)+y\}) = 0$ $dy$-a.e., the second term of r.h.s. of equation (11) reads

$$\frac{1}{h}\int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(0), W_\theta(0)+y))\, dy,$$

which is less or equal than

$$(W'_\theta(0)+o(1))\,\mu_f((W_\theta(0), W_{\theta+h}(0))).$$

Since $W_\theta(0)$ is continuous in the neighborhood of $\theta$, this goes to zero with $h$. The third term converges to 0 for the same reasons. So we have in this case:

$$\lim_{h\to 0+}\frac{1}{h}\Phi(\theta,h) = [f(W_\theta(0)) - f(W_\theta(T_1-))]W'_\theta(0).$$

**Case 2:** here $W_{\theta+h}(0) < W_\theta(0)$ and $\varphi = -1$. Due to the order of $W_{\theta+h}(0)$ and $W_\theta(0)$, we find a formula different from equation (11)—see Figure 5:

$$\begin{aligned}\frac{1}{h}\Phi(\theta,h) &= -\Big\{\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big]\frac{-\Delta W_{\theta,h}}{h} \\ &\quad - \frac{1}{h}\int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(0)-y, W_\theta(0)])\, dy \\ &\quad + \frac{1}{h}\int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(T_1-)-y, W_\theta(T_1-)])\, dy\Big\}. \qquad (12)\end{aligned}$$

The first term is the same as in case 1, but the second is equal to

$$\frac{1}{h}\int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(0)-y, W_\theta(0)))\,dy \;-\; \mu_f(\{W_\theta(0)\})\frac{\Delta W_{\theta,h}}{h}$$

and its limit is $\mu_f(\{W_\theta(0)\})W'_\theta(0)$. The last term of $\Phi(\theta,h)/h$ is computed in a similar way. Finally:

$$\begin{aligned}\lim_{h\to 0+}\frac{1}{h}\Phi(\theta,h) &= \Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - \mu_f(\{W_\theta(0)\}) + \mu_f(\{W_\theta(T_1-)\})\Big]W'_\theta(0).\end{aligned}$$

We can summarize the above cases in the following formula:

$$\begin{aligned}\lim_{h\to 0+}\frac{1}{h}\Phi(\theta,h) &= W'_\theta(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - \mathbb{1}_{\{W'_\theta(0)<0\}}[\mu_f(\{W_\theta(0)\}) - \mu_f(\{W_\theta(T_1-)\})]\Big].\end{aligned}$$

Now that we have $\lim_{h\to 0+}\Phi(\theta,h)/h$, the next step is to find a bound for $\Phi(\theta,h)/h$ which has a finite mean with respect to $\mathrm{P}^0$. The formulas for each case give:

$$\begin{aligned}\Big|\frac{1}{h}\Phi(\theta,h)\Big| &\le \Big(f(W_\theta(0)) - f(W_\theta(T_1-))\Big)\Big|\frac{\Delta W_{\theta,h}}{h}\Big| \\ &\quad + |f(W_{\theta+h}(0)) - f(W_\theta(0))|\cdot\Big|\frac{\Delta W_{\theta,h}}{h}\Big| \\ &\quad + |f(W_{\theta+h}(T_1-)) - f(W_\theta(T_1-))|\cdot\Big|\frac{\Delta W_{\theta,h}}{h}\Big| \\ &\le 3f(W_{\theta^*}(0))K_\theta^W(0).\end{aligned}$$

The last inequality takes advantage of the fact that $f$ is non-decreasing and of the domination property (3). $K_\theta^W(t)$ is a Lipschitz coefficient for $W(t)$ w.r.t. $\theta$. Finally,

$$\Big|\frac{1}{h}\Phi(\theta,h)\Big| \le 3f(W_{\theta^*}(0))K_\theta^W(0).$$

The latter expression is independent from $h$. Moreover, it has a finite mean under $\mathrm{P}^0$: from Cauchy-Schwartz inequality,

$$\mathbb{E}^0\Big[f(W_{\theta^*}(0))K_\theta^W(0)\Big] \le \sqrt{\mathbb{E}^0[f(W_{\theta^*}(0))]^2}\sqrt{\mathbb{E}^0[K_\theta^W(0)]^2}.$$

The first mean is finite from assumption **A2**-$(iii)$. To prove that the second one is also finite, we must first give an expression of $K_\theta^W(0)$:

$$
\begin{aligned}
\Big|\frac{W_{\theta+h}(0)-W_\theta(0)}{h}\Big| &\le \Big|\frac{W_{\theta+h}(T_{-1})-W_\theta(T_{-1})}{h}\Big| + \Big|\frac{\sigma_0(\theta+h)-\sigma_0(\theta)}{h}\Big| \\
&\le \sum_{n\in\mathrm{Z}}\Big|\frac{\sigma_n(\theta+h)-\sigma_n(\theta)}{h}\Big|\mathbb{1}_{[R^*_-(T_0),0)}(T_n) \\
&\le \sum_{n\in\mathrm{Z}} K^\sigma(\xi^\sigma_n)\mathbb{1}_{[R^*_-(0),R^*_+(0))}(T_n) \\
&\stackrel{\text{def}}{=} K^W_\theta(0).
\end{aligned}
$$

The first inequality comes from equation (1) and inequality $|a^+ - b^+| \le |a-b|$; then we use the boundary property (4) and last the Lipschitz property **A1**-$(i)$. To prove that $\mathbb{E}^0[K^W_\theta(0)]^2$ is finite, we can use the inequality $(x_1+\cdots+x_n)^p \le n^{p-1}(x_1^p+\cdots+x_n^p)$ and

$$
\begin{aligned}
&\mathbb{E}^0\Big[\sum_{n\in\mathrm{Z}} K^\sigma(\xi^\sigma_n)\mathbb{1}_{[R^*_-(0),R^*_+(0))}(T_n)\Big]^2 \\
&\quad\le \mathbb{E}^0\Big[\sum_{n\in\mathrm{Z}} N([R^*_0,R^*_1))[K^\sigma(\xi^\sigma_n)]^2\mathbb{1}_{[R^*_-(0),R^*_+(0))}(T_n)\Big] \\
&\quad\le \mathbb{E}^0[N([R^*_0,R^*_1))]^2[K^\sigma(\xi^\sigma_0)]^2 \\
&\quad\le \sqrt{[N([R^*_0,R^*_1))]^4}\sqrt{\mathbb{E}^0[K^\sigma(\xi^\sigma_0)]^4},
\end{aligned}
$$

which is finite from **A2**-$(i)$ and **A2**-$(ii)$. Here, the second inequality uses Lemma 1.

Summing up our results, we can apply the Dominated Convergence Theorem:

$$
\begin{aligned}
J'_r(\theta) &\stackrel{\text{def}}{=} \lim_{h\to 0+}\mathbb{E}[f(W_{\theta+h}(0))-f(W_\theta(0))] \\
&= \lim_{h\to 0+}\lambda\,\mathbb{E}^0\,\frac{1}{h}\Phi(\theta,h) \\
&= \lambda\,\mathbb{E}^0\lim_{h\to 0+}\frac{1}{h}\Phi(\theta,h)
\end{aligned}
$$

This gives equation (8). The case of $h<0$ is handled in the same way and gives equation (9)—loosely speaking, the above cases used the sign of $W_{\theta+h}(0)-W_\theta(0)$; this sign is inverted if $h<0$.

This concludes the proof of the theorem. ■

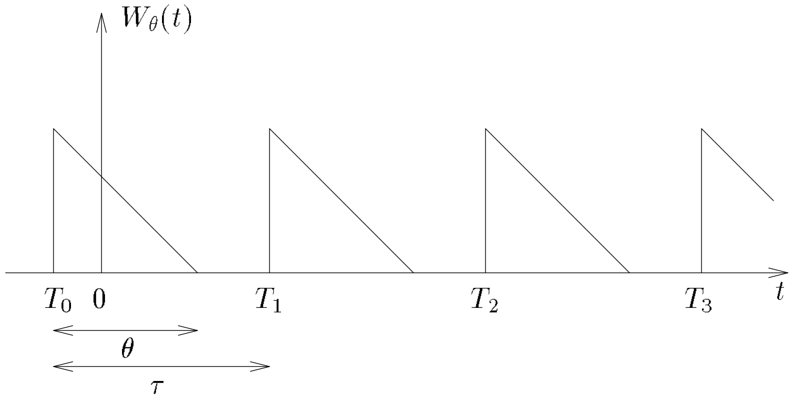



Figure 6: workload of the D/D/1 queue.

**Remark** Assumption **A2** ensures that

$$\mathbb{E}^0\left[f(W_{\theta^*}(0))K_\theta^W(0)\right] < \infty$$

If we know that $f$ is bounded, for example, the only assumptions we will need are

$(i)$ $\mathbb{E}^0[K^\sigma(\xi_0^\sigma)]^2 < \infty$;

$(ii)$ $\mathbb{E}^0[N([R_0^*, R_1^*))]^2 < \infty$.

This reduced set of assumptions can for instance be used in Example 1. □

It is important to point out that Corollary 3 cannot always be applied. We show such a case in next example :

**Example 2** Consider a D/D/1 queue, that is with deterministic inter-arrival time $\tau$ and service time $\theta < \tau$. In order to have a stationary queue, $T_0$ must be uniformly spread in $[-\tau, 0]$. As we can see in Figure 6, we have

$$W(T_n) = \theta,\ W(T_n-) = 0,\ W_\theta'(0) = 1\ \mathrm{P}^0\text{-a.s.}$$

For $x > 0$, take $f(w) = \mathbb{1}_{\{w \geq x\}}$ as in Example 1. Then if $\theta \leq x$, $J(\theta) = 0$, else

$$J(\theta) = \int_{-\tau}^{0} \mathbb{1}_{\{\theta + t \geq x\}} \frac{dt}{\tau} = \frac{\theta - x}{\tau}.$$

Finally $J(\theta) \equiv \mathrm{P}(W_\theta(0) \geq x) = \left(\frac{\theta - x}{\tau}\right)^+$, which is not differentiable at point $\theta = x$. Besides,

$$
\begin{aligned}
J_r'(\theta) &= \lambda\, \mathbb{E}^0[\mathbb{1}_{\{\theta \geq x\}} - \mathbb{1}_{\{0 \geq x\}}] \\
&= \frac{1}{\tau} \mathbb{1}_{\{\theta \geq x\}} \\
J_l'(\theta) &= \lambda\, \mathbb{E}^0[\mathbb{1}_{\{\theta \geq x\}} - \mathbb{1}_{\{0 \geq x\}} + \mathbb{1}_{\{0 = x\}} - \mathbb{1}_{\{\theta = x\}}] \\
&= \frac{1}{\tau} \mathbb{1}_{\{\theta > x\}}
\end{aligned}
$$

□

## 4 Second order derivative.

The method used in Section 3 can be used for higher-order derivatives. However, we need assumptions on the properties of our system and some new moment conditions:

**A3** *$G^\sigma$ and $f$ verify the following:*

*(i)* *$\theta \mapsto G^\sigma(\xi, \theta)$ is twice differentiable and has a bounded second derivative, that is*

$$|G^\sigma(\xi, \theta + 2h) - 2G^\sigma(\xi, \theta + h) + G^\sigma(\xi, \theta)| \leq h^2 K_0^{\sigma\prime}(\xi)$$

*(ii)* *$w \mapsto f(w)$ is non-decreasing and differentiable.*

**A4** *The following inequalities hold:*

*(i)* $\mathbb{E}^0[K^\sigma(\xi_0^\sigma)]^8 < \infty$;

*(ii)* $\mathbb{E}^0[K^{\sigma\prime}(\xi_0^\sigma)]^4 < \infty$;

*(iii)* $\mathbb{E}^0[N([R_0^*, R_1^*))]^8 < \infty$;

*(iv)* $\mathbb{E}^0[f(W_{\theta^*}(0))]^2 < \infty$;

*(v)* $\mathbb{E}^0[\sup_\theta f'(W_\theta(0))]^2 < \infty$;

*(vi)* $\mathbb{E}^0[\sup_\theta f'(W_\theta(T_1-))]^2 < \infty$.

The main result of this section is:

**Theorem 4** *Assume* **A3** *and* **A4** *hold; Then $J$ admits a right second derivative with respect to $\theta$ given by*

$$
\begin{aligned}
J_r''(\theta) \quad = \quad & \lambda\, \mathbb{E}^0 \Big[ W_\theta''(0) \big[ f(W_\theta(0)) - f(W_\theta(T_1-)) \big] \\
& + [W'(0)]^2 \big[ f'(W_\theta(0)) - f'(W_\theta(T_1-)) \\
& - \mathbb{1}_{\{W_\theta'(0)<0\}} [\mu_{f'}(\{W_\theta(0)\}) - \mu_{f'}(\{W_\theta(T_1-)\})] \big] \Big], \quad (13)
\end{aligned}
$$

*and its left second derivative is*

$$
\begin{aligned}
J_l''(\theta) \quad = \quad & \lambda\, \mathbb{E}^0 \Big[ W_\theta''(0) \big[ f(W_\theta(0)) - f(W_\theta(T_1-)) \big] \\
& + [W'(0)]^2 \big[ f'(W_\theta(0)) - f'(W_\theta(T_1-)) \\
& - \mathbb{1}_{\{W_\theta'(0)>0\}} [\mu_{f'}(\{W_\theta(0)\}) - \mu_{f'}(\{W_\theta(T_1-)\})] \big] \Big]. \quad (14)
\end{aligned}
$$

**Corollary 5** *Assume* **A3** *and* **A4** *hold; if $f$ is continuous or if $W_\theta(0)$ and $W_\theta(T_1-)$ admit densities w.r.t.* $\mathrm{P}^0$ *then $J(\theta)$ is differentiable twice and*

$$
\begin{aligned}
J''(\theta) \quad = \quad & \lambda\, \mathbb{E}^0 \Big[ W_\theta''(0) \big[ f(W_\theta(0)) - f(W_\theta(T_1-)) \big] \\
& + [W'(0)]^2 \big[ f'(W_\theta(0)) - f'(W_\theta(T_1-)) \big] \Big].
\end{aligned}
$$

**Proof of Theorem 4** As this proof is very similar to that of Theorem 2, we will omit the parts of it which are not new. We want to compute the limit as $h \to 0$ of

$$
\frac{1}{h^2}\, \mathbb{E} \big[ f(W_{\theta+2h}(0)) - 2f(W_{\theta+h}(0)) + f(W_\theta(0)) \big] = \frac{\lambda}{h^2}\, \mathbb{E}^0\, \Phi_2(\theta, h),
$$

with

$$
\begin{aligned}
\Phi_2(\theta, h) \quad \stackrel{\text{def}}{=} \quad & \int_{\mathbf{R}_+} \int_0^{T_1} \Big[ [\mathbb{1}_{\{W_{\theta+2h}(t)>x\}} - \mathbb{1}_{\{W_{\theta+h}(t)>x\}}] \\
& - [\mathbb{1}_{\{W_{\theta+h}(t)>x\}} - \mathbb{1}_{\{W_\theta(t)>x\}}] \Big]\, dt\, \mu_f(dx).
\end{aligned}
$$

We will once more distinguish two important cases among all possible ones, depending on the sign of $W_\theta'(0)$. Suppose first that $h > 0$.

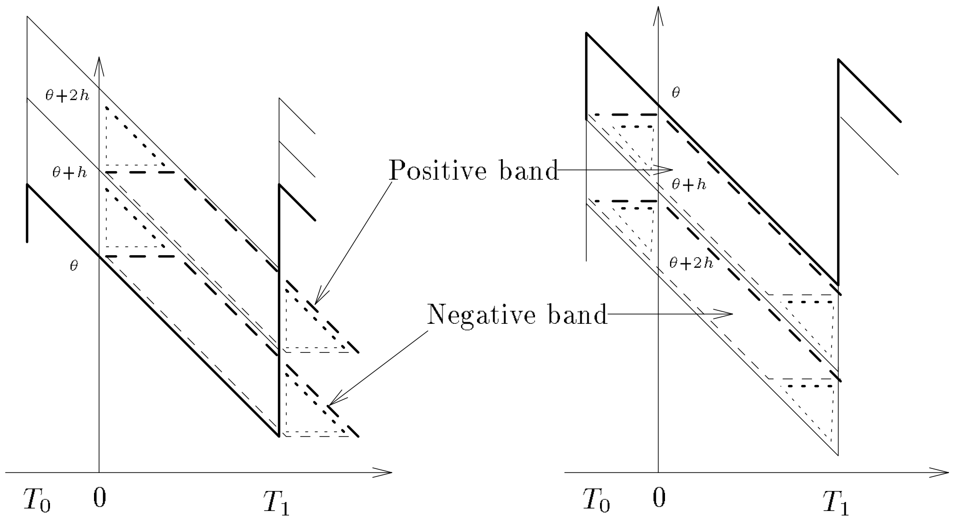



Figure 7: computation of $\Phi_2$ in cases 1 and 2

**Case 1:** $W'_\theta(0) > 0$; for $h$ small enough, $W_\theta(0) < W_{\theta+h}(0) < W_{\theta+2h}(0)$—see Figure 7. We have here to subtract the areas of two bands which are of the same sort as in Theorem 2:

$$\begin{aligned}\Phi_2(\theta, h) &= \Delta W_{\theta+h,h}\Big[f(W_{\theta+h}(0)) - f(W_{\theta+h}(T_1-))\Big] \\ &\quad - \Delta W_{\theta,h}\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big] \\ &\quad + A_{\theta,h}(0) - A_{\theta,h}(T_1-)\end{aligned}$$

where

$$\begin{aligned}A_{\theta,h}(t) &= \int_0^{\Delta W_{\theta+h,h}} \mu_f((W_{\theta+h}(t), W_{\theta+h}(t)+y])\, dy \\ &\quad - \int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(t), W_\theta(t)+y])\, dy.\end{aligned}$$

The main term is equal to

$$\begin{aligned}&\Delta W_{\theta+h,h}\Big[f(W_{\theta+h}(0)) - f(W_{\theta+h}(T_1-)) - f(W_\theta(0)) + f(W_\theta(T_1-))\Big] \\ &+ \Delta^2 W_{\theta,h}\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big].\end{aligned}$$

Moreover,

$$A_{\theta,h}(0) = \int_0^{\Delta W_{\theta,h}} \Big[f(W_{\theta+h}(0)+y) - f(W_{\theta+h}(0))$$

$$
\begin{aligned}
& \quad - f(W_\theta(0)+y) + f(W_\theta(0))\Big] dy + o(h^2) \\
= & \int_0^{\Delta W_{\theta,h}} h W'(0) \mu_{f'}((W_\theta(0), W_\theta(0)+y]) dy + o(h^2)
\end{aligned}
$$

As in Theorem 2-case 1, $\lim_{h\to 0} A_{\theta,h}(0)/h^2 = 0$; the limit is the same for $A_{\theta,h}(T_1-)$. Consequently,

$$
\begin{aligned}
\lim_{h\to 0+} \frac{1}{h^2}\Phi_2(\theta,h) &= [W'_\theta(0)]^2\Big[f'(W_\theta(0)) - f'(W_\theta(T_1-))\Big] \\
&\quad + W''_\theta(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big].
\end{aligned}
$$

**Case 2:** $W'_\theta(0) < 0$; for $h$ small enough, $W_\theta(0) > W_{\theta+h}(0) > W_{\theta+2h}(0)$ and

$$
\begin{aligned}
\Phi_2(\theta,h) &= -1 \cdot \Big\{ -\Delta W_{\theta+h,h}\Big[f(W_{\theta+h}(0)) - f(W_{\theta+h}(T_1-))\Big] \\
&\quad - \Delta W_{\theta,h}\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big] \\
&\quad - B_{\theta,h}(0) + B_{\theta,h}(T_1-)\Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
B_{\theta,h}(t) &\stackrel{\text{def}}{=} \int_0^{-\Delta W_{\theta+h,h}} \mu_f((W_{\theta+h}(t)-y, W_{\theta+h}(t)])\, dy \\
&\quad - \int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(t)-y, W_\theta(t)])\, dy.
\end{aligned}
$$

While the main part has the same limit as in case 1, we have

$$
\begin{aligned}
B_{\theta,h}(0) &= \int_0^{\Delta W_{\theta,h}} h W'(0) \mu_{f'}((W_\theta(0), W_\theta(0)+y]) dy + o(h^2) \\
&= -h^2 [W'_\theta(0)]^2 \mu_{f'}(\{W_\theta(0)\}) + o(h^2).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\lim_{h\to 0-} \frac{1}{h^2}\Phi_2(\theta,h) &= [W'_\theta(0)]^2\Big[f'(W_\theta(0)) - f'(W_\theta(T_1-))\Big] \\
&\quad + W''_\theta(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big] \\
&\quad - [W'_\theta(0)]^2\Big[\mu_{f'}(\{W_\theta(0)\}) - \mu_{f'}(\{W_\theta(T_1-)\})\Big].
\end{aligned}
$$

Besides,

$$\begin{aligned}\left|\frac{1}{h^2}\Phi_2(\theta,h)\right| &\le 3\Big[\sup_\theta f'(W_\theta(0)) + \sup_\theta f'(W_\theta(T_1-))\Big]\Big[K_\theta^W(0)\Big]^2 \\ &\quad + f(W_{\theta^*}(0))K_\theta^{W'}(0),\end{aligned}$$

where $K_\theta^W(0)$ is the same as in Theorem 2 and

$$K_\theta^{W'}(0) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} K^{\sigma'}(\xi_n^\sigma)\mathbb{1}_{[R_-^*(0),R_+^*(0))}(T_n).$$

As in theorem 2, we use Lemma 1, Cauchy-Schwartz inequality and assumption **A4** to prove that $|\Phi_2(\theta,h)/h^2|$ has a finite mean under $\mathrm{P}^0$. Using the Dominated Convergence Theorem, we find expressions (13) and (14) for the second derivatives of $J$. ■

# 5 IPA for queues with parameterized speed.

Let us consider a setting slightly different from the original one: we still deal with a GI/G/1 queue, but now working at speed $\nu$. Neither the inter-arrival times $\{\tau_n\}_{n\in\mathbb{Z}}$ nor the service times $\{\sigma_n\}_{n\in\mathbb{Z}}$ depend on $\nu$ and when the queue is stationary, Lindley's equation for the workload of the queue reads:

$$W_\nu(t) = \Big(W_\nu(T_n-) + \sigma_n - \nu(t-T_n)\Big)^+,\ t\in[T_n,T_{n+1}).$$

We address the same problem as in Section 3 in this new setting: estimate the derivative of $J(\nu) \stackrel{\text{def}}{=} \mathbb{E}\, f(W_\nu(0))$ w.r.t. $\nu$, where $f$ is any real non-decreasing function. Our method can apply in this case in the same way as for variable service times; we will try to keep the notations as close as possible to those of Section 2 to point out the similitudes, replacing $\theta$ with $\nu$ when necessary.

The construction of the queue follows Section 2, except that we do not need the inversion representation to define our queue on a probability space independent from $\nu$. To get a domination property, assume that $\nu \ge \nu^* > 0$, where $\nu^*$ is the minimal speed; then for all $\nu\ge\nu^*$ and $t\in\mathbb{R}$:

$$W_\nu(t) \le W_{\nu^*}(t) \tag{15}$$

$$R_-^*(t) \le R_-(\nu) \le t < R_+(\nu)(t) \le R_+^*(t), \tag{16}$$

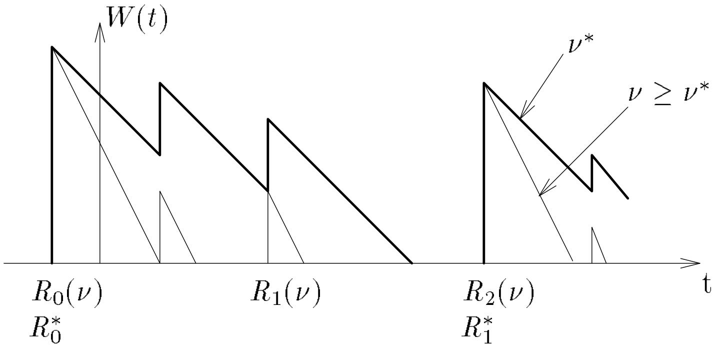



Figure 8: the domination property for the speed

with the notation $R_n^* \stackrel{\text{def}}{=} R_n(\nu^*)$—see Figure 8. Note that Lemma 1 is still valid in this setting.

The assumption on moments we need is much like **A2**:

**A5** *The following moments are finite:*

$(i)$ $\mathbb{E}^0[\tau_0]^4 < \infty$;

$(ii)$ $\mathbb{E}^0[N([R_0^*, R_1^*))]^4 < \infty$;

$(iii)$ $\mathbb{E}^0[f(W_{\nu^*}(0))]^2 < \infty$.

The first real difference with the results of Section 3 is that the expression for the derivative use a primitive of $f$, whereas only $f$ appeared in Theorem 2.

**Theorem 6** *Let $F$ be a primitive of $f$; if* **A5** *holds, then $J$ has a right-hand derivative equal to:*

$$\begin{aligned} J_r'(\nu) &= \frac{\lambda}{\nu^2} \mathbb{E}^0 \Big\{ \nu W_\nu'(0) \Big[ f(W_\nu(0)) - f(W_\nu(T_1-)) \Big] \\ &\quad + F(W_\nu(0)) - F(W_\nu(T_1-)) \\ &\quad - [W_\nu(0) - W_\nu(T_1-)] f(W_\nu(T_1-)) \Big\} \end{aligned} \tag{17}$$

*and its left hand derivative is*

$$J_l'(\nu) = \frac{\lambda}{\nu^2} \mathbb{E}^0 \Big\{ \nu W_\nu'(0) \Big[ f(W_\nu(0)) - f(W_\nu(T_1-)) \Big]$$

$$
\begin{aligned}
&+ F(W_\nu(0)) - F(W_\nu(T_1-)) \\
&- [W_\nu(0) - W_\nu(T_1-)] f(W_\nu(T_1-)) \\
&+ \nu W_\nu'(0)[\mu_f(\{W_\nu(0)\}) - \mu_f(\{W_\nu(T_1-)\})] \\
&- [W_\nu(0) - W_\nu(T_1-)]\mu_f(\{W_\nu(T_1-)\})\Big\}.
\end{aligned}
$$

*Moreover, if $f$ is continuous or if both $W_\nu(0)$ and $W_\nu(T_1-)$ admit densities w.r.t.* $\mathrm{P}^0$, *then $J$ is differentiable and its derivative is equal to $J_r'$.*

**Remark** The expressions in Theorem 6 seem really complicated when compared to those obtained in Theorem 2; in fact, in the case where $f$ is differentiable, the inversion formula applied to (17) gives the classic IPA formula

$$J'(\nu) = \mathbb{E}\, W_\nu'(0) f(W_\nu(0)).$$

The complexity of (17) comes from the fact that $W_\nu'(t)$ is not constant on $[T_0, T_1)$. □

**Proof of Theorem 6** We once more proceed as in the proof of Theorem 2: define

$$\Phi(\nu, h) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} \int_0^{T_1} [\mathbb{1}_{\{W_{\nu+h}(t)>x\}} - \mathbb{1}_{\{W_\nu(t)>x\}}]\, dt\, \mu_f(dx)$$

and remark that

$$\frac{1}{h}\, \mathbb{E}[f(W_{\nu+h}(0)) - f(W_\nu(0))] = \frac{\lambda}{h}\, \mathbb{E}^0\, \Phi(\nu, h).$$

We will consider separately the right-hand and the left-hand derivatives.

**Case 1:** $h > 0$

Figure 9 shows how $\Phi$ can be computed: the main area is equal to the area of the trapezium on the right. As $W_\nu$ is linear in $\nu$, we have

$$\Delta W_{\nu,h}(T_1-) - \Delta W_{\nu,h}(0) = hT_1',$$

where

$$T_1' \stackrel{\text{def}}{=} \min\Big[\frac{W_\nu(0)}{\nu}, T_1\Big] = \frac{W_\nu(0) - W_\nu(T_1-)}{\nu}.$$

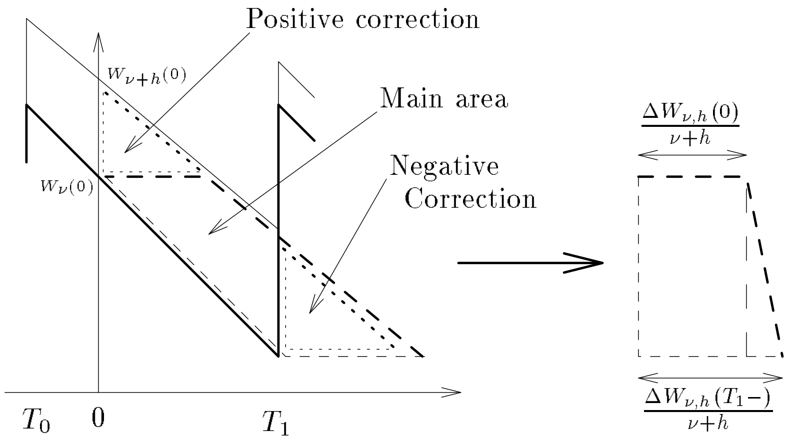



Figure 9: computation of $\Phi$ in case 1.

The area of the trapezium of Figure 9 is equal to

$$\begin{aligned}
\mathcal{A} &= \frac{\Delta W_{\nu,h}(0)}{\nu+h}\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big] \\
&\quad + \int_0^{\frac{hT_1'}{\nu+h}} \mu_f\Big(\Big(W_\nu(T_1-), W_\nu(0) - \frac{\nu(\nu+h)}{h}y\Big]\Big)\,dy \\
&= \frac{h}{\nu+h}\Big\{\frac{\Delta W_{\nu,h}(0)}{h}\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big] \\
&\quad + \frac{1}{\nu}\Big[F(W_\nu(0)) - F(W_\nu(T_1-))\Big] \\
&\quad - \frac{W_\nu(0) - W_\nu(T_1-)}{\nu} f(W_\nu(T_1-))\Big\}.
\end{aligned}$$

The additional terms read:

$$\begin{aligned}
&\int_0^{\frac{\Delta W_{\nu,h}(0)}{\nu+h}} \mu_f((W_\nu(0), W_\nu(0) + (\nu+h)y])\,dy \\
&- \int_0^{\frac{\Delta W_{\nu,h}(T_1-)}{\nu+h}} \mu_f((W_\nu(T_1-), W_\nu(T_1-) + (\nu+h)y])\,dy.
\end{aligned}$$

As we have shown in the proof of Theorem 2, this kind of expression is an $o(h)$ and

$$\lim_{h\to 0+} \Phi(\nu,h) = \frac{W_\nu'(0)}{\nu}\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big]$$

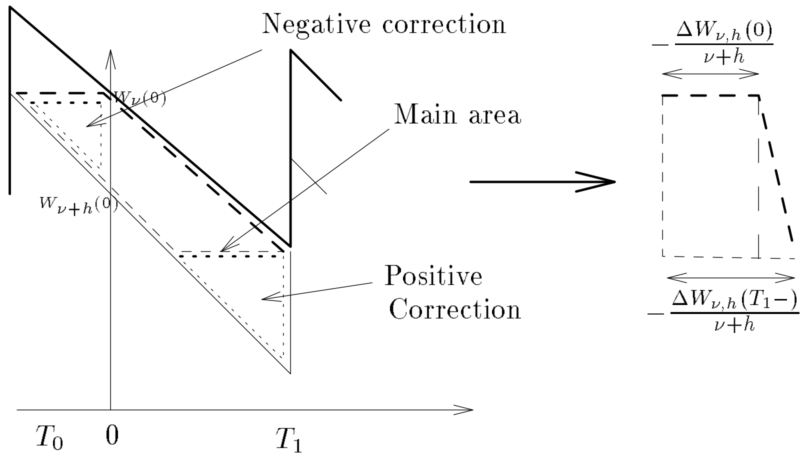



Figure 10: computation of $\Phi$ in case 2.

$$
\begin{aligned}
&+\frac{1}{\nu^2}\Big[F(W_\nu(0)) - F(W_\nu(T_1-))\Big] \\
&-\frac{W_\nu(0) - W_\nu(T_1-)}{\nu} f(W_\nu(T_1-)).
\end{aligned}
$$

**Case 2:** $h < 0$

The main trapezium—see Figure 10—has exactly the same area as in case 1; the difference lies in the corrective terms, which are

$$
\begin{aligned}
& - \int_0^{-\frac{\Delta W_{\nu,h}(0)}{\nu+h}} \mu_f((W_\nu(0) - (\nu+h)y, W_\nu(0)])\, dy \\
& + \int_0^{-\frac{\Delta W_{\nu,h}(T_1-)}{\nu+h}} \mu_f((W_\nu(T_1-) - (\nu+h)y, W_\nu(T_1-)])\, dy \\
= \;& \frac{\Delta W_{\nu,h}(0)}{\nu+h}\mu_f(\{W_\nu(0)\}) - \frac{\Delta W_{\nu,h}(T_1-)}{\nu+h}\mu_f(\{W_\nu(T_1-)\}) + o(h).
\end{aligned}
$$

The limit is:

$$
\begin{aligned}
\lim_{h\to 0-} \Phi(\nu,h) \;=\; & \lim_{h\to 0+} \Phi(\nu,h) \\
& + \frac{W_\nu'(0)}{\nu}[\mu_f(\{W_\nu(0)\}) - \mu_f(\{W_\nu(T_1-)\})] \\
& - \frac{W_\nu(0) - W_\nu(T_1-)}{\nu^2}\mu_f(\{W_\nu(T_1-)\}).
\end{aligned}
$$

Moreover,

$$\begin{aligned}
\left|\frac{1}{h}\Phi(\nu,h)\right| &\leq \frac{|\Delta W_{\nu,h}(0)|}{h\nu^*}\Big|f(W_\nu(0)) - f(W_\nu(T_1-))\Big| \\
&\quad + \frac{T_1'}{\nu^*}\Big|f(W_\nu(0)) - f(W_\nu(T_1-))\Big| \\
&\quad + \frac{|\Delta W_{\nu,h}(0)|}{h\nu^*}\Big|f(W_{\nu+h}(0)) - f(W_\nu(0))\Big| \\
&\quad + \frac{|\Delta W_{\nu,h}(T_1-)|}{h\nu^*}\Big|f(W_{\nu+h}(T_1-)) - f(W_\nu(T_1-))\Big| \\
&\leq \frac{1}{\nu}[3K_\nu^W(0) + 2\tau_0]f(W_{\nu^*}(0)),
\end{aligned}$$

where $K_\nu^W(0)$ is a Lipschitz coefficient for $W_\nu(0)$ w.r.t. $\nu$, which can be expressed as in Theorem 2 as

$$K_\nu^W(0) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \tau_n \mathbb{1}_{[R_-^*(0),R_+^*(0))}(T_n).$$

One can easily check that assumption **A5** suffices to prove that $|\Phi/h|$ is bounded by an integrable variable. Consequently, we can apply the Dominated Convergence Theorem and find the expected result. ■

# 6 Change of time scale.

The method used so far does not apply to the case where the parameter of interest is a parameter of the inter-arrival times; in this case, the Palm measure associated to the arrival process depends on the parameter and the method fails. In this section, we show how a change of time scale can be used in some cases.

Consider a G/G/1 queue which inter-arrival times depend on a parameter, say $\alpha \geq \alpha^* > 0$, and with a server working at speed 1. Given a mean performance $J(\alpha) = \mathbb{E}\, f(W_\alpha(0))$, we want to find an estimator of the derivative of $J$ w.r.t. $\alpha$. We will restrict our attention to the following case:

**A6** *$\alpha$ is a scale parameter for $\tau_n(\alpha)$, that is $\tau_n(\alpha) = \alpha\eta_n$.*

Lindley's equation takes the form

$$W_\alpha(t) = \Big(W_\alpha(T_n(\alpha)-) + \sigma_n - (t - T_n(\alpha))\Big)^+,\ t \in [T_n(\alpha), T_{n+1}(\alpha)) \quad (18)$$

Now define a G/G/1 queue *with speed* $\alpha$ which inter-arrival times, service times and arrival process are given by:

$$\begin{aligned} \tilde{\tau}_n &\stackrel{\text{def}}{=} \frac{\tau_n(\alpha)}{\alpha} = \xi_n^\tau \\ \tilde{\sigma}_n &\stackrel{\text{def}}{=} \sigma_n \\ \widetilde{T}_n &\stackrel{\text{def}}{=} \frac{T_n(\alpha)}{\alpha}. \end{aligned}$$

These processes are stationary with respect to the measurable flow $\tilde{\theta}_t \stackrel{\text{def}}{=} \theta_{\alpha t}$ and the queue they define is stable whenever the original one is; this queue will be referred to as the "auxiliary system". In the remaining of the section, we will use the same notations as for the main system, but with a tilde. Lindley's equation for the auxiliary system reads:

$$\widetilde{W}_\alpha(t) = \left(\widetilde{W}_\alpha(\widetilde{T}_n-) + \tilde{\sigma}_n - \alpha(t - \widetilde{T}_n)\right)^+, \ t \in [\widetilde{T}_n, \widetilde{T}_{n+1}). \tag{19}$$

Comparing equations (18) and (19) and noting that the process $W_\alpha(\alpha t)$ is stationary with respect to the flow $\tilde{\theta}_t$, uniqueness in Loynes' Stability Theorem—see Baccelli and Brémaud [1]— yields

$$W_\alpha(t) = \widetilde{W}_\alpha(t/\alpha).$$

The effect of the change of time scale can be seen on Figure 11. Moreover,

$$\begin{aligned} \tilde{\lambda} &= \mathbb{E}\,\widetilde{N}((0,1]) \\ &= \mathbb{E}\,N((0,\alpha]) = \alpha\lambda(\alpha) \\ \widetilde{W}'_\alpha(0) &= W'_\alpha(0). \end{aligned}$$

In the computation of $\tilde{\lambda}$, we use the fact that the auxiliary system is defined on the same probability space than the main one. However, it has its own Palm measure associated to $\{\widetilde{T}_n\}_{n\in\mathbb{Z}}$, say $\widetilde{\mathrm{P}}^0$. The way to switch between probability measures $\mathrm{P}^0_\alpha$ and $\widetilde{\mathrm{P}}^0$ will be shown in the proof of Theorem 7.

Formula (5) allows us to derive an expression for the derivative of the workload:

$$W'_\alpha(0) = [R_-(\alpha)(0)]' = \frac{R_0(\alpha)}{\alpha}$$

Before proceeding, we need a set of **A5**-like conditions:

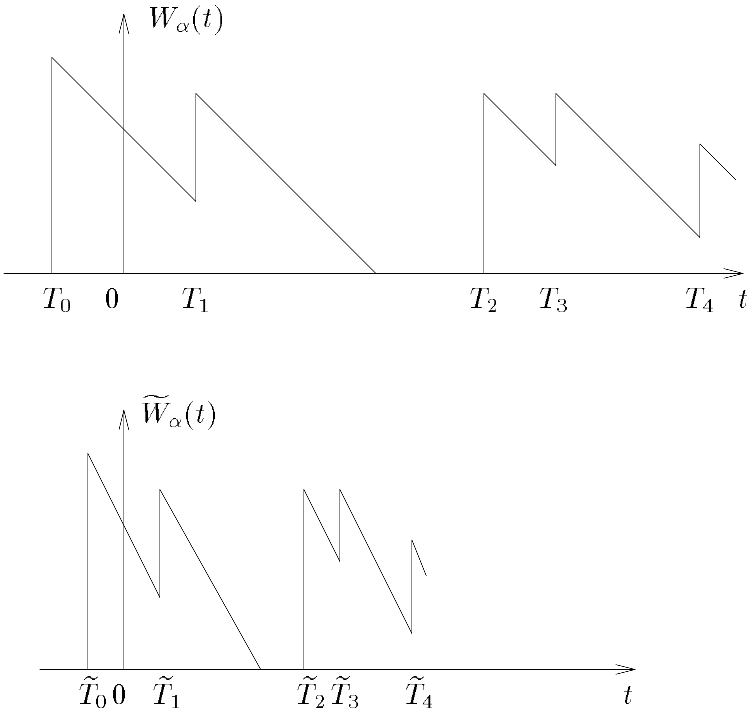



Figure 11: Change of time scale for $\alpha = 2$.

**A7** *The following conditions hold:*

$(i)$ $\mathbb{E}^0_\alpha[\tau_0]^4 < \infty$;

$(ii)$ $\mathbb{E}^0_{\alpha^*}[N([R^*_0, R^*_1))]^4 < \infty$;

$(iii)$ $\mathbb{E}^0_{\alpha^*}[f(W_{\alpha^*}(0))]^2 < \infty$;

Using this model, we find the following result:

**Theorem 7** *Assume* **A6** *and* **A7** *hold; then*

$$\begin{aligned}
J'_r(\alpha) &= \frac{\lambda}{\alpha}\mathbb{E}^0_\alpha\Big\{\alpha W'_\alpha(0)\Big[f(W_\alpha(0)) - f(W_\alpha(T_1-))\Big] \\
&\quad + \Big[F(W_\alpha(0)) - F(W_\alpha(T_1-))\Big] \\
&\quad - [W_\alpha(0) - W_\alpha(T_1-)]f(W_\alpha(T_1-))\Big\} \\
J'_l(\alpha) &= \frac{\lambda}{\alpha}\mathbb{E}^0_\alpha\Big\{\alpha W'_\alpha(0)\Big[f(W_\alpha(0)) - f(W_\alpha(T_1-))\Big] \\
&\quad + \Big[F(W_\alpha(0)) - F(W_\alpha(T_1-))\Big] \\
&\quad - [W_\alpha(0) - W_\alpha(T_1-)]f(W_\alpha(T_1-)) \\
&\qquad + \frac{W_\alpha(T_1-)}{\alpha}\mu_f(\{W_\alpha(T_1-)\}) - \frac{W_\alpha(0)}{\alpha}\mu_f(\{W_\alpha(0)\})\Big\}.
\end{aligned}$$

*Moreover if $f$ is continuous or if $W_\alpha(0)$ and $W_\alpha(T_1(\alpha)-)$ admit densities with respect to $\mathrm{P}^0_\alpha$ then $J$ is differentiable w.r.t. $\alpha$ and its derivative is equal to $J'_r$.*

**Proof** We have

$$J(\alpha) = \mathbb{E}\, f(W_\alpha(0)) = \mathbb{E}\, f(\widetilde{W}_\alpha(0))$$

where $\widetilde{W}_\alpha(0)$ is the workload of the auxiliary queue with speed $\alpha$. We aim to apply Theorem 6 to this queue and then adapt the result to the main queue. The three conditions of **A5** correspond to the three ones of **A7**: for condition **A5**-$(i)$, note that

$$\begin{aligned}
\widetilde{\mathbb{E}}^0[\tilde{\tau}_0]^4 &= \frac{1}{\tilde{\lambda}}\mathbb{E}\sum_{n\in\mathrm{Z}}[\tilde{\tau}_n]^4 \mathbb{1}_{\{\widetilde{T}_n\in(0,1]\}} \\
&= \frac{1}{\alpha\lambda(\alpha)}\mathbb{E}\sum_{n\in\mathrm{Z}}\Big[\frac{\tau_n(\alpha)}{\alpha}^4\Big]\mathbb{1}_{\{T_n(\alpha)\in(0,\alpha]\}} \\
&= \frac{1}{\alpha^4}\mathbb{E}^0_\alpha[\tau_0]^4 < \infty
\end{aligned}$$

and for **A5**-$(ii)$,

$$\begin{aligned}
\widetilde{\mathbb{E}}^0[\widetilde{N}([\widetilde{R}_0^*, \widetilde{R}_1^*))]^4 &= \frac{1}{\tilde{\lambda}}\,\mathbb{E}\sum_{n\in\mathbb{Z}}[\widetilde{N}([\widetilde{R}_-^*(\widetilde{T}_n), \widetilde{R}_+^*(\widetilde{T}_n)))]^4 \mathbb{1}_{\{\widetilde{T}_n\in(0,1]\}} \\
&= \frac{1}{\alpha\lambda(\alpha)}\,\mathbb{E}\sum_{n\in\mathbb{Z}}[N([R_-^*(T_n), R_+^*(T_n)))]^4 \mathbb{1}_{\{T_n(\alpha^*)\in(0,\alpha^*]\}} \\
&= \mathbb{E}^0_{\alpha^*}[N([R_0^*, R_1^*))]^4 < \infty.
\end{aligned}$$

Finally, for **A5**-$(iii)$,

$$\widetilde{\mathbb{E}}^0[f(\widetilde{W}_{\alpha^*}(0))]^2 = \mathbb{E}^0_{\alpha^*}[f(W_{\alpha^*}(0))]^2 < \infty.$$

So we apply Theorem 6 and find:

$$\begin{aligned}
J_r'(\alpha) &= \frac{\tilde{\lambda}}{\alpha^2}\,\widetilde{\mathbb{E}}^0\Big\{\alpha\widetilde{W}_\alpha'(0)\Big[f(\widetilde{W}_\alpha(0)) - f(\widetilde{W}_\alpha(\widetilde{T}_1-))\Big] \\
&\quad + \Big[F(\widetilde{W}_\alpha(0)) - F(\widetilde{W}_\alpha(\widetilde{T}_1-))\Big] \\
&\quad - [\widetilde{W}_\alpha(0) - \widetilde{W}_\alpha(\widetilde{T}_1-)]f(\widetilde{W}_\alpha(\widetilde{T}_1-))\Big\} \\
&= \frac{\lambda}{\alpha}\,\mathbb{E}^0_\alpha\Big\{\alpha W_\alpha'(0)\Big[f(W_\alpha(0)) - f(W_\alpha(T_1-))\Big] \\
&\quad + \Big[F(W_\alpha(0)) - F(W_\alpha(T_1-))\Big] \\
&\quad - [W_\alpha(0) - W_\alpha(T_1-)]f(W_\alpha(T_1-))\Big\}
\end{aligned}$$

The expression for $J_l'$ can be found in the same way. ∎

# 7 Conclusion.

We have given a method to derive IPA estimates non-smooth functions of the workload of GI/G/1 queues. We have shown that our method can apply in various settings; it is worth noting that the same method could be applied to other quantities than the workload of the queue, for instance the number of customers in the system.

The method can work for a general G/G/1 queue, except that the inversion representation used in Section 2 does not always give a probability measure independent of $\theta$, which is a prerequisite for IPA. While $\xi_n^\tau$ and $\xi_n^\sigma$ are uniformly spread on $[0,1]$, the joint probability $P(\xi_n^\tau > t, \xi_n^\sigma > s)$ can depend on $\theta$. Note that this representation is not used in Sections 5 and 6 so that Theorems 6 and 7 are valid even for a G/G/1 queue.

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