Bayesian image modeling by generalized sparse Markov random fields and loopy belief propagation

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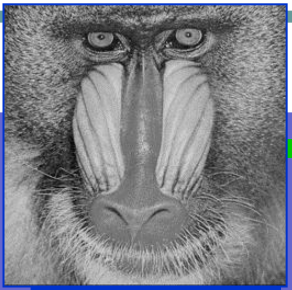
Outline

1. Introduction
2. Bayesian Image Modeling by Generalized Sparse Prior
3. Noise Reductions by Generalized Sparse Prior
4. Expansions to Segmentations
5. Concluding Remarks

Noise Reduction by Bayesian Image Modeling



Original Image

Noise

Transmission

Posterior

Estimate

Degraded Image

Assumption 2: Degraded images are randomly generated from the original image by according to a conditional probability of degradation process.

Assumption 1: Original images are randomly generated by according to a prior probability.

Bayes Formula

$$\overbrace{\Pr\{\text{Original Image} \mid \text{Degraded Image}\}}^{\text{Posterior}}$$

$$\propto \overbrace{\Pr\{\text{Degraded Image} \mid \text{Original Image}\}}^{\text{Degradation Process}} \overbrace{\Pr\{\text{Original Image}\}}^{\text{Prior}}$$

Prior in Bayesian Image Modeling

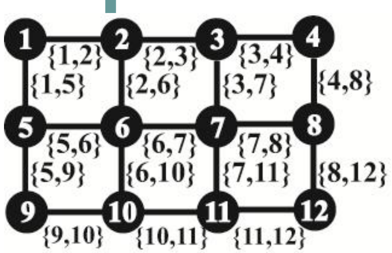
<u>Assumption</u>: Prior Probability is given as the following Gibbs distribution with the interaction α between every nearest neighbour pair of pixels:

$$\overbrace{\Pr\{\text{Original Image} \mid \text{Degraded Image}\}}^{\text{Posterior}}$$

$$\propto \overbrace{\Pr\{\text{Degraded Image} \mid \text{Original Image}\}}^{\text{Degradation Process}}$$

$$\times \overbrace{\Pr\{\text{Original Image}\}}^{\text{Prior}}$$

$$\Pr\{\boldsymbol{F} = \boldsymbol{f} \mid p, \alpha\} = P(\boldsymbol{f} \mid p, \alpha) = \frac{1}{Z(p,\alpha)} \exp\left(-\frac{1}{2} \alpha \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)$$



i : *i*-th pixel

f_i : State Variable of Light Intensity at *i*-pixel

$f = (f_1, f_2, \cdots, f_{12})^{\mathrm{T}}$ State Vector of Original Image

$V = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ Set of All the pixels

$E = \{\{1,2\},\{2,3\},\{3,4\},\{5,6\},\{6,7\},\{7,8\},\{9,10\},\{10,11\},\{11,12\},$
$\{1,5\},\{2,6\},\{3,7\},\{4,8\},\{5,9\},\{6,10\},\{7,11\},\{8,12\}\}$

Set of All the Nearest Neighbour Pairs of Pixels

$$f_i \in \{0, 1, \cdots, q-1\}$$

***p*=0: *q*-state Potts model**

***p*=2: *q*-Ising model (Discrete Gaussian Graphical Model)**

Prior in Bayesian Image Modeling

$$\Pr\{\boldsymbol{F}=\boldsymbol{f}\mid p,\alpha\}=P(\boldsymbol{f}|p,\alpha)=\frac{1}{Z(p,\alpha)}\exp\left(-\frac{1}{2}\alpha\sum_{\{i,j\}\in E}|f_i-f_j|^p\right)$$

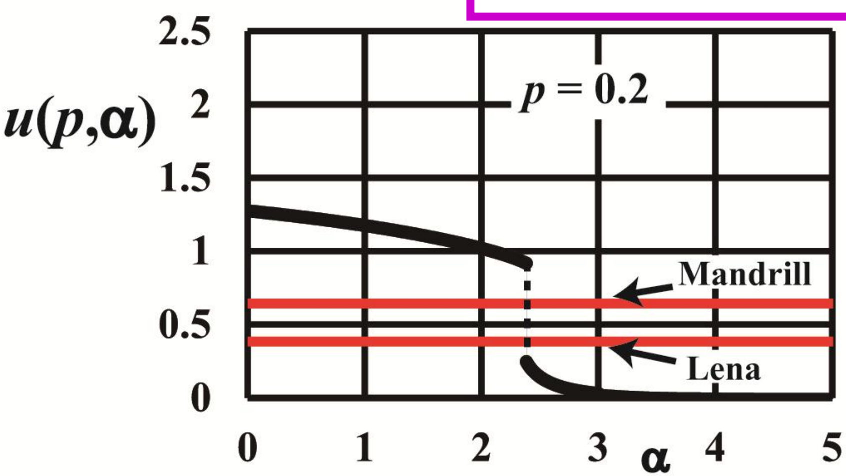
$$\alpha^*=\arg\max_{\alpha}\ln\Pr\{\boldsymbol{F}=\boldsymbol{f}^*\mid p,\alpha\}$$

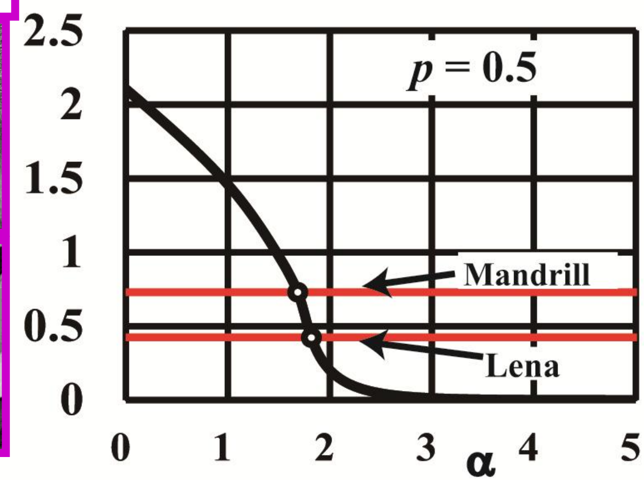
$$u(p,\alpha^*)=u^*$$

$$u(p,\alpha)\equiv\sum_{\{i,j\}\in E}\sum_{\boldsymbol{f}}|f_i-f_j|^p P(\boldsymbol{f}|p,\alpha)$$

$$u^*\equiv\frac{1}{|E|}\sum_{\{i,j\}\in E}|f_i^*-f_j^*|^p$$

Loopy Belief Propagation





In the region of $0<p<0.3504\ldots$, the first order phase transition appears and the solution α^* does not exist.

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1. Introduction
2. Bayesian Image Modeling by Generalized Sparse Prior in Noise Reductions
3. Noise Reduction Procedure by Generalized Sparse Prior
4. Expansions to Segmentations
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Prior by Conditional Maximization of Entropy in Bayesian Image Modeling

DEX-SMI Statistical-Mechanical Approach to Probabilistic Information Processing

Assumption: Prior Probability is given by the conditional maximization of entropy under constraints.

$$\Pr\{\boldsymbol{F}=\boldsymbol{f}\mid p,u\}=\arg\max_{P(\boldsymbol{f})}\left\{-\sum_{\boldsymbol{z}}P(\boldsymbol{z})\ln P(\boldsymbol{z})\;\middle|\;\sum_{\vec{z}}\sum_{\{i,j\}\in E}|z_i-z_j|^p P(\boldsymbol{z})=u|E|\right\}$$

Lagrange Multiplier

$$\Pr\{\vec{F}=\vec{f}\mid p,u\}=P\big(f\big|p,\alpha(p,u)\big)=\frac{1}{Z(p,\alpha(p,u))}\exp\left(-\frac{1}{2}\alpha(p,u)\sum_{\{i,j\}\in E}|f_i-f_j|^p\right)$$

Repeat until *A* converges

$$\alpha(p,u)=A\Leftarrow A\times\sqrt{\frac{1}{u|E|}\sum_{\{i,j\}\in E}\sum_{\boldsymbol{f}}|f_i-f_j|^p P\big(\boldsymbol{f}\big|p,A\big)}$$

Loopy Belief Propagation

Prior Analysis by LBP and Conditional Maximization of Entropy in Bayesian Image Modeling

Prior Probability

$$\Pr\{\boldsymbol{F}=\boldsymbol{f}\mid p,u\}=P\left(\boldsymbol{f}\middle|p,\alpha(p,u)\right)=\frac{1}{Z\left(p,\alpha(p,u)\right)}\exp\left(-\frac{1}{2}\alpha(p,u)\sum_{\{i,j\}\in E}\left|f_i-f_j\right|^p\right)$$

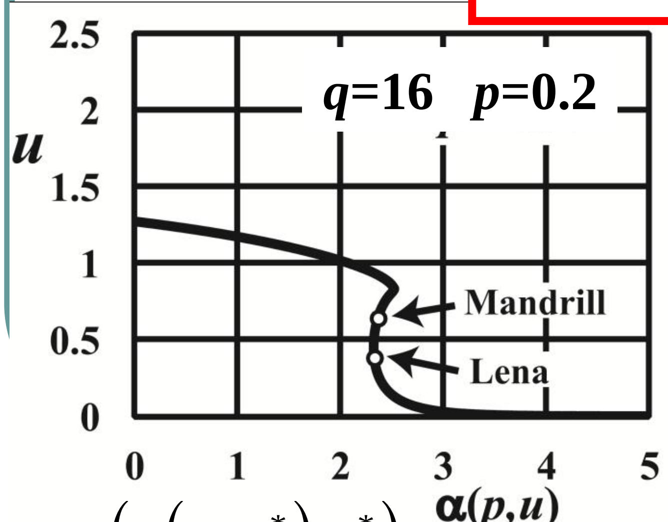
Repeat until *A* converges

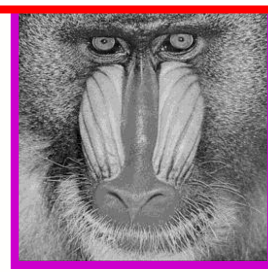
$$\boldsymbol{M}_{j\to i}\Leftarrow\sum_{f_j}\left(\prod_{k\in\partial j\setminus i}M_{k\to j}\right)\exp\left(-\frac{1}{2}A\left|f_i-f_j\right|^p\right)\ (\forall\{i,j\}\in E)$$

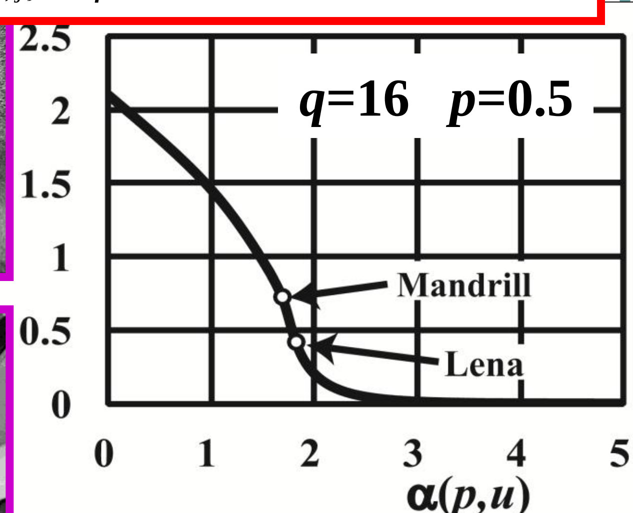
Compute marginals $P_{\{i,j\}}\left(f_i,f_j\middle|p,C\right)$ in LBP by messages $\boldsymbol{M}$

$$\alpha(p,u)=A\Leftarrow A\times\sqrt{\frac{1}{u|E|}\sum_{\{i,j\}\in E}\sum_{\boldsymbol{f}}\left|f_i-f_j\right|^p P_{\{i,j\}}\left(f_i,f_j\middle|p,A\right)}$$

LBP







$\bigcirc\leftarrow\left(\alpha\left(p,\underline{u^*}\right),\underline{u^*}\right)$

$$u^*=\frac{1}{|E|}\sum_{\{i,j\}\in E}\left(f_i^*-f_j^*\right)^p$$

Prior in Bayesian Image Modeling

DEX-SMI

Statistical-Mechanical Approach to Probabilistic Information Processing

Log-Likelihood for *p* when the original image f^* is given

$$\ln \Pr\{\boldsymbol{F} = \boldsymbol{f}^* \mid p, u^*\} = -\frac{1}{2}\alpha(p,u^*) \sum_{\{i,j\}\in E} \left| f_i^* - f_j^* \right|^p - \ln Z\big(p, \alpha(p,u^*)\big)$$

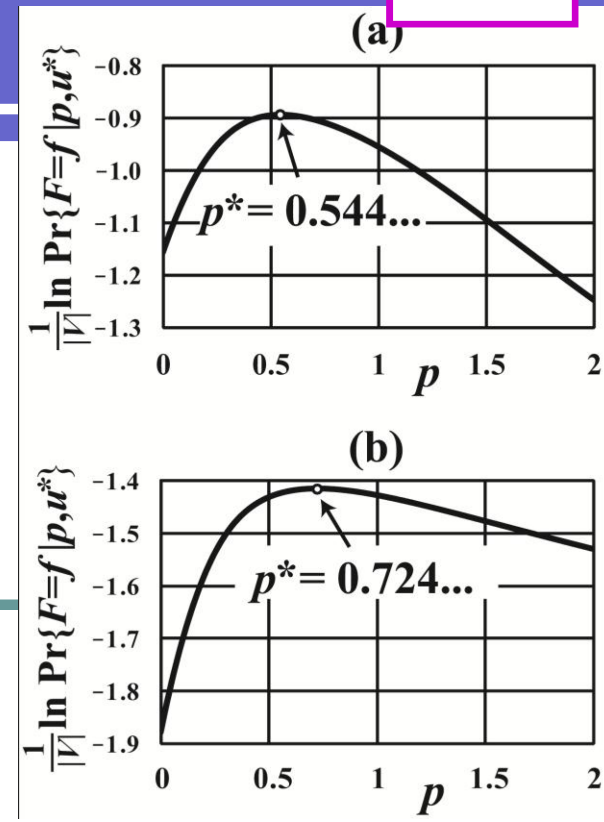
Free Energy of Prior

LBP

$$p^* = \arg\max_{p} \ln \Pr\{\boldsymbol{F} = \boldsymbol{f}^* \mid p, u^*\}$$

$$u^* = \frac{1}{|E|} \sum_{\{i,j\}\in E} \left| f_i^* - f_j^* \right|^p$$

***q*=16**



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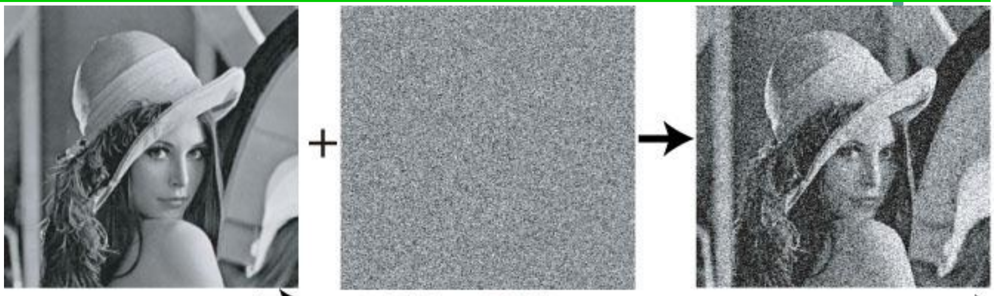
Degradation Process in Bayesian Image Modeling

Assumption: Degraded image is generated from the original image by Additive White Gaussian Noise.

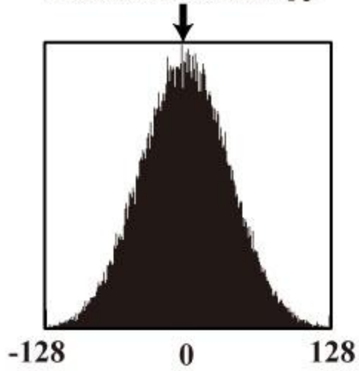
$$\overbrace{\Pr\{\text{Original Image} \mid \text{Degraded Image}\}}^{\text{Posterior}}$$

$$\propto \overbrace{\Pr\{\text{Degraded Image} \mid \text{Original Image}\}}^{\text{Degradation Process}}$$

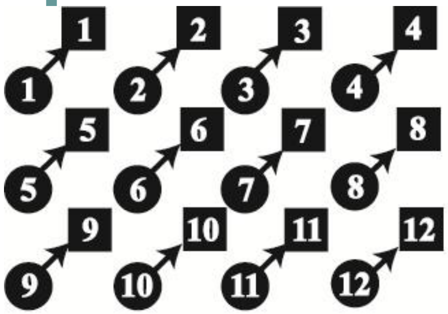
$$\times \overbrace{\Pr\{\text{Original Image}\}}^{\text{Prior}}$$



Original Image $\vec{f}$ + Additive White Gaussian Noise $\vec{n}$ → Degraded Image $\vec{g}$



Histogram of Noise



$$\Pr\{\boldsymbol{G}=\boldsymbol{g} \mid \boldsymbol{F}=\boldsymbol{f},\sigma\} \propto \prod_{i\in V} \exp\left(-\frac{1}{2\sigma^2}(g_i - f_i)^2\right)$$

$$\sigma > 0$$

● :Pixel of Original Image

$$f = (f_1, f_2, \ldots, f_{12})^{\mathrm{T}}$$

■ :Pixel of Degraded Image

$$g = (g_1, g_2, \ldots, g_{12})^{\mathrm{T}}$$

Posterior Probability and Conditional Maximization of Entropy in Bayesian Image Modeling

Posterior Probability

$$\overbrace{\Pr\{\text{Original Image}\,|\,\text{Degraded Image}\}}^{\text{Posterior}}$$

$$\propto \overbrace{\Pr\{\text{Degraded Image}\,|\,\text{Original Image}\}}^{\text{Degradation Process}} \times \overbrace{\Pr\{\text{Original Image}\}}^{\text{Prior}}$$

$$\Pr\{\boldsymbol{F}=\boldsymbol{f}\,|\,\boldsymbol{G}=\boldsymbol{g},p,\sigma,u\} = \arg\max_{P(f)}\left\{-\sum_{\mathbf{z}}P(\mathbf{z})\ln P(\mathbf{z})\,\middle|\, \begin{array}{l}\sum_{\mathbf{z}}\sum_{\{i,j\}\in E}|z_i-z_j|^p P(\mathbf{z})=u|E| \\ \sum_{\mathbf{z}}\sum_{i\in V}(z_i-g_i)^2 P(\mathbf{z})=\sigma^2|V|\end{array}\right\}$$

Lagrange Multipliers

$$\Pr\{\boldsymbol{F}=\boldsymbol{f}\,|\,\boldsymbol{G}=\boldsymbol{g},p,\sigma,u\} = P(\boldsymbol{f}\,|\,\boldsymbol{g},p,B,C)$$

$$=\frac{1}{Z(\boldsymbol{g},p,B,C)}\exp\left(-\frac{1}{2B}\sum_{i\in V}(f_i-g_i)^2-\frac{1}{2}C\sum_{\{i,j\}\in E}|f_i-f_j|^p\right)$$

$$\sigma^2=B \text{ and } \alpha(p,u)=C$$

Bayes Formula

Posterior Probability and Conditional Maximization of Entropy in Bayesian Image Modeling

$$\overbrace{\Pr\{\text{Original Image} \mid \text{Degraded Image}\}}^{\text{Posterior}}$$

$$\propto \overbrace{\Pr\{\text{Degraded Image} \mid \text{Original Image}\}}^{\text{Degradation Process}} \times \overbrace{\Pr\{\text{Original Image}\}}^{\text{Prior}}$$

$$P(\boldsymbol{f} \mid \boldsymbol{g}, p, B, C) \propto \exp\left(-\frac{1}{2B} \sum_{i \in V} (f_i - g_i)^2 - \frac{1}{2} C \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)$$

$$P(\boldsymbol{f} \mid p, C) \propto \exp\left(-\frac{1}{2} C \sum_{\{i,j\} \in E} |f_i - f_j|^p \right)$$

Deterministic Equations for *B*, *C* and *u*

$$\frac{1}{|E|} \sum_{\{i,j\} \in E} \sum_{\boldsymbol{f}} |f_i - f_j|^p P(\boldsymbol{f} \mid \boldsymbol{g}, p, B, C) = \frac{1}{|E|} \sum_{\{i,j\} \in E} \sum_{\boldsymbol{f}} |f_i - f_j|^p P(\boldsymbol{f} \mid p, C) = u$$

$$\frac{1}{|V|} \sum_{\boldsymbol{f}} (f_i - g_i)^2 P(\boldsymbol{f} \mid \boldsymbol{g}, p, B, C) = B$$

Noise Reduction Procedures Based on LBP and Conditional Maximization of Entropy

Input p and data g

Repeat until C and M converge

$$P_{\{i,j\}}(f_i,f_j|p,C) \Leftarrow \boldsymbol{M}_{j\to i} \Leftarrow \sum_{f_j}\left(\prod_{k\in\partial j\backslash i} M_{k\to j}\right)\exp\left(-\frac{1}{2}C|f_i-f_j|^p\right)$$

$$C \Leftarrow C\times\sqrt{\frac{1}{u|E|}\sum_{\{i,j\}\in E}\sum_{f_i}\sum_{f_j}|f_i-f_j|^p P_{\{i,j\}}(f_i,f_j|p,C)}$$

Repeat until u, B and L converge

$$\boldsymbol{L}_{j\to i} \Leftarrow \sum_{f_j}\left(\prod_{k\in\partial j\backslash i}\boldsymbol{L}_{k\to j}\right)\exp\left(-\frac{1}{2|\partial j|B}(f_j-g_j)^2-\frac{1}{2}C|f_i-f_j|^p\right)\begin{pmatrix}\forall i\in V\\ \forall j\in\partial i\end{pmatrix}$$

Compute marginals $P_{ij}(f_i,f_j|\boldsymbol{g},p,B,C)$ and $P_i(f_i|\boldsymbol{g},p,B,C)$ by messages $\boldsymbol{L}$

$$u \Leftarrow \frac{1}{|E|}\sum_{\{i,j\}\in E}\sum_{f_i}\sum_{f_j}|f_i-f_j|^p P_{ij}(f_i,f_j|\boldsymbol{g},p,B,C)$$

$$B \Leftarrow \frac{1}{|V|}\sum_{f_i}(f_i-g_i)^2 P_i(f_i|\boldsymbol{g},p,B,C)$$

Marginals and Free Energy in LBP

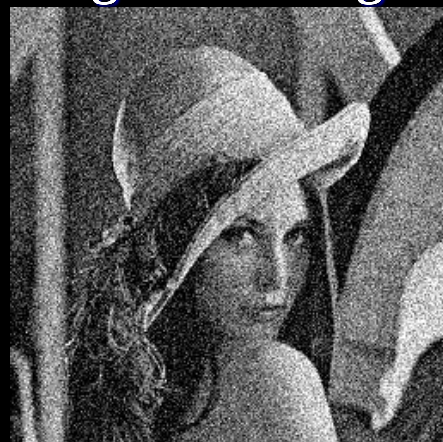
$$\ln\Pr\{\boldsymbol{G}=\boldsymbol{g}\,|\,p,\hat{\sigma},\hat{u}\}$$
$$=\ln Z(\boldsymbol{g},p,\hat{\sigma}^2,\alpha(p,\hat{u}))$$
$$-\ln Z(p,\alpha(p,\hat{u}))-\frac{1}{2}|V|\ln(2\pi\hat{\sigma})$$
$$\hat{p}=\arg\max_p\Pr\{\boldsymbol{G}=\boldsymbol{g}\,|\,p,\hat{\sigma},\hat{u}\}$$

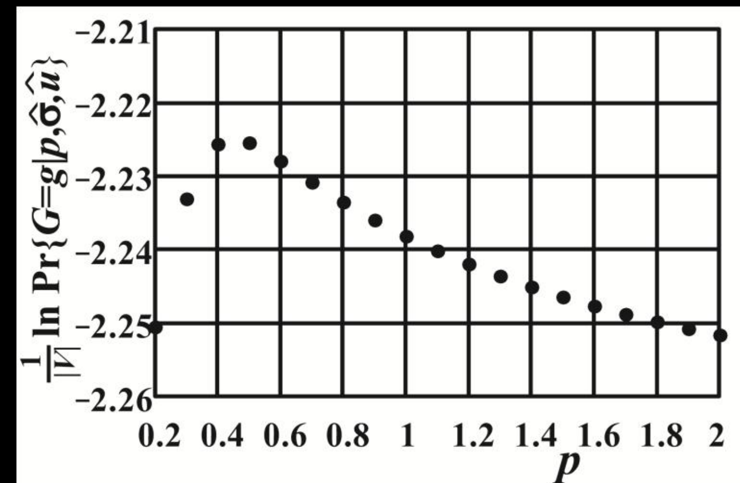
$$\hat{\sigma}\Leftarrow\sqrt{B},\ \hat{u}\Leftarrow u,\ \alpha(p,\hat{u})\Leftarrow C$$
$$\hat{f}_i(\boldsymbol{g},p)\Leftarrow\arg\max P_i(f_i|\boldsymbol{g},p,B,C)$$

Noise Reductions by Generalized Sparse Priors and Loopy Belief Propagation

Original Image

Degraded Image





***p*=0.2**

***p*=0.5**

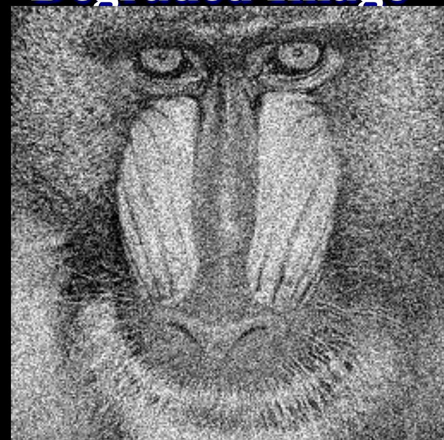
***p*=1**

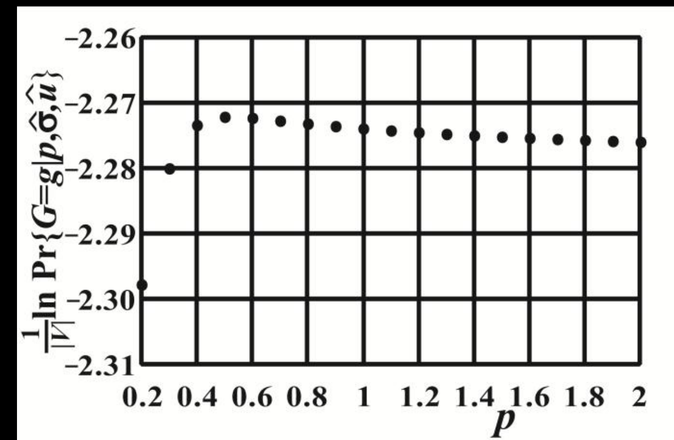
Restored Image

Noise Reductions by Generalized Sparse Priors and Loopy Belief Propagation

Original Image

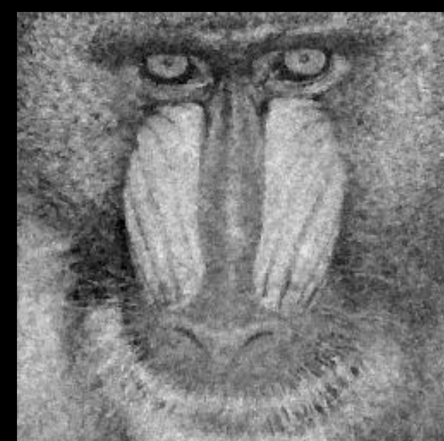
Degraded Image





***p*=0.2**

***p*=0.5**



***p*=1**

Restored Image

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Segmentation based on Bayesian Image Modeling by Gaussian Mixture Model

$$\left.\begin{matrix} P\left(\boldsymbol{f}\middle|\boldsymbol{g},\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\gamma}\right) \\ P\left(\boldsymbol{f},\boldsymbol{g}\middle|\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\gamma}\right) \end{matrix}\right\} \propto \prod_{i\in V}\gamma_{f_i}\exp\left(-\frac{1}{2}\left(\frac{g_i-\mu_{f_i}}{\sigma_{f_i}}\right)^2\right)$$

$$\boldsymbol{\mu}=\begin{pmatrix}\mu(0)\\ \mu(1)\\ \vdots\\ \mu(q-1)\end{pmatrix},\ \boldsymbol{\sigma}=\begin{pmatrix}\sigma(0)\\ \sigma(1)\\ \vdots\\ \sigma(q-1)\end{pmatrix},\ \boldsymbol{\gamma}=\begin{pmatrix}\gamma(0)\\ \gamma(1)\\ \vdots\\ \gamma(q-1)\end{pmatrix}$$

$$\left(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\gamma}}\right)=\arg\max_{\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\gamma}} P\left(\boldsymbol{g}\middle|\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\gamma}\right)$$

$$P\left(\boldsymbol{g}\middle|\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\gamma}\right)=\sum_{\boldsymbol{f}}P\left(\boldsymbol{f},\boldsymbol{g}\middle|\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\gamma}\right)$$

$$=\prod_{i\in V}\left(\gamma_0\rho_0(g_i)+\gamma_1\rho_1(g_i)+\cdots+\gamma_{q-1}\rho_{q-1}(g_i)\right)$$

$$\rho_l(g_i)=\frac{1}{\sqrt{2\pi\sigma_l}}\exp\left(-\frac{1}{2}\left(\frac{g_i-\mu_l}{\sigma_l}\right)^2\right)$$

***g*:Original Image**

***f*:Segmented Image**

Extension of Gaussian Mixture Model based on Generalized Sparse Prior in Bayesian Segmentation

DEX-SMI

Statistical-Mechanical Approach to Probabilistic Information Processing

$$\boldsymbol{\mu} = \left(\mu_0, \mu_1, \cdots, \mu_{q-1}\right)^{\mathrm{T}}$$

$$\boldsymbol{\sigma} = \left(\sigma_0, \sigma_1, \cdots, \sigma_{q-1}\right)^{\mathrm{T}}$$

$$P\left(\boldsymbol{f} \middle| \boldsymbol{g}, p, C, \boldsymbol{\mu}, \boldsymbol{\sigma}\right) \propto \exp\left(-\frac{1}{2}\sum_{i\in V}\left(\frac{g_i - \mu_{f_i}}{\sigma_{f_i}}\right)^2 - \frac{1}{2}C\sum_{\{i,j\}\in E}\left|f_i - f_j\right|^p \right)$$

$$P\left(\boldsymbol{f} \middle| p, C\right) \propto \exp\left(-\frac{1}{2}C\sum_{\{i,j\}\in E}\left|f_i - f_j\right|^p \right)$$

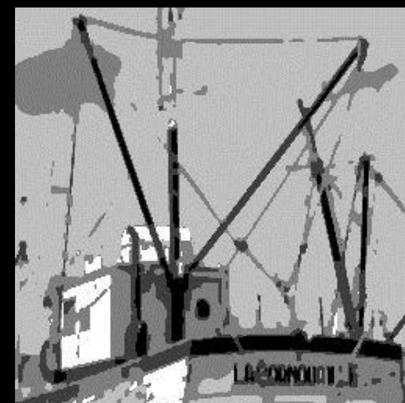
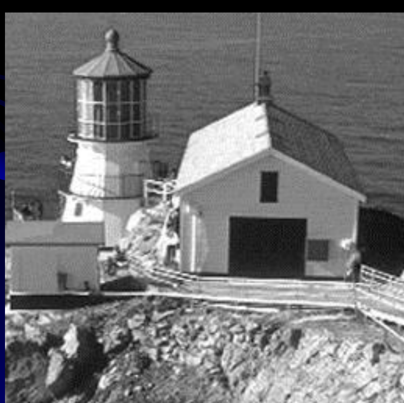
Deterministic Equations for *C* and *u*

$$\frac{1}{|E|}\sum_{\{i,j\}\in E}\sum_{\boldsymbol{f}}\left|f_i - f_j\right|^p P\left(\boldsymbol{f} \middle| \boldsymbol{g}, p, C, \boldsymbol{\mu}, \boldsymbol{\sigma}\right) = \frac{1}{|E|}\sum_{\{i,j\}\in E}\sum_{\boldsymbol{f}}\left|f_i - f_j\right|^p P\left(\boldsymbol{f} \middle| p, C\right) = u$$

***g*:Original Image**

***f*:Segmented Image**

Segmentation by Generalized Sparse Prior and Loopy Belief Propagation

Original Image g	Segmented Image q=3, p=0.2	Segmented Image q=4, p=0.2	Segmented Image q=5, p=0.2
			
			

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Summary

- Formulation of Bayesian image modeling for image processing by means of generalized sparse priors and loopy belief propagation are proposed.
- Our formulation is based on the conditional maximization of entropy with some constraints.
- In our sparse priors, although the first order phase transitions often appear, our algorithm works well also in such cases.
- Numerical experimental results in noise reductions and segmentations have been shown.

References

1. J. Inoue and K. Tanaka: Mean Field Theory of EM Algorithm for Bayesian Gray Scale Image Restoration, Journal of Physics A: Mathematical and General, Vol.36, No.43, pp.10997-11010,2003.
2. K. Tanaka, J. Inoue and D. M. Titterington: Probabilistic Image Processing by Means of Bethe Approximation for Q-Ising Model, Journal of Physics A: Mathematical and General, Vol. 36, No. 43 (October 2003), pp.11023-11036, 2003.
3. K. Tanaka: Mathematical Structures of Loopy Belief Propagation and Cluster Variation Method, Journal of Physics: Conference Series, vol.143, article no.012023, pp.1-18, 2009
4. S. Kataoka, M. Yasuda and K. Tanaka: Statistical Performance Analysis in Probabilistic Image Processing, Journal of the Physical Society of Japan, vol.79, no.2, article no.025001, 2010.
5. S. Kataoka, M. Yasuda, K. Tanaka and D. M. Titterington: Statistical Analysis of the Expectation-Maximization Algorithm with Loopy Belief Propagation in Bayesian Image Modeling, Philosophical Magazine: The Study of Condensed Matter, Vol.92, Nos.1-3, pp.50-63,2012.
6. S. Kataoka, M. Yasuda and K. Tanaka: Statistical Analysis of Gaussian Image Inpainting Problems, Journal of the Physical Society of Japan, Vol.81, No.2 (February 2012), Article No.025001, pp.1-2, 2012.