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Birth and Death Processes on Certain Random Trees: Classification and Stationary Laws

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Abstract: The main substance of the paper concerns the growth rate and the classification (ergodicity, transience) of a family of random trees. In the basic model, new edges appear according to a Poisson process, and leaves can be deleted at a rate μ. The main results lay the stress on the famous number e. In the case of a pure birth process, i.e. $\mu = 0$, the height of the tree at time t grows linearly at the rate e, in mean and almost surely as $t \to \infty$. When deletions of leaves are permitted, a complete classification of the process is given in terms of the intensity factor $\rho = \lambda/\mu$: it is ergodic if $\rho \leq e^{-1}$, and transient if $\rho > e^{-1}$. There is a phase transition phenomenon: the usual region of null recurrence (in the parameter space) here does not exist. This fact is rare for countable Markov chains with exponentially distributed jump. Bounds are obtained for the transient regime. Some basic stationary laws are computed, e.g. the number of nodes and the height. An extension to the so-called multiclass case is presented, with a more complex classification.

Key-words: random trees, ergodicity, transience, nonlinear differential equations, Schröder equation.

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Processus de naissance et de mort sur certains arbres aléatoires : classification et lois stationnaires

Résumé : Cet article donne le taux de croissance et la classification (ergodicité, transience) d'une famille d'arbres aléatoires. Dans le modèle de base, il y a adjonction d'arcs suivant un Processus de Poisson et suppression des feuilles selon un taux μ. Les résultats principaux mettent en vedette le nombre e. Dans le cas de naissance pure, i.e. $\mu = 0$, la hauteur d'arbre à l'instant t croît linéairement avec une pente asymptotiquement égale à e, en moyenne et presque sûrement, lorsque $t \to \infty$. Quand les suppressions de feuilles sont également autorisées, on obtient une classification complète du processus selon les valeurs du facteur d'intensité $\rho = \lambda/\mu$: ergodicité si $\rho \leq e^{-1}$ et transience si $\rho > e^{-1}$. Un phénomène de transition de phase apparaît : dans l'espace des paramètres, la région correspondant habituellement à la récurrence nulle n'existe pas. Cette situation est rare pour des chaînes de Markov dénombrables dont les lois des sauts sont exponentielles. Des bornes sont également proposées dans le cas de transience. On determine quelques lois stationnaires (nombre de nœuds, hauteur). Enfin, une extension à un modèle multi-classes est donnée, la classification étant alors plus complexe.

Mots-clés : arbres aléatoires, ergodicité, transience, équations différentielles non linéaires, équation de Schröder.

1 Introduction

So far, few results seem to exist for random trees as soon as insertions and deletions are simultaneously permitted (see e.g. [10]). Let $G = \{G(t), t \geq 0\}$ be a continuous time Markov chain with state space the set of finite directed trees rooted at some fixed vertex v_0. This paper is a self-contained continuation of [8, 4] and presents exact results about the growth rate and the complete classification of G (ergodicity, transience) under some evolution rules, which correspond to quite natural models, since one may think of a vertex as being the node of a network or of some general data structure.

1.1 Notation

Throughout the study, the *distance* between two vertices is the number of edges in the path joining them, and the *height* $h(v)$ of a vertex v is the distance from the root. The set of vertices having the same height k form the k-th *level* of the tree, the root v_0 being at level 0. Hence the height of G is a stochastic process $\{H_G(t), t \geq 0\}$, where

$$H_G(t) \stackrel{\text{def}}{=} \max_{v \in G(t)} h(v).$$

$N_G(t)$ will stand for the *volume* of $G(t)$ (*i.e.* its total number of vertices).

Wherever the meaning is clear from the context, we shall omit the subscript G and simply write H or N. The *indegree* of a vertex v is the number of edges starting at v and a vertex with indegree 0 is a *leaf*. Finally, we will also need the classical notion of *subtree* with root v, which goes without saying.

1.2 The basic model

At time $t = 0$, $G(0)$ consists of the single vertex v_0. Then at time $t > 0$, the transitions on G are of two types:

- **Adjunction.** At each vertex v, a new edge having its origin at v can be appended to the tree at the epochs of a Poisson process with parameter $\lambda > 0$. In this case, the *indegree* of v is increased by one and the new edge produces a new leaf.

- **Deletion.** From its birth, any leaf (but the root) can be deleted at a rate μ. In other words, a vertex *as long as it has no descendant* has an exponentially distributed lifetime with parameter $\mu \geq 0$.

In section 2, one computes the growth rate of $H_G(t)$ in the case $\mu = 0$. Section 3 is devoted to the complete classification of G (ergodicity and transience conditions) in the general situation $\mu > 0$, and a phase transition phenomenon is enlightened, which corresponds exactly to the absence of a null recurrence region (see [4]). Some important stationary laws are also obtained (total number of nodes, height of the tree). The last section 4 deals with the extension to a multiclass case.

2 Pure growth: $\mu = 0$

In this section we shall only consider the pure birth process, that is $\mu = 0$. The main outcome is quoted now.

Theorem 2.1.

$$\begin{cases} \lim_{t\to\infty} \dfrac{H(t)}{t} & = \lambda e, \ \textit{with probability one}, \\ \lim_{t\to\infty} \dfrac{\mathsf{E}H(t)}{t} & = \lambda e. \end{cases}$$

The proof is a direct consequence of the four forthcoming lemmas.

Denote by $X_n(t)$ the number of vertices at the n-th level.

Lemma 2.1.

$$\mathsf{E}X_n(t) = \frac{(\lambda t)^n}{n!}. \tag{2.1}$$

Proof. Since

$$\mathsf{P}\big\{X_n(t+dt) = X_n(t) + 1 \big| X_n(t), X_{n-1}(t)\big\} = \lambda X_{n-1}(t)dt + o(dt),$$

we obtain

$$\begin{cases} \dfrac{d}{dt}\mathsf{E}X_n(t) = \lambda \mathsf{E}X_{n-1}(t), \quad n \geq 1, \\ \mathsf{E}X_0(t) = 1, \end{cases}$$

and the result is immediate by induction. ∎

Lemma 2.2.

$$\mathsf{E}H(t) \leq \lambda e t + \frac{\sqrt{\lambda e t}}{\sqrt{2\pi}(e-1)}. \tag{2.2}$$

Proof. Clearly, $\mathsf{P}\{H(t) \geq k\} = \mathsf{P}\{X_k(t) > 0\} \leq \mathsf{E}X_k(t)$, and therefore

$$\mathsf{E}H(t) = \sum_{k=1}^{\infty} \mathsf{P}\{H(t) \geq k\} \leq \sum_{k=1}^{m} 1 + \sum_{k=m+1}^{\infty} \frac{(\lambda t)^k}{k!}.$$

Taking $m = \lceil \lambda e t \rceil$ and using Stirling's formula we have

$$\begin{aligned} \mathsf{E}H(t) &\leq m + \frac{(\lambda t)^{m+1}}{(m+1)!} \sum_{k=0}^{\infty} \Big(\frac{\lambda t}{m+1} \Big)^k \leq \lambda e t + \frac{(\lambda t)^{\lambda e t + 1}}{(\lambda t)^{\lambda e t} \sqrt{2\pi \lambda e t}} \frac{e}{e-1} \\ &\leq \lambda e t + \frac{\sqrt{\lambda e t}}{\sqrt{2\pi}(e-1)}. \end{aligned}$$

■

Lemma 2.3. *With probability one,*

$$\liminf_{t \to \infty} \frac{H(t)}{t} \geq \lambda e. \tag{2.3}$$

Proof. We introduce a so-called *restricted tree*, which means that any vertex of G can possibly be *frozen*, according to some given policy. The main point is that a frozen vertex *does not produce new edges.* Then the corresponding restricted tree G' is a random process defined on the same probability space as G. From the obvious inequality $H_{G'}(t) \leq H_G(t)$, the height of a restricted tree appears to be a lower estimate for $H_G(t)$.

Fix an integer $n \geq 1$ and some time $\tau > 0$. The policy used to freeze vertices will be the following: in each time interval $[\tau k, \tau(k+1)]$, $k = 0, 1, 2, \ldots$, only levels $nk, nk+1, \ldots, nk+(n-1)$ are allowed to produce new edges.

Let $Y_k^{(n)}$ be the number of vertices at the (nk)-th level of the restricted tree obtained at time τk. Clearly $Y_k^{(n)}$ is a branching process with offspring distribution $X_n(\tau)$. Taking τ and n subject to the constraint

$$\mathsf{E}X_n(\tau) = \frac{(\lambda \tau)^n}{n!} > 1,$$

i.e. $n/\tau < \lambda e$ for n sufficiently large, we get

$$y(\tau,n) \stackrel{\text{def}}{=} \mathsf{P}\{Y_k^{(n)} > 0 \text{ for all } k\Big\} > 0,$$

which implies

$$\mathsf{P}\left\{\liminf_{t\to\infty} \frac{H(t)}{t} \geq \frac{n}{\tau}\right\} \geq \mathsf{P}\{H(k\tau) \geq kn \text{ for all } k\} \geq y(\tau,n) > 0.$$

$N(t)$ being the number of vertices of $G(t)$, one can find N_0 and T, such that, for a given $\varepsilon > 0$,

$$\big[1 - y(\tau,n)\big]^{N_0} > 1-\varepsilon, \quad P\big\{N(T) > N_0\big\} > 1-\varepsilon.$$

Thus at time T there will be at least N_0 vertices with probability $(1-\varepsilon)$, and, among the restricted trees rooted at each vertex, one of them will survive with the same probability. Accordingly,

$$\mathsf{P}\big\{H(T+k\tau) \geq kn \text{ for all } k\big\} > (1-\varepsilon)^2$$

and

$$\mathsf{P}\left\{\liminf_{t\to\infty} \frac{H(t+T)}{t} \geq \frac{n}{\tau}\right\} = \mathsf{P}\left\{\liminf_{t\to\infty} \frac{H(t)}{t} \geq \frac{n}{\tau}\right\} > (1-\varepsilon)^2.$$

Keeping in mind that the last inequality holds for all $\varepsilon > 0$ and all n, τ with $n/\tau < \lambda e$, we can write

$$\mathsf{P}\left\{\liminf_{t\to\infty} \frac{H(t)}{t} \geq \lambda e\right\} = 1.$$

The proof of the lemma is terminated. ■

Lemma 2.4. *With probability one,*

$$\limsup_{t\to\infty} \frac{H(t)}{t} \leq \lambda e. \tag{2.4}$$

Proof. Choose now τ and n such that $n/\tau = \gamma\lambda e$ with $\gamma > 1$. Then,

$$\begin{aligned}
\mathsf{P}\left\{\limsup_{t\to\infty}\frac{H(t)}{t} < \gamma\lambda e\right\} &\geq \mathsf{P}\big\{H(k\tau) < kn,\ k = 1,2,\dots\big\} \\
&\geq 1 - \sum_{k=1}^{\infty}\mathsf{P}\big\{H(k\tau) \geq kn\big\} \geq 1 - \sum_{k=1}^{\infty}\frac{(k\lambda\tau)^{kn}}{(kn)!} \\
&\geq 1 - \sum_{k=1}^{\infty}\frac{\gamma^{kn}}{\sqrt{2\pi kn}} \geq 1 - \frac{1}{(1-\gamma^{-n})\sqrt{2\pi n}}.
\end{aligned}$$

Letting n tend to infinity, we obtain

$$\mathsf{P}\left\{\limsup_{t\to\infty}\frac{H(t)}{t} < \gamma\lambda e\right\} = 1.$$

Hence, since γ can be arbitrary close to 1,

$$\mathsf{P}\left\{\limsup_{t\to\infty}\frac{H(t)}{t} \leq \lambda e\right\} = 1,$$

which is the announced result. ∎

Finally the second assertion of theorem 2.1 is a straight consequence of the super-additivity of the function $\mathsf{E}H(t)$,

$$\mathsf{E}H(s+t) \geq \mathsf{E}H(s) + \mathsf{E}H(t),$$

which, by a well known lemma, implies the existence of $\lim_{t\to\infty}\frac{H(t)}{t}$.

Theorem 2.1 is completely proved. ∎

3 The general case: $\lambda > 0, \mu > 0$

In this section leaves can be deleted and the main problem is to find the exact conditions for the process to be ergodic or transient.

With this in view, define the *lifetime* τ_v of an arbitrary vertex v, which measures the length of the time interval between the birth and the death of v (for consistency $\tau_v = \infty$ if v is never erased).

Lemma 3.1. *All vertices, but the root, have the same lifetime distribution* $p(t)$, *which satisfies the following system (S)*

$$\beta(t) = \mu \exp\Big\{-\lambda \int_0^t (1-p(x))dx\Big\}, \tag{3.1}$$

$$\beta(t) = \frac{dp(t)}{dt} + \int_0^t \beta(t-y)dp(y), \tag{3.2}$$

with the initial condition $p(0) = 0$.

Proof. Let v be a particular vertex of $G(t)$ and consider the related random subtree with root v. Its evolution does not depend on anything below v, as long as v exists. Therefore all these subtrees are identically distributed and, accordingly, their vertices have the same lifetime distribution.

To capture more precisely the evolution of the process, we introduce two important random variables associated with each vertex v:

- t_v, the *proper time* of v, such that v appears at $t_v = 0$;
- $X(t_v)$, the number of direct descendants of v (i.e. who are located at a distance 1 from v).

At rate λ, a vertex v produces descendants whose lifetimes are independent, with the common distribution $p(t)$. As soon as $X(t_v) = 0$, v can die at rate μ, in which case the process of production stops.

It is actually useful to extend $X(t_v)$ for all $t_v \geq 0$ by deciding that, instead of deleting v, a μ*-event* occurs without stopping the production of descendants. With this convention, the number of descendants of the root vertex v_0 evolves as $X(t)$, for all $t \geq 0$. Let τ_v denote the random epoch of the first μ-event, which is distributed according to $p(t_v)$.

Clearly the process X is regenerative with respect to the μ-events. Thus that the random variables $X(t_v)$ and $X(\tau_v + t_v)$ have the same distribution.

For any fixed t_v, we write down a sum of conditional probabilities, expressing the fact that v had exactly k descendants, who all have died in $[0, t_v]$, their birth-times being independent and uniformly spread over $[0, t_v]$). This yields at once equation

(3.1), since

$$\begin{aligned}\mathsf{P}\{X(t_v)=0\} &= \sum_{k=0}^{\infty}\frac{e^{-\lambda t_v}(\lambda t_v)^k}{k!}\Big(\int_0^{t_v}\frac{p(x)dx}{t_v}\Big)^k \\ &= \exp\Big\{-\lambda\int_0^{t_v}(1-p(x))dx\Big\}. \end{aligned} \tag{3.3}$$

By means of a regenerative argument, it is also possible to rewrite the above probability in another way, starting from the decomposition

$$\mathsf{P}\{X(t_v)=0\}=\mathsf{P}\{X(t_v)=0,\tau_v\ge t_v\}+\mathsf{P}\{X(t_v)=0,\tau_v<t_v\}. \tag{3.4}$$

In fact, we have the trite relations

$$\begin{cases}\dfrac{dp(t_v)}{dt}=\mu\mathsf{P}\{X(t_v)=0,\tau_v\ge t_v\},\\ \mathsf{P}\{X(t_v)=0,\tau_v<t_v\}=\mathsf{P}\{X(t_v-\tau_v)=0,\tau_v<t_v\},\\ \mathsf{P}\{\tau_v\in(y,y+dy)\}=dp(y),\end{cases}$$

which yield in particular,

$$\mathsf{P}\{X(t_v)=0,\tau_v<t_v\}=\int_0^{t_v}\mathsf{P}\{X(t_v-y)=0\}dp(y).$$

Hence, putting $\beta(t)\stackrel{\text{def}}{=}\mu\mathsf{P}\{X(t_v)=0\}$, one sees that (3.4) corresponds term by term to (3.2). The proof of the lemma is concluded. ■

Theorem 3.1.

(A) The Markov chain G is ergodic if, and only if,

$$\rho\stackrel{def}{=}\frac{\lambda}{\mu}\le\frac{1}{e}\,. \tag{3.5}$$

(B) When the system is ergodic, the mean lifetime $m\stackrel{def}{=}\mathsf{E}(\tau_v)$ is given by

$$m=\frac{r}{\lambda},$$

where $r \leq 1$ denotes the smallest root of the equation

$$re^{-r} = \rho \tag{3.6}$$

and represents the mean number of descendants of an arbitrary vertex at steady state.

(C) When $\rho > \dfrac{1}{e}$, then the system is transient. More precisely,

$$\lim_{t\to\infty} p(t) \stackrel{def}{=} \ell < 1,$$

where ℓ might be extracted from the solution of the non local integral equation (3.15).

The proof of the theorem is spread over the next two sections.

3.1 Ergodicity

Relying on the standard theory of Markov chains with countable state space (see [5, vol. I]), we claim the system ergodic if, and only if, $m < \infty$. As a matter of fact, the μ-events are regeneration points for the process $X(t)$, which represents exactly the number of descendants of the *root* v_0. Hence when $\mathsf{E}(\tau_v) < \infty$ (i.e. $\beta(\infty) > 0$), the event $\{X(t) = 0\}$ has a positive probability, so that G is ergodic. Conversely, if $\mathsf{E}(\tau_v) = \infty$ then $X(t)$ is transient and so is G.

To show the necessity of condition (3.5), suppose G is ergodic. Then, by (3.1), there exists the limit

$$\lim_{t\to\infty} \beta(t).$$

For an arbitrary positive function f, denote by f^* its ordinary Laplace transform

$$f^*(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} f(t)dt, \quad \Re(s) \geq 0.$$

By (3.1), $e^{-m} \leq \alpha(t) \leq 1$, so that we can apply a limiting relation of Abelian type for Laplace transforms (see e.g. [6]). Hence equations (3.1) and (3.2) – the latter belonging to the Volterra class – yield respectively

$$\begin{cases} \lim\limits_{t\to\infty} \beta(t) = \lim\limits_{s\to 0} s\beta^*(s) = \lim\limits_{s\to 0} \dfrac{s^2 p^*(s)}{1 - sp^*(s)} = \dfrac{1}{m}, \\ \lim\limits_{t\to\infty} \beta(t) = \mu e^{-\lambda m}, \end{cases} \tag{3.7}$$

whence the equality

$$\rho = \lambda m e^{-\lambda m}.$$

As the function xe^{-x} reaches its maximum e^{-1} at $x = 1$, we conclude that $\rho \leq e^{-1}$.

In order to prove the sufficiency of (3.5), we have to get a deeper insight into system (S). There will be done along two main steps.

(a) Although (S) reduces to a second order nonlinear integro-differential equation, this does not help much. What is more useful is that all derivatives $p^{(n)}(0)$, $\beta^{(n)}(0)$, taken at the the origin in the complex t-plane, can be recursively computed for all n. This can be checked at once, rewriting (3.1) in the differential form

$$\frac{d\beta(t)}{dt} + \lambda(1 - p(t))\beta(t) = 0. \tag{3.8}$$

Noticing the derivatives $p^{(n)}(0)$ [resp. $\beta^{(n)}(0)$] have alternate signs when n varies, it is direct to verify that β and p are analytic functions around the origin, and that their respective power series have a non-zero radius of convergence. The first singularities of p and β are on the negative real axis, but not easy to locate precisely. Hence (S) has a solution, which is unique, remarking also that uniqueness is a mere consequence of the Lipschitz character of $dp(t)/dt$ with respect to β in the Volterra integral equation (3.2) (see e.g. [2]). En passant, it is worth noting that the solution in the whole complex plane –which is not really needed for our purpose– could be obtained by analytic continuation directly on system (S).

(b) When (3.5) holds, the next stage consists in exhibiting a *non-defective* probabilistic solution $p(t)$ [necessarily unique by step (a)], with a finite mean $m < \infty$. This is more intricate and will be achieved by constructing a converging iterative scheme.

Consider the system

$$
\begin{cases}
\beta_0(t) & = \mu, \quad t \geq 0\,,\\
\beta_k(t) & = \dfrac{dp_k(t)}{dt} + \displaystyle\int_0^t \beta_k(t-y)dp_k(y),\\
\beta_{k+1}(t) & = \mu \exp\Big\{-\lambda \displaystyle\int_0^t \big(1-p_k(y)\big)\,dy\Big\},\\
p_k(0) & = 0,\ \forall k \geq 0\,.
\end{cases}
\tag{3.9}
$$

The second equation in (3.9) is equivalent to

$$
sp_k^*(s) = \frac{\beta_k^*(s)}{1+\beta_k^*(s)},
\tag{3.10}
$$

allowing to derive p_k from β_k, by means of any classical inversion formula for Laplace transforms. Hence, the computational algorithm becomes simple:

1. $p_0(t) = 1 - e^{-\mu t}$.
2. Compute $\beta_1(t) = \mu \exp\big[-\rho(1-e^{-\mu t})\big]$.
3. Compute $p_1(t)$, then $\beta_2(t), p_2(t)$, etc.

At each step, the successive p_k's are non-defective probability distributions, with finite means denoted by m_k. Indeed, one has to check first that the right-hand side of (3.10) is the Laplace transform of a positive measure, since a priori it does not correspond to a *completely monotone* function, according to the classical definition of [5]. The following easy lemma does answer this question and might be of intrinsic interest.

Lemma 3.2. *Let Q be a measure concentrated on $[0,\infty[$ and define*

$$
\psi(s) \stackrel{def}{=} \int_0^\infty \mu e^{-(\lambda Q(t)+st)}dt, \quad \Re(s) \geq 0.
$$

Then $\omega(s) \stackrel{def}{=} \dfrac{\psi(s)}{1+\psi(s)}$ is the Laplace transform of a positive measure on $[0,\infty[$, which in addition is a decreasing functional of Q.

Proof. Take $\Re(s) \geq \mu$. Then the function $\widetilde{\psi}(s) \stackrel{\text{def}}{=} \psi(s+\mu)$ can be viewed as the the Laplace transform of a positive random variable U having the probability density

$\mu e^{-(\lambda Q(t)+\mu t)}$. Thus

$$\mathsf{P}\{U \leq t\} = \int_0^t \mu e^{-(\lambda Q(t)+\mu t)} dt \leq 1 - e^{-\mu t} \leq 1, \tag{3.11}$$

and the following expansion holds

$$\omega(s) = \sum_{k=0}^{\infty} (-1)^k \widetilde{\psi}^{(k+1)}(s-\mu),$$

where the modulus of the k-th term of the series stands for the transform of the $(k+1)$-fold convolution of U. A function being uniquely determined [up to values in a set of measure zero] by the values of its Laplace transform in the region $\Re(s) \geq \mu$, the conclusion of the lemma follows immediately by using elementary properties of alternate series, together with the contraction (3.11). ■

The scheme (3.9) enjoys two nice properties.

(i) It is monotone *decreasing*. Suppose $p_k(t) \leq p_{k-1}(t)$, which is in particular true for $k = 1$. Then the third equation of (3.9) implies $\beta_{k+1}(t) \leq \beta_k(t)$, so that, by lemma 3.2, $p_{k+1}(t) \leq p_k(t)$. The probabilistic interpretation is clear, remarking that, in the third equation of (3.9), $\beta_k(t)/\mu$ is simply the probability of being empty for an M/G/∞ queue with service time distribution p_k.

So, the positive sequences $\{p_k(t), \beta_k(t), k \geq 0\}$ are uniformly bounded and non-increasing for each fixed t. Consequently,

$$p(t) = \lim_{k\to\infty} \downarrow p_k(t) \quad \text{and} \quad \beta(t) = \lim_{k\to\infty} \beta_k(t)$$

form the unique solutions of (S).

(ii) Letting $r_k \stackrel{\text{def}}{=} \lambda m_k$ and combining the two main equations of (3.9), we get

$$r_{k+1} = \rho e^{r_k}, \ \forall k \geq 0, \quad \text{with } r_0 = \rho.$$

For $\rho \leq e^{-1}$, the r_k's form an increasing sequence of positive real numbers, with a finite positive limit r satisfying equation (3.6). Since $1 - p_k(t)$ is an increasing sequence of positive functions, the theorem of Beppo Levi ensures the equality

$$\int_0^\infty (1-p(t))dt = \lim_{k\to\infty} \int_0^\infty (1-p_k(t))dt = \lim_{k\to\infty} m_k = \frac{r}{\lambda}. \tag{3.12}$$

It is worth to point out that scheme (3.9), is equivalent to the construction of a sequence of trees $\{G_k, k \geq 0\}$, such that, for any finite k, G_k is ergodic and has a height not greater than k.

This completes the proof of points (A) and (B) of the theorem. ■

Remarks In the scheme (3.9), the initial condition $\beta_0(t) = \mu$ is tantamount to say that here was implicitly a fictitious function, say p_{-1} satisfying $p_{-1}(t) = 1,\ \forall t \geq 0$.

But we could have also considered the scheme

$$\begin{cases} \gamma_0(t) & = \mu e^{-\lambda t}, \quad t \geq 0\,, \\ \gamma_k(t) & = \dfrac{dq_k(t)}{dt} + \displaystyle\int_0^t \gamma_k(t-y)dq_k(y), \\ \gamma_{k+1}(t) & = \mu \exp\Big\{-\lambda \displaystyle\int_0^t \big(1-q_k(y)\big)dy\Big\}, \\ q_k(0) & = 0,\ \forall k \geq 0\,, \end{cases} \tag{3.13}$$

which differs from (3.9) only by its first equation, but this difference is crucial and corresponds to a fictitious function $q_{-1}(t) = 0,\ \forall t \geq 0$.

The scheme (3.13), which will be analyzed in some details in the next section, enjoys the following properties: it is *increasing*; the distribution q_k are *defective*, and they dominate defective exponential distributions with Laplace transforms of the form $\dfrac{a_k}{b_k + s}$; under condition (3.5),

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \frac{\lambda}{r}$$

and q_k converges in L_1 to the proper distribution p. Actually, the scheme (3.13) produces a sequence of trees $\{L_k, k \geq 0\}$, and the leaves of L_k at level k in fact never die.

3.2 Transience

After needlessly complicated attempts, it turns out that the classification of the process when $\rho > e^{-1}$ can be obtained rather straightforwardly from analytic arguments.

Recalling that $\ell = \lim_{t\to\infty} p(t)$, we define

$$\varepsilon(t) \stackrel{\text{def}}{=} \lambda \int_0^t (\ell - p(x))dx, \quad \text{with } \frac{d\varepsilon(t)}{dt}_{|t=0} = \lambda\ell. \tag{3.14}$$

By a direct computation, we get

$$\beta^*(s) = \mu \int_0^\infty \exp\left[-(\varepsilon(t) + (\lambda(1-\ell)+s)t)\right] dt,$$

together with the functional equation

$$\frac{s^2\varepsilon^*(s)}{\lambda} = \frac{\ell + (\ell-1)\beta^*(s)}{1+\beta^*(s)}. \tag{3.15}$$

Assume now $\ell = 1$. Then (3.15) yields

$$s^2\varepsilon^*(s)\left[1 + \mu\int_0^\infty \exp\left[-\varepsilon(t) - st\right] dt\right] = \lambda. \tag{3.16}$$

Letting now s tend to 0 in (3.16) and applying classical properties of Laplace transforms [already used in (3.7)], we get the equality

$$\lim_{t\to\infty}\left[\varepsilon(t)\exp(-\varepsilon(t))\right] = \rho,$$

which is possible only if $\rho \le e^{-1}$. Thus when $\rho > e^{-1}$ one has necessarily $\ell < 1$, and the system is transient.

The last point is the exact computation of ℓ, which turns out to be an intricate problem. Without presenting a complete solution in this report [although one can hardly expect more than an approximate formula], we shall nonetheless present the main lines of two possible ways of tackling this question.

The first one might be to proceed by analytic continuation on the functional equation (3.15). Indeed, from the definition (3.14) the right-hand side member of (3.15) can be analytically continued to the region $\Re(s) < -\lambda(1-\ell)$, thus rendering possible an analysis of singularities, from where the values ℓ and $L \stackrel{\text{def}}{=} \lim_{t\to\infty}\varepsilon(t)$ could be obtained.

The second method is more constructive and relies on the second iterative scheme (3.13), which is convergent for all ρ. In this scheme, the functions $q_k(t)$, $k \ge 0$, are defective, their limit being proper if and only if $\rho \leq e^{-1}$. When $\rho > e^{-1}$, the limiting function $p(t)$ is still defective and

$$\lim_{t\to\infty} p(t) = \lim_{k\to\infty} \lim_{t\to\infty} q_k(t) = \ell.$$

Hereafter we derive a lower bound on ℓ, and an upper bound on L in showing by induction that, for each k, $q_k(t)$ dominates an exponential distribution.

Take the variable s real positive and assume

$$q_k(t) \geq \frac{a_k}{b_k}\Big[1 - e^{-b_k t}\Big], \tag{3.17}$$

which is in particular true for $k = 0$, since $q_0(t) = \dfrac{\mu}{\lambda+\mu}\Big[1 - e^{-(\lambda+\mu)t}\Big]$. Then the third equation of (3.13) yields, after some calculus,

$$\begin{aligned}
\gamma_{k+1}(t) &\geq \mu \exp\Big[-\lambda\Big(1 - \frac{a_k}{b_k}\Big)t + \frac{\lambda a_k}{b_k^2}(e^{-b_k t} - 1)\Big] \\
&\geq \mu \exp\Big(\frac{-\lambda a_k}{b_k^2}\Big) \exp\Big[-\lambda\Big(1 - \frac{a_k}{b_k}\Big)t\Big],
\end{aligned}$$

so that

$$s q^*_{k+1}(s) = \frac{\gamma^*_{k+1}(s)}{1 + \gamma^*_{k+1}(s)} \geq \frac{\mu \exp\Big(\frac{-\lambda a_k}{b_k^2}\Big)}{\mu \exp\Big(\frac{-\lambda a_k}{b_k^2}\Big) + \lambda\Big(1 - \frac{a_k}{b_k}\Big) + s}.$$

Hence the induction hypothesis holds, provided the sequences $a_k, b_k, k \geq 0$, satisfy the recursive relationships

$$\begin{cases}
a_{k+1} = \mu \exp\Big(\dfrac{-\lambda a_k}{b_k^2}\Big) \\[2ex]
b_{k+1} = a_{k+1} + \lambda\Big(1 - \dfrac{a_k}{b_k}\Big).
\end{cases} \tag{3.18}$$

Setting $a \stackrel{\text{def}}{=} \lim_{k\to\infty} a_k$ and $b \stackrel{\text{def}}{=} \lim_{k\to\infty} b_k$ in (3.18), it is not difficult to see that

$$\frac{a}{b} = \begin{cases} 1, \text{ if } & \rho \leq e^{-1}, \\ x, \text{ if } & \rho \geq e^{-1}, \end{cases}$$

where $x \le 1$ is the root of the equation

$$xe^x = \frac{1}{\rho}. \tag{3.19}$$

Thus, in the transient case, the following bounds hold.

$$\begin{cases} \ell \le x, \\ \displaystyle\int_0^\infty t dp(t) = \int_0^\infty \big(\ell - p(t)\big) dt = \frac{L}{\lambda} \le \frac{a}{b^2} = \frac{x}{\lambda}. \end{cases}$$

It is interesting to compare (3.19) with (3.6), and to check that $x = r = 1$ when $\rho = e^{-1}$.

The proof of theorem 3.1 is terminated. ∎

Subsidiary comments The method of schemes to analyze nonlinear operators in a probabilistic context is extremely powerful (see e.g. [3] for problems related to systems in thermodynamical limit), and in some sense deeply related to the construction of Lyapounov functions. Up to sharp technicalities, the schemes (3.9) and (3.13) can be exploited to derive precise information about the speed of convergence when $t \to \infty$, for any ρ, $0 < \rho < \infty$, and when pushing exact computations slightly farther, one perceives underlying relationships with some complicated continued fractions. Finally, we note that the question of transience could be studied from a large deviation point of view, by considering $\varepsilon(t)$ as the member of a family indexed by the parameter $(\rho - e^{-1})$.

3.3 Some stationary distributions

In this sections, we derive the stationary laws of some performance measures of interest when the system is ergodic, i.e. $\rho \le e^{-1}$. As an aside, note that the only process that has been studied so far, that is the number X of vertices attached to the root, behaves like the number of customers in a M/G/∞ queue with arrival rate λ and service time distribution p, so that

$$\lim_{t\to\infty} \mathsf{P}\{X(t) = k\} = e^{-r}\frac{r^k}{k!}, \ \forall k \ge 0.$$

Let $N \stackrel{\text{def}}{=} \lim_{t\to\infty} N(t)$, where $N(t)$ has been introduced in section 1.1 as the volume of $G(t)$.

Theorem 3.2. *When $\rho \le e^{-1}$, the distribution of the stationary volume N is given by*

$$\mathsf{P}\{N = k\} = \frac{1}{r}\frac{k^{k-1}}{k!}\rho^k, \tag{3.20}$$

where r is given by (3.6). Moreover, the mean value of N is given by

$$\mathsf{E}N = \frac{1}{1-r}. \tag{3.21}$$

Proof. We proceed as in lemma 3.1, saying the number of vertices in the tree at time t is equal to 1 plus the numbers of vertices in all the descendants that have appeared in $[0, t]$ and are not yet dead. The construction of the volume of a subtree is done as for the process $X(\cdot)$: the volume of a subtree rooted at some vertex v is distributed as $N(t_v)$ for $t_v \le \tau_v$.

For any complex number $z, |z| < 1$, we have therefore

$$\begin{aligned}
\mathsf{E}z^{N(t)} &= z\sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\left\{\int_0^t \frac{dx}{t}\mathsf{E}\left[z^{N(x)}\mathbb{1}_{\{x\le\tau\}}\right]\right\}^k \\
&= z\exp\left\{\lambda\mathsf{E}\Big[\int_0^t \big(z^{N(x)}-1\big)\mathbb{1}_{\{x\le\tau\}}dx\Big]\right\},
\end{aligned}$$

and, letting $t \to \infty$,

$$\mathsf{E}z^N = z\exp\left\{\lambda\mathsf{E}\Big[\int_0^\tau \big(z^{N(x)}-1\big)dx\Big]\right\}.$$

Consider the sequence $0 = t_0 < t_1 < t_2 ...$ of μ-events for the process X, so that $\tau_n \stackrel{\text{def}}{=} t_{n+1} - t_n$ is distributed as τ, and let $N_n(x) = N(t_n + x)$, for $x > 0$. Then for any functional f of N, a routine application of the ergodic theorem leads to

$$\begin{aligned}
\mathsf{E}f(N) &= \lim_{t\to\infty}\frac{1}{t}\int_0^t f(N(x))dx \\
&= \lim_{n\to\infty}\frac{1}{t_n}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} f(N(x))dx \\
&= \lim_{n\to\infty}\frac{n}{t_n}\frac{1}{n}\sum_{k=0}^{n}\int_0^{\tau_n} f(N_n(x))dx \\
&= \frac{1}{\mathsf{E}\tau}\mathsf{E}\Big[\int_0^{\tau} f(N(x))dx\Big],
\end{aligned}$$

so that $\phi(z) \stackrel{\text{def}}{=} \mathsf{E}z^N$ is a solution of the equation

$$\rho z = r\phi(z)\exp\big\{-r\phi(z)\big\}. \tag{3.22}$$

Hence $r\phi(z) = T(\rho z)$, where T stands for the classical *tree generating function* (see e.g. [11]). Combining with (3.6), we get the series expansion (derived by means of Lagrange's inversion formula)

$$\phi(z) = \frac{1}{r}\sum_{k=0}^{\infty}\frac{k^{k-1}}{k!}(\rho z)^k,$$

which yields (3.20). The mean (3.21) is obtained by differentiating (3.22) with respect to z and taking $z = 1$. ■

The analysis of the asymptotics of (3.20) with respect to k confirms an interesting change of behaviour (already mentioned in [4]) when $\rho = e^{-1}$. Indeed, for k sufficiently large, Stirling's formula yields

$$\mathsf{P}\{N = k\} = \frac{1}{r}\frac{k^{k-1}}{k!}\rho^k \approx \frac{1}{r}\frac{k^{k-1}}{\sqrt{2\pi k}\,k^k\,e^{-k}}\rho^k = \frac{1}{r}\frac{(\rho e)^k}{\sqrt{2\pi}\,k^{\frac{3}{2}}}.$$

Moreover, a straightforward Taylor expansion of (3.6) gives the following estimate of $\mathsf{E}N$, as $\rho \to e^{-1}$:

$$\mathsf{E}N = \frac{1}{1-r} \approx \frac{1}{\sqrt{2(1-\rho e)}}.$$

Thus, while all moments of N exist for $\rho < e^{-1}$, there is no finite mean as soon as $\rho = e^{-1}$. We note in passing that this phenomenon appears sometimes in branching processes and can be viewed as a phase transition inside the parameter region.

Let $H \stackrel{\text{def}}{=} \lim_{t\to\infty} H(t)$, where $H(t)$ has been introduced in section 1.1 as the height of $G(t)$. The distribution of H is given by the following theorem.
Theorem 3.3.

1. *For $\rho \le e^{-1}$, the following relations hold:*

$$\begin{aligned} \mathsf{P}\{H = 0\} &= e^{-r}, & (3.23)\\ \mathsf{P}\{H > h+1\} &= 1 - \exp\Big[-r\mathsf{P}\{H > h\}\Big], \quad \forall h \ge 0\,. & (3.24) \end{aligned}$$

2. *If $\rho < e^{-1}$, then there exists a positive constant $\theta(r,1)$, such that*

$$\mathsf{P}\{H > h\} = \theta(r,1)r^h + O\Big(\frac{r^{2h}}{1-r}\Big) \tag{3.25}$$

where the function $\theta(r,z)$ is the locally analytic solution of the functional Schröder equation

$$\theta(r, 1 - e^{-rz}) = r\theta(r,z).$$

3. *When $\rho = e^{-1}$,*

$$\mathsf{P}\{H > h\} = \frac{2}{h} + O\Big(\frac{\log h}{h^2}\Big)\,. \tag{3.26}$$

Proof. As in the previous proof, one writes the height of the tree at time t is less than $h+1$ if, and only if, all the descendants that have appeared in $[0,t]$ are either dead or have a height smaller than h:

$$\begin{aligned} \mathsf{P}\big\{H(t) \le h+1\big\} &= \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\Big\{\int_0^t \frac{dx}{t}\big[1 - \mathsf{P}\big\{H(x) > h, x \le \tau\big\}\big]\Big\}^k \\ &= \exp\Big\{-\lambda \int_0^t \mathsf{P}\big\{H(x) > h, x \le \tau\big\}dx\Big\}. \end{aligned}$$

Letting $t \to \infty$ and arguing as in theorem 3.2, we can write

$$\mathsf{P}\big\{H \le h+1\big\} = \exp\Big[-r\mathsf{P}\big\{H > h\big\}\Big],$$

which proves (3.24). On the other hand, equation (3.23) is immediate, since it is in fact a plain rewriting of (3.3).

To prove the remainder of the theorem, let $d_0 \stackrel{\text{def}}{=} x$, where x is a positive real number, and consider the sequence

$$d_{h+1} = 1 - e^{-rd_h}, \ h = 0, 1, ... \tag{3.27}$$

When $x = 1$, note that we have exactly $d_{h+1} = \mathsf{P}\{H > h\}$. The question that faces us now is to compute and to estimate the iterates of an analytic function, in the circumstances $1 - e^{-rx}$. This subject concerns a wide branch of mathematics (including functional equations, automorphic functions, boundary value problems), and it has received considerable attention since the nineteen twenties. We shall employ the main results without further comment, referring the reader e.g. to [9] and [1] for a more extensive treatment.

A Taylor expansion up to second order in (3.27) gives

$$rd_h \geq d_{h+1} = 1 - e^{-rd_h} \geq rd_h - \frac{r^2 d_h^2}{2},$$

which implies that $r^{-h}d_h$ is a decreasing sequence with $\lim\limits_{h\to\infty} \downarrow d_h = 0$ (that we already knew!) and

$$d_h \leq xr^h. \tag{3.28}$$

As $h \to \infty$, the asymptotic behaviour of d_h has a twofold nature, depending on whether $r = 1$ or $r < 1$.

Case $r = 1$. This is the easy part. Writing

$$\frac{1}{d_{h+1}} = \frac{1}{1 - e^{-d_h}} = \frac{1}{d_h} + \frac{1}{2} + O(d_h),$$

we obtain

$$\frac{1}{d_{h+1}} - \frac{1}{d_h} = \frac{1}{2} + O\left(\frac{1}{h}\right), \quad \text{so that} \quad \frac{1}{d_h} = \frac{h}{2} + O(\log h),$$

leading immediately to (3.26).

Case $r < 1$. The analysis is less direct. From (3.27) and (3.28), we infer that, when $h \to \infty$, $r^{-h} d_h$ has a limit denoted by $\theta(r,z)$, with

$$0 \le r^{-h} d_h - \theta(r,z) \le \frac{r^h}{1-r}.$$

First let us show that $\theta(r,z)$ is strictly positive. Indeed,

$$\frac{d_{h+1}}{r^{h+1}} = x \prod_{m=0}^{h} \big(1 - \varphi_m(r,x)\big), \tag{3.29}$$

where, $\forall m \ge 0$, the quantity $\varphi_m(r,x) = O(r^{m+1})$ is an analytic function of the pair of real variables (r,x) in the region $[0,1[\times[0,A]$, with $0 \le A < \infty$. Hence, as $h \to \infty$, the infinite product in (3.29) converges uniformly to a strictly positive value, $\forall x > 0$, so that $\theta(r,x)$ is also analytic of (r,x) in the aforementioned region. To summarize,

$$\lim_{h\to\infty} r^{-h} d_h \stackrel{\text{def}}{=} \theta(r,x) > 0.$$

The interesting fact is that θ, taken as a function of x, satisfies the so-called Schröder equation

$$\theta(r, 1 - e^{-rx}) = r\theta(r,x), \tag{3.30}$$

with $\theta(r,0) = 0$ and $\dfrac{\partial\theta}{\partial x}(r,0) = 1$. We have taken the variable x on the positive real half-line to get sharper bounds, e.g. (3.28). Actually, arguing as above, it is immediate to check that θ has an analytic continuation in the complex x-plane in a a neighborhood of the origin. In this respect, without going into a full discussion, we mention the relationships with automorphic functions and boundary value problems, which would allow integral representations. For our purpose, simply writing

$$\theta(r,x) = \sum_{i\ge 0} c_i x^i, \quad c_0 = 0, \; c_1 = 1,$$

we see that all the c_i's can be computed recursively. Furthermore the iteration of (3.30) yields

$$\theta(r, d_h) = r^h \theta(r,x).$$

from which we obtain

$$d_h = \omega\big(r, r^h\theta(r,x)\big),$$

where $\omega(r, x)$ denotes the inverse function of θ with respect to the variable x and satisfies of the functional relation

$$1 - \exp\{-r\omega(r, y)\} = \omega(r, ry). \tag{3.31}$$

We have

$$\omega(r, y) = \sum_{i \geq 0} d_i y^i,$$

and again the d_i's are evaluated recursively. Choosing $x = 1$ concludes the proof of (3.25) . ■

4 Extension to the multiclass case

The extension of the study to models encompassing several classes of vertices is very tempting, although not quite evident. We solve hereafter a case where the birth and death parameters depend on classes in a reasonably general way.

Let $\mathcal{C}$ be a finite set of classes. Then the multiclass Markov chain $G_{\mathcal{C}}$ has the following evolution.

- The root v_0 of the tree is of class $c \in \mathcal{C}$ with probability π_c, with $\sum_{c \in \mathcal{C}} \pi_c = 1$.
- At any given vertex of class c, a new edge of class $c' \in \mathcal{C}$ can be added at the epochs of a Poisson process with parameter $\lambda_{cc'} \geq 0$.
- Any leaf attached to an edge of class c can be deleted at rate $\mu_c > 0$.

Setting

$$\rho_{cc'} \stackrel{\text{def}}{=} \frac{\lambda_{cc'}}{\mu_{c'}}, \ \forall c, c' \in \mathcal{C},$$

we denote by $\rho \geq 0$ the Perron-Frobenius eigenvalue (see [7]) of the non-negative matrix

$$R \stackrel{\text{def}}{=} \left(\rho_{cc'}\right)_{c, c' \in \mathcal{C}}.$$

This eigenvalue is also the spectral radius of R, so that

$$\rho = \lim_{n \to \infty} \|R^n\|^{1/n},$$

the limit being independent of the choice of the matrix-norm.

Let p_c be the lifetime distribution of a vertex of class c. The following lemma is the analogous of lemma 3.1.

Lemma 4.1. *The lifetime distributions $p_c, c \in \mathcal{C}$ satisfy the following set of equations.*

$$\beta_c(t) = \mu_c \exp\Big\{-\sum_{c'\in\mathcal{C}} \lambda_{cc'} \int_0^t (1 - p_{c'}(x))dx\Big\}, \tag{4.1}$$

$$\beta_c(t) = \frac{dp_c(t)}{dt} + \int_0^t \beta_c(t-y)dp_c(y), \tag{4.2}$$

with the initial conditions $p_c(0) = 0, \forall c \in \mathcal{C}$.

Proof. It suffices to mimic the proof of lemma 3.1 with

$$\beta_c(t) \stackrel{\text{def}}{=} \mu_c \mathsf{P}\{X_c(t) = 0\},$$

and therefore we omit the details. ■

It is actually not easy to come up with a natural explicit extension of theorem 3.1 in the setting of this section. However, the following theorem provides a necessary and sufficient condition for ergodicity.

Theorem 4.1.

1. *The Markov chain $G_{\mathcal{C}}$ is ergodic if, and only if, the nonlinear system*

$$y_c = \sum_{d\in\mathcal{C}} \rho_{cd} \exp\{y_d\}, \ \forall c \in \mathcal{C}, \tag{4.3}$$

has at least one non-negative solution. In this case, the mean lifetime of a node of class c can be written as

$$m_c = \frac{1}{\mu_c} \exp\{r_c\}, \tag{4.4}$$

where the r_c form the smallest non-negative solution of (4.3), that is $r_c \le y_c$, $\forall c \in \mathcal{C}$. Note that (4.3) implies that $r_c \ge \rho_c$.

2. *A simple sufficient condition for ergodicity is*

$$\rho_c \stackrel{def}{=} \sum_{c' \in \mathcal{C}} \rho_{cc'} \leq \frac{1}{e}, \tag{4.5}$$

in which case $\rho_c \leq r_c \leq \rho_c e$, $\forall c \in \mathcal{C}$.

3. *Whenever (4.3) has a real solution, one has necessarily*

$$\rho \leq \frac{1}{e}. \tag{4.6}$$

Remark Before stating the proof of the theorem, it is worth pointing out that equation (4.3) may in general have several real solutions (as in dimension 1). Therefore, there is no guarantee that the solution y_c is the correct one. However, its sole existence proves ergodicity and (4.4).

Proof. Assume first that $G_{\mathcal{C}}$ is ergodic. Then, as in theorem 3.1, we take the limit $t \to \infty$ in (4.1)–(4.2) to obtain the relation

$$\frac{1}{\mu_c m_c} = \exp\Big\{ - \sum_{c' \in \mathcal{C}} \lambda_{cc'} m_{c'} \Big\},$$

which in its turn yields (4.3) and (4.4), just choosing

$$y_c = r_c \stackrel{\text{def}}{=} \sum_{c' \in \mathcal{C}} \lambda_{cc'} m_{c'}.$$

As for the proof of sufficiency in item 1, we introduce the following modified version of scheme (3.9):

$$\begin{cases} p_{c;0}(t) & = 1, \ t \geq 0\,, \\ \beta_{c;k+1}(t) & = \ \mu_c \exp\Big\{ - \displaystyle\sum_{c' \in \mathcal{C}} \lambda_{cc'} \int_0^t \big(1 - p_{c';k}(y)\big) dy \Big\}, \ k \geq 0, \\ \beta_{c;k}(t) & = \ \dfrac{dp_{c;k}(t)}{dt} + \displaystyle\int_0^t \beta_{c;k}(t-y) dp_{c;k}(y), \ k \geq 1, \\ p_{c;k}(0) & = \ 0, \ k \geq 1\,. \end{cases} \tag{4.7}$$

Then, for any $c \in \mathcal{C}$, we have

$$\beta_{c;1}(t) = \mu_c \quad \text{and} \quad p_{c;1}(t) = 1 - e^{-\mu_c t} \ \leq \ p_{c;0}(t).$$

Here again, the positive sequences $\{p_{c;k}(t); k \geq 0\}$ and $\{\beta_{c;k}(t); k \geq 0\}$ are uniformly bounded and non-increasing, for each fixed $t > 0$. Consequently,

$$p_c(t) = \lim_{k\to\infty} \downarrow p_{c;k}(t) \quad \text{and} \quad \beta_c(t) = \lim_{k\to\infty} \beta_{c;k}(t)$$

form the unique solution of (4.1)–(4.2), uniqueness resulting from the Lipschitz character of equation (4.2).

Letting $m_{c;k}$ denote the finite mean associated with each distribution $p_{c;k}$ and

$$r_{c;k} \stackrel{\text{def}}{=} \sum_{c'\in\mathcal{C}} \lambda_{cc'} m_{c';k} \ ,$$

we can write the following recurrence equation

$$r_{c;k+1} = \sum_{c'\in\mathcal{C}} \rho_{cc'} \exp\{r_{c';k}\}, \ \forall c \in \mathcal{C}.$$

The $p_{c;k}$'s are decreasing sequences, and hence the $r_{c;k}$'s are non-decreasing, with $r_{c;0} = 0, \forall c \in \mathcal{C}$. If (4.3) has a solution, then the relation

$$r_{c;k+1} - y_c \ = \sum_{c'\in\mathcal{C}} \rho_{cc'} \left[\exp\{r_{c';k}\} - \exp\{y_{c'}\}\right]$$

yields $r_{c;k} \leq y_c$, for all $c \in \mathcal{C}$. Therefore, each sequence $r_{c;k}$ converges as $k \to \infty$ to a finite value $r_c \leq y_c$, and $G_{\mathcal{C}}$ is ergodic since, by (4.4), the m_c's are also finite. When (4.5) holds, the same line of argument shows that the sequences $r_{c;k}$ are non-decreasing and bounded from above by $\rho_c e$.

Finally, to prove (4.6), we use the following inequality (see [7]), valid for any $x_c > 0$, $c \in \mathcal{C}$:

$$\rho \leq \max_{c\in\mathcal{C}} \sum_{c'\in\mathcal{C}} \frac{\rho_{cc'} x_{c'}}{x_c}.$$

When the r_c's satisfy (4.3), the choice $x_c = \exp\{r_c\}$ implies

$$\rho \leq \max_{c\in\mathcal{C}} \left[\sum_{c'\in\mathcal{C}} \rho_{cc'} \exp\{r_{c'}\} \exp\{-r_c\}\right] = \max_{c\in\mathcal{C}} \left[r_c \exp\{-r_c\}\right] \leq \frac{1}{e},$$

which concludes the proof of the theorem. ■

It is possible to extend the results of section 3.3 to the multiclass case. We will only sketch the proofs in what follows, since they are very similar to their monoclass counterparts. At time $t > 0$, let $N_{cd}(t)$ be the number of vertices of class d inside a tree, the root of which is of class c. Then, as in proof of theorem 3.2, if z_c is a complex number such that $|z_c| < 1$, $\forall c \in \mathcal{C}$,

$$\mathsf{E}\Big[\prod_{d\in\mathcal{C}} z_d^{N_{cd}(t)}\Big] = z_c \exp\Big\{\sum_{c'\in\mathcal{C}} \lambda_{cc'} \mathsf{E}\Big[\int_0^t \Big(\prod_{d\in\mathcal{C}} z_d^{N_{c'd}(x)} - 1\Big) \mathbb{1}_{\{x\le\tau_{c'}\}} dx\Big]\Big\}$$

Assume that the system is ergodic, and let $N_{cd} \stackrel{\text{def}}{=} \lim_{t\to\infty} N_{cd}(t)$ and

$$\phi_c(\vec{z}) \stackrel{\text{def}}{=} \mathsf{E}\Big[\prod_{d\in\mathcal{C}} z_d^{N_{cd}}\Big].$$

Then computations similar to the ones in theorem 3.2 yield

$$\phi_c(\vec{z}) = z_c \exp\Big\{\sum_{c'\in\mathcal{C}} \lambda_{cc'} m_{c'}(\phi_{c'}(\vec{z}) - 1)\Big\}, \quad c \in \mathcal{C}. \tag{4.8}$$

Unfortunately, no closed form solution is known for ϕ_c from this equation. It is however possible, as for (3.21), to write down a system of equations for the expectations of the N_{cd}'s.

$$\mathsf{E}\big[N_{cd}\big] = \mathbb{1}_{\{c=d\}} + \sum_{c'\in\mathcal{C}} \rho_{cc'} \exp\{r_{c'}\} \mathsf{E}\big[N_{c'd}\big].$$

This system admits of a non-negative matrix solution if, and only if, the Perron-Frobenius eigenvalue of the matrix

$$M \stackrel{\text{def}}{=} \Big(\rho_{cc'} \exp\{r_{c'}\}\Big)_{c,c'\in\mathcal{C}}$$

is smaller than 1. A simple necessary condition for this to hold is (4.5).

Finally, the same line of argument allows to extend (3.24). If H_c is the height in stationary regime of a tree which root is of class $c \in \mathcal{C}$, then

$$\begin{aligned} \mathsf{P}\{H_c = 0\} &= e^{-r_c}, \\ \mathsf{P}\{H_c > h+1\} &= 1 - \exp\Big[-\sum_{c'\in\mathcal{C}} \rho_{cc'} \exp\{r_{c'}\} \mathsf{P}\{H_{c'} > h\}\Big], \quad \forall h \ge 0\,. \end{aligned}$$

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