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A CLASS OF EFFICIENT LOCALLY CONSTRUCTED PRECONDITIONERS BASED ON COARSE SPACES*

HUSSAM AL DAAS† AND LAURA GRIGORI†

Abstract. In this paper we present a class of robust and fully algebraic two-level preconditioners for symmetric positive definite (SPD) matrices. We introduce the notion of algebraic local symmetric positive semidefinite splitting of an SPD matrix and we give a characterization of this splitting. This splitting leads to construct *algebraically and locally* a class of efficient coarse spaces which bound the spectral condition number of the preconditioned system by a number defined a priori. We also introduce the τ-filtering subspace. This concept helps compare the dimension minimality of coarse spaces. Some PDEs-dependant preconditioners correspond to a special case. The examples of the algebraic coarse spaces in this paper are not practical due to expensive construction. We propose a heuristic approximation that is not costly. Numerical experiments illustrate the efficiency of the proposed method.

Key words. preconditioners, iterative linear solvers, domain decomposition, condition number, coarse spaces

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1. Introduction.

The conjugate gradient (CG) method [8] is a widely known Krylov iterative method for solving large linear systems of equations of the form

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (SPD) matrix, $b \in \mathbb{R}^n$ is the right-hand side, and $x \in \mathbb{R}^n$ is the vector of unknowns. It finds at iteration j the approximate solution $x_j \in x_0 + K_j(A, r_0)$ that minimizes the A-norm of the error $\|x_* - x_j\|_A$, where x_0 is the initial guess, $r_0 = b - Ax_0$, $K_j(A, r_0)$ is the Krylov subspace of dimension j related to A and r_0, x_* is the exact solution of (1.1), and $\|.\|_A$ is the A-norm. The convergence of this method is well studied in the literature [16]. The rate of convergence depends on the condition number of the matrix A. Letting $\kappa = \frac{\lambda_n}{\lambda_1}$ be the spectral condition number of A, where λ_n and λ_1 are the largest and the smallest eigenvalues of A, respectively, the error at iteration j satisfies the following inequality:

$$\|x_* - x_j\|_A \le \|x_* - x_0\|_A \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^j. \tag{1.2}$$

We suppose that the graph of the matrix is partitioned into a number of subdomains by using a k-way partitioning method [10]. To enhance the convergence, it is common to solve the preconditioned system

$$M^{-1}Ax = M^{-1}b. \tag{1.3}$$

Block Jacobi, additive Schwarz, restricted additive Schwarz, etc., are widely used preconditioners. These preconditioners are called one-level preconditioners. They

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†ALPINES, INRIA, Paris 75012, France (hussam.al-daas@inria.fr, https://who.rocq.inria.fr/Hussam.Al-Daas/index.html, laura.grigori@inria.fr, https://who.rocq.inria.fr/Laura.Grigori/index.html).

correspond to solving subproblems on subdomains. In [2, 3] the authors prove that the largest eigenvalue of the preconditioned system by the additive Schwarz preconditioner is bounded by a number that is independent of the number of subdomains. However, no control is guaranteed for the smallest eigenvalue of the preconditioned matrix. Furthermore, when the number of subdomains increases, the smallest eigenvalue might become even smaller. Thus, the number of iterations to reach convergence typically increases. This occurs since this type of preconditioner employs only local information and does not include global information. For this reason, these preconditioners are usually combined with a second-level preconditioner, which corresponds to a coarse space correction or deflation. In principle, it is meant to annihilate the impact of the smallest eigenvalues of the operator. Different strategies exist in the literature to add this level. In [20], the authors compare different strategies of applying two-level preconditioners. In [2, 21, 12, 18, 3, 6, 11], the authors propose different methods for constructing a coarse space correction. Coarse spaces can be categorized into two types, analytic and algebraic. Analytic coarse spaces depend on the underlying problem from which the matrix A is issued. Algebraic coarse spaces depend only on the coefficient matrix A and do not require information from the underlying problem from which A arises. Based on the underlying partial differential equation (PDE) and its discretization, several methods that propose analytic coarse spaces are described in the literature [3, 2, 21, 12, 18].

In most cases, a generalized (or standard) eigenvalue problem is solved in each subdomain. Every subdomain then contributes to the construction of the coarse space by adding certain eigenvectors. These methods are efficient in several applications. Nevertheless, the dependence on the analytic information makes it impossible to be made in a pure algebraic way. Algebraic coarse space correction can be found in the literature [6, 11]. However, the construction of the coarse space can be even more costly than solving the linear system (1.1). In this paper we discuss a class of robust preconditioners that are based on locally constructed coarse spaces. We characterize the local eigenvalue problems that allow us to construct an efficient coarse space related to the additive Schwarz preconditioner. The paper is organized as follows. In section 2 we review general theory of one- and two-level preconditioners, and in section 3 we present our main result. We introduce the notion of *algebraic local symmetric positive semidefinite (SPSD) splitting of an SPD matrix*. For a simple case, given the block SPD matrix

$$B = \begin{pmatrix} B_{11} & B_{12} & \\ B_{21} & B_{22} & B_{23} \\ & B_{32} & B_{33} \end{pmatrix},$$

the local SPSD splitting of B with respect to the first block means finding two SPSD matrices B_1, B_2 of the form

$$B_1 = \begin{pmatrix} B_{11} & B_{12} & \\ B_{21} & * & \\ & & \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} & & \\ & * & B_{23} \\ & B_{32} & B_{33} \end{pmatrix},$$

where $*$ represents a nonzero block matrix such that $B = B_1 + B_2$. We characterize all possible local SPSD splittings. Then we introduce the τ-filtering subspace. Given two SPSD matrices A, B, a τ-filtering subspace Z makes the following inequality hold:

$$(u - Pu)^\top B (u - Pu) \le u^\top A u \quad \forall u,$$

where P is an orthogonal projection on Z. Based on the local SPSD splitting and the τ-filtering subspace, we propose in section 4 an efficient coarse space, which bounds the spectral condition number by a given number defined a priori. Furthermore, we show how the coarse space can be chosen such that its dimension is minimal. The resulting spectral condition number depends on three parameters. The first parameter depends on the sparsity of the matrix, namely, the minimum number of colors k_c needed to color subdomains such that two subdomains of the same color are disjoint; see Lemma 2.7 [2, Theorem 12]. The second parameter k_m depends on the algebraic local SPSD splitting. It is bounded by the number of subdomains. For a special case of splitting it can be chosen to be the maximal number of subdomains that share a degree of freedom (DOF). The third parameter is chosen such that the spectral condition number is bounded by the user-defined upper bound. In all stages of the construction of this coarse space, no information is necessary but the coefficient matrix A and the desired bound on the spectral condition number. We show how the coarse space constructed analytically by the method GenEO [17, 3] corresponds to a special case of our characterization. We also discuss the extreme cases of the algebraic local SPSD splitting and the corresponding coarse spaces. We explain how these two choices are expensive to construct in practice. Afterward, we propose a practical strategy to compute efficiently an approximation of the coarse space. In section 5 we present numerical experiments to illustrate the theoretical and practical impact of our work. At the end, we give our conclusion in section 6.

To facilitate the comparison with GenEO we follow the presentation in [3, Chapter 7].

Notation. Let $A \in \mathbb{R}^{n\times n}$ denote a symmetric positive definite matrix. We use MATLAB notation. Let $S_1, S_2 \subset \{1, \dots, n\}$ be two sets of indices. The concatenation of S_1 and S_2 is represented by $[S_1, S_2]$. We note that the order of the concatenation is important. $A(S_1, :)$ is the submatrix of A formed by the rows whose indices belong to S_1. $A(:, S_1)$ is the submatrix of A formed by the columns whose indices belong to S_1. $A(S_1, S_2) := (A(S_1, :))(:, S_2)$. The identity matrix of size n is denoted I_n. We suppose that the graph of A is partitioned into N nonoverlapping subdomains, where $N \ll n$. The coefficient matrix A is represented as $(a_{ij})_{1\le i,j\le n}$. Let $\mathcal{N} = \{1, \dots, n\}$ and let $\mathcal{N}_{i,0}$ for $i \in \{1, \dots, N\}$ be the subsets of $\mathcal{N}$ such that $\mathcal{N}_{i,0}$ stands for the subset of the DOF in the subdomain i. We refer to $\mathcal{N}_{i,0}$ as the *interior DOF* in the subdomain i. Let Δ_i for $i \in \{1, \dots, N\}$ be the subset of $\mathcal{N}$ that represents the neighbors DOF of the subdomain i, i.e., the DOFs of distance $= 1$ from the subdomain i through the graph of A. We refer to Δ_i as the *overlapping DOF* in the subdomain i. We denote $\mathcal{N}_i = [\mathcal{N}_{i,0},\, \Delta_i]\ \forall i \in \{1, \dots, N\}$, the concatenation of the interior and the overlapping DOF of the subdomain i. We denote $\mathcal{C}_i\ \forall i \in \{1, \dots, N\}$, the complementary of $\mathcal{N}_i$ in $\mathcal{N}$, i.e., $\mathcal{C}_i = \mathcal{N} \setminus \mathcal{N}_i$. We note $n_{i,0}$ the cardinality of the set $\mathcal{N}_{i,0}$, δ_i the cardinality of Δ_i, and n_i the cardinality of the set $\mathcal{N}_i\ \forall i \in \{1, \dots, N\}$. Let $R_{i,0} \in \mathbb{R}^{n_{i,0}\times n}$ be defined as $R_{i,0} = I_n(\mathcal{N}_{i,0}, :)$. Let $R_{i,\delta} \in \mathbb{R}^{\delta_i\times n}$ be defined as $R_{i,\delta} = I_n(\Delta_i, :)$. Let $R_i \in \mathbb{R}^{n_i\times n}$ be defined as $R_i = I_n(\mathcal{N}_i, :)$. Let $R_{i,c} \in \mathbb{R}^{(n-n_i)\times n}$ be defined as $R_{i,c} = I_n(\mathcal{C}_i, :)$. Let $\mathcal{P}_i = I_n([\mathcal{N}_{i,0}, \Delta_i, \mathcal{C}_i], :) \in \mathbb{R}^{n\times n}$ be a permutation matrix associated to the subdomain $i\ \forall i \in \{1, \dots, N\}$. We denote $D_i \in \mathbb{R}^{n_i,\times n_i}$, $i = 1, \dots, N$, any nonnegative diagonal matrix such that

$$I_n = \sum_{i=1}^{N} R_i^\top D_i R_i. \tag{1.4}$$

We refer to $(D_i)_{1\le i\le N}$ as the algebraic partition of unity. Let n_0 be a positive integer, $n_0 \ll n$. Let $V_0 \in \mathbb{R}^{n\times n_0}$ be a tall and skinny matrix of full rank. We denote $\mathcal{S}$ the subspace spanned by the columns of V_0. This subspace will stand for the coarse space. We denote R_0 the projection operator on $\mathcal{S}$. We denote $R_0^\top$ the interpolation operator from $\mathcal{S}$ to the global space. Let $\mathscr{R}_1$ be the operator defined by

$$\begin{aligned} \mathscr{R}_1 \,:\, \prod_{i=1}^{N} \mathbb{R}^{n_i} &\to \mathbb{R}^n, \\ (u_i)_{1\le i\le N} &\mapsto \sum_{i=1}^{N} R_i^\top u_i. \end{aligned} \tag{1.5}$$

In the same way we define $\mathscr{R}_2$ by taking into account the coarse space correction

$$\begin{aligned} \mathscr{R}_2 \,:\, \prod_{i=0}^{N} \mathbb{R}^{n_i} &\to \mathbb{R}^n, \\ (u_i)_{0\le i\le N} &\mapsto \sum_{i=0}^{N} R_i^\top u_i. \end{aligned} \tag{1.6}$$

We note that the subscripts 1 and 2 in $\mathscr{R}_1$ and $\mathscr{R}_2$ refer to one-level and two-level interpolation operators, respectively. The following example of two-subdomains-partitioned A illustrates our notation. Let A be given as

$$A = \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{pmatrix}.$$

Then, $\mathcal{N} = \{1,2,3,4\}$. The sets of interior DOF of subdomains are $\mathcal{N}_{1,0} = \{1,2\}$, $\mathcal{N}_{2,0} = \{3,4\}$. The sets of overlapping DOF of subdomains are $\Delta_1 = \{3\}$, $\Delta_2 = \{2\}$. The sets of concatenation of the interior DOF and the overlapping DOF of subdomains are $\mathcal{N}_1 = \{1,2,3\}$, $\mathcal{N}_2 = \{3,4,2\}$. The restriction operator on the interior DOF of subdomains is

$$R_{1,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad R_{2,0} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The restriction operator on the overlapping DOF of subdomains is

$$R_{1,\delta} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \qquad R_{2,\delta} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

The restriction operator on the concatenation of the interior DOF and the overlapping DOF is

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The permutation matrix associated with each subdomain is

$$\mathcal{P}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{P}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The permuted matrix associated with each subdomain is

$$\mathcal{P}_1 A \mathcal{P}_1^\top = \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{pmatrix}, \qquad \mathcal{P}_2 A \mathcal{P}_2^\top = \begin{pmatrix} a_{33} & a_{34} & a_{32} & \\ a_{43} & a_{44} & & \\ a_{23} & & a_{22} & a_{21} \\ & & a_{12} & a_{11} \end{pmatrix}.$$

Finally, the algebraic partition of unity can be defined as

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad D_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

We note that the reordering of lines in the partition of unity matrices $(D_i)_{1\le i\le N}$ has to be adapted with the lines reordering of $(R_i)_{1\le i\le N}$ such that (1.4) holds.

2. Background. In this section, we start by presenting three lemmas that help compare two symmetric positive definite (or semidefinite) matrices. Then, we review generalities of one- and two-level additive Schwarz preconditioners.

2.1. Auxiliary lemmas. Lemma 2.1 can be found in [3, Lemma 7.3, p. 164]. This lemma helps prove the effect of the additive Schwarz preconditioner on the largest eigenvalues of the preconditioned operator.

LEMMA 2.1. *Let A_1, $A_2 \in \mathbb{R}^{n\times n}$ be two symmetric positive definite matrices. Suppose that there is a constant $c_u > 0$ such that*

$$v^\top A_1 v \le c_u\, v^\top A_2 v \quad \forall v \in \mathbb{R}^n. \tag{2.1}$$

Then the eigenvalues of $A_2^{-1}A_1$ are strictly positive and bounded from above by c_u.

Lemma 2.2 is widely known in the community of domain decomposition by the *fictitious subspace lemma*. We announce it following an analog presentation as in [3, Lemma 7.4, p. 164].

LEMMA 2.2 (fictitious subspace lemma). *Let $A \in \mathbb{R}^{n_A\times n_A}$, $B \in \mathbb{R}^{n_B\times n_B}$ be two symmetric positive definite matrices. Let $\mathscr{R}$ be an operator defined as*

$$\begin{aligned} \mathscr{R} \,:\, \mathbb{R}^{n_B} &\to \mathbb{R}^{n_A}, \\ v &\mapsto \mathscr{R}v, \end{aligned} \tag{2.2}$$

and let $\mathscr{R}^\top$ be its transpose. Suppose that the following conditions hold:

1. *The operator $\mathscr{R}$ is surjective.*
2. *There exists $c_u > 0$ such that*
$$(\mathscr{R}v)^\top A\,(\mathscr{R}v) \le c_u\; v^\top B v \quad \forall v \in \mathbb{R}^{n_B}. \tag{2.3}$$
3. *There exists $c_l > 0$ such that $\forall v_{n_A} \in \mathbb{R}^{n_A}, \exists v_{n_B} \in \mathbb{R}^{n_B} | v_{n_A} = \mathscr{R}v_{n_B}$ and*
$$c_l\; v_{n_B}^\top B v_{n_B} \le (\mathscr{R}v_{n_B})^\top A\,(\mathscr{R}v_{n_B}) = v_{n_A}^\top A v_{n_A}. \tag{2.4}$$

Then, the spectrum of the operator $\mathscr{R}B^{-1}\mathscr{R}^\top A$ is contained in the segment $[c_l,\, c_u]$.

Proof. We refer the reader to [3, Lemma 7.4, p. 164] or [14, 13, 4] for a detailed proof. ∎

We note that there is a general version of Lemma 2.2 for infinite dimensions. This lemma plays a crucial role in bounding the condition number of our preconditioned operator. The operator $\mathscr{R}$ will stand for the interpolation operator. The matrix B will stand for the block diagonal operator of local subdomain problems. It is important to note that in the finite dimension the existence of the constants c_u and c_l are guaranteed. This is not the case in the infinite dimension spaces. In the finite dimension case, the hard part in the fictitious subspace lemma is to find $\mathscr{R}$ such that c_u/c_l is independent of the number of subdomains. When $\mathscr{R}$ and B are chosen to form the one- or two-level additive Schwarz operator, the first two conditions are satisfied for an upper bound c_u independent of the number of subdomains. An algebraic proof which depends only on the coefficient matrix can be found in [3]. However, the third condition is still an open question if no information from the underlying PDE is used. In this paper we address the problem of defining algebraically a surjective interpolation operator of the two-level additive Schwarz operator such that the third condition holds for a c_l independent of the number of subdomains. This is related to the *stable decomposition property*, which was introduced in [9]. Later, in [3], the authors proposed a stable decomposition with the additive Schwarz. This decomposition was based on the underlying PDE. Thus, when only the coefficient matrix A is known, this decomposition is not possible to be computed.

The two following lemmas will be applied to choose the local vectors that contribute to the coarse space. They are based on low rank corrections. In [3], the authors present two lemmas [3, Lemma 7.6, p. 167, and Lemma 7.7, p. 168] similar to the following lemmas. The rank correction proposed in their version is not of minimal rank. We modify these two lemmas to obtain the smallest rank correction.

LEMMA 2.3. *Let A, $B \in \mathbb{R}^{m\times m}$ be two symmetric positive semidefinite matrices. Let $ker(A)$, $range(A)$ denote the null space and the range of A, respectively. Let $ker(B)$ denote the kernel of B. Letting $L = ker(A) \cap ker(B)$, we note $L^{\perp_{ker(A)}}$ the orthogonal complementary of L in $ker(A)$. Let P_0 be an orthogonal projection on $range(A)$. Let τ be a strictly positive real number. Consider the generalized eigenvalue problem,*

$$\begin{aligned} &P_0 B P_0 u_k = \lambda_k A u_k, \\ &u_k \in range(A), \\ &\lambda_k \in \mathbb{R}. \end{aligned} \tag{2.5}$$

Let P_τ be an orthogonal projection on the subspace

$$Z = L^{\perp_{ker(A)}} \oplus span\{u_k \,|\, \lambda_k > \tau\},$$

and then the following inequality holds:

$$(u - P_\tau u)^\top B (u - P_\tau u) \le \tau u^\top A u \ \forall u \in \mathbb{R}^m. \tag{2.6}$$

Furthermore, Z is the subspace of smallest dimension such that (2.6) *holds.*

Proof. Let $m_A = dim(range(A))$. Let

$$\lambda_1 \le \cdots \le \lambda_{m_\tau} \le \tau < \lambda_{m_\tau+1} \le \cdots \le \lambda_{m_A}$$

be the eigenvalues of the generalized eigenvalue problem (2.5). Let

$$u_1, \ldots, u_{m_\tau}, u_{m_\tau+1}, \ldots, u_{m_A}$$

be the corresponding eigenvectors, A-orthonormalized. Let $k_B = \dim(\ker(B) \cap \ker(A))$, $k_A = \dim(\ker(A)) = m - m_A$. Let $v_1, \ldots, v_{k_B}$ be an orthogonal basis of L and let $v_{k_B+1}, \ldots, v_{k_A}$ be an orthogonal basis of $L^{\perp_{ker(A)}}$ such that $v_1, \ldots, v_{k_A}$ is an orthogonal basis of $ker(A)$. The symmetry of A and B permits one to have

$$\begin{aligned}
u_i^\top A u_j &= \delta_{ij}, \quad 1 \le i, j \le m_A, \\
u_i^\top B u_j &= \lambda_i \delta_{ij}, \quad 1 \le i, j \le m_A, \\
v_i^\top v_j &= \delta_{ij}, \quad 1 \le i, j \le k_A, \\
L &= \operatorname{span}\left\{v_1, \ldots, v_{k_B}\right\}, \\
L^{\perp_{ker(A)}} &= \operatorname{span}\left\{v_{k_B+1}, \ldots, v_{k_A}\right\},
\end{aligned}$$

where δ_{ij} stands for the Kronecker symbol. For a vector $u \in \mathbb{R}^m$ we can write

$$P_0 u = \sum_{k=1}^{m_A} (u_k^\top A P_0 u) u_k.$$

Then, we have

$$P_\tau u = u - P_0 u - \sum_{k=1}^{k_B} (v_k^\top u) v_k + \sum_{k=m_\tau+1}^{m_A} (u_k^\top A P_0 u) u_k.$$

Thus,

$$u - P_\tau u = \sum_{k=1}^{k_B} (v_k^\top u) v_k + \sum_{k=1}^{m_\tau} (u_k^\top A P_0 u) u_k.$$

Hence, the left side of (2.6) can be written as

$$\begin{aligned}
&(u - P_\tau u)^\top B (u - P_\tau u) \\
&= \left(\sum_{k=1}^{k_B} (v_k^\top u) v_k + \sum_{k=1}^{m_\tau} (u_k^\top A P_0 u) u_k \right)^\top B \left(\sum_{k=1}^{k_B} (v_k^\top u) v_k + \sum_{k=1}^{m_\tau} (u_k^\top A P_0 u) u_k \right), \\
&= \left(\sum_{k=1}^{k_B} (v_k^\top u) v_k + \sum_{k=1}^{m_\tau} (u_k^\top A P_0 u) u_k \right)^\top \left(\sum_{k=1}^{m_\tau} \lambda_k (u_k^\top A P_0 u) A u_k \right), \\
&= \left(\sum_{k=1}^{k_B} (v_k^\top u) A v_k + \sum_{k=1}^{m_\tau} (u_k^\top A P_0 u) A u_k \right)^\top \left(\sum_{k=1}^{m_\tau} \lambda_k (u_k^\top A P_0 u) u_k \right), \\
&= \left(\sum_{k=1}^{m_\tau} (u_k^\top A P_0 u) A u_k \right)^\top \left(\sum_{k=1}^{m_\tau} \lambda_k (u_k^\top A P_0 u) u_k \right), \\
&= \left(\sum_{k | \lambda_k \le \tau} (u_k^\top A P_0 u) u_k \right)^\top \left(\sum_{k | \lambda_k \le \tau} \lambda_k (u_k^\top A P_0 u) A u_k \right), \\
&= \left(\sum_{k | \lambda_k \le \tau} \sum_{j | \lambda_j \le \tau} (u_k^\top A P_0 u) u_k^\top \left(\lambda_j (u_j^\top A P_0 u) A u_j \right) \right), \\
&= \sum_{k | \lambda_k \le \tau} (u_k^\top A P_0 u)^2 \lambda_k.
\end{aligned}$$

We obtain (2.6) by remarking that

$$\begin{aligned}\sum_{k|\lambda_k\leq\tau}(u_k^\top AP_0u)^2\lambda_k &\leq \tau\sum_{k=1}^{m_A}(u_k^\top AP_0u)^2,\\ &= \tau\sum_{k=1}^{m_A}(u_k^\top AP_0u)(u_k^\top AP_0u),\\ &= \tau(P_0u)^\top AP_0u,\\ &= \tau u^\top Au.\end{aligned}$$

There remains the minimality of the dimension of Z. First, note that

$$u^\top Bu > \tau u^\top Au \quad \forall u \in Z.$$

To prove the minimality, suppose that there is a subspace Z_1 of dimension less than the dimension of Z. By this assumption, there is a nonzero vector $w \in (Z\cap Z_1)^{\perp_Z}$, where $(Z\cap Z_1)^{\perp_Z}$ is the orthogonal complementary of $(Z\cap Z_1)$ in Z, such that $w \perp Z_1$. By construction, we have

$$w^\top Bw > \tau w^\top Aw.$$

This contradicts (2.6) and the minimality is proved. ∎

LEMMA 2.4. *Let $A \in \mathbb{R}^{m\times m}$ be a symmetric positive matrix and $B \in \mathbb{R}^{m\times m}$ be an SPD matrix. Let* $\ker(A)$, *range*(A) *denote the null space and the range of A, respectively. Let P_0 be an orthogonal projection on range(A). Let τ be a strictly positive real number. Consider the following generalized eigenvalue problem:*

$$Au_k = \lambda_k Bu_k. \tag{2.7}$$

Let P_τ be an orthogonal projection on the subspace

$$Z = span\left\{u_k \mid \lambda_k < \frac{1}{\tau}\right\},$$

and then the following inequality holds:

$$(u - P_\tau u)^\top B\,(u - P_\tau u) \leq \tau u^\top Au \;\forall u \in \mathbb{R}^m. \tag{2.8}$$

Z is the subspace of smallest dimension such that (2.8) *holds.*

Proof. Let $u_1, \ldots, u_{m_0}$ be an orthogonal basis vectors of $\ker(A)$. Let

$$0 < \lambda_{m_0+1} \leq \cdots \leq \lambda_{m_\tau} < \frac{1}{\tau} \leq \lambda_{m_\tau+1} \leq \cdots \leq \lambda_m$$

be the eigenvalues strictly larger than 0 of the generalized eigenvalue problem (2.7). Let

$$u_{m_0+1}, \ldots, u_{m_\tau}, u_{m_\tau+1}, \ldots, u_m$$

be the corresponding eigenvectors A-orthonormalized. We can suppose that

$$\begin{aligned}u_i^\top Au_j &= \delta_{ij}, \quad m_0+1 \leq i,j \leq m,\\ u_i^\top Bu_j &= \frac{1}{\lambda_i}\delta_{ij}, \quad m_0+1 \leq i,j \leq m,\\ u_i^\top u_j &= \delta_{ij}, \quad 1 \leq i,j \leq m_0,\end{aligned}$$

where δ_{ij} stands for the Kronecker symbol. We can write

$$P_0 u = \sum_{k=m_0+1}^{m} (u_k^\top A P_0 u) u_k.$$

Then, we have

$$P_\tau u = u - P_0 u + \sum_{k=m_0+1}^{m_\tau} (u_k^\top A P_0 u) u_k.$$

Thus,

$$\begin{aligned} u - P_\tau u &= \sum_{k=m_\tau+1}^{m} (u_k^\top A P_0 u) u_k, \\ &= \sum_{k \mid \lambda_k \geq \frac{1}{\tau}} (u_k^\top A P_0 u) u_k. \end{aligned}$$

Hence, the left side of (2.8) can be written

$$\begin{aligned} (u - P_\tau u)^\top B (u - P_\tau u) &= \left(\sum_{k \mid \lambda_k \geq \frac{1}{\tau}} (u_k^\top A P_0 u) u_k \right)^\top B \left(\sum_{k \mid \lambda_k \geq \frac{1}{\tau}} (u_k^\top A P_0 u) u_k \right), \\ &= \left(\sum_{k \mid \lambda_k \geq \frac{1}{\tau}} (u_k^\top A P_0 u) u_k \right)^\top \left(\sum_{k \mid \lambda_k \geq \frac{1}{\tau}} \frac{1}{\lambda_k} (u_k^\top A P_0 u) A u_k \right), \\ &= \left(\sum_{k \mid \lambda_k \geq \frac{1}{\tau}} \sum_{j \mid \lambda_j \geq \frac{1}{\tau}} (u_k^\top A P_0 u) u_k^\top \left(\frac{1}{\lambda_j} (u_j^\top A P_0 u) A u_j \right) \right), \\ &= \sum_{k \mid \lambda_k \geq \frac{1}{\tau}} (u_k^\top A P_0 u)^2 \frac{1}{\lambda_k}. \end{aligned}$$

We obtain (2.8) by remarking that

$$\begin{aligned} \sum_{k \mid \lambda_k \geq \frac{1}{\tau}} (u_k^\top A P_0 u)^2 \frac{1}{\lambda_k} &\leq \tau \sum_{k=1}^{m} (u_k^\top A P_0 u)^2, \\ &= \tau \sum_{k=m_0+1}^{m} (u_k^\top A P_0 u)(u_k^\top A P_0 u), \\ &= \tau (P_0 u)^\top A P_0 u, \\ &= \tau u^\top A u. \end{aligned}$$

There remains the minimality of Z. First, note that

$$u^\top B u > \tau u^\top A u \quad \forall u \in Z.$$

To prove the minimality, suppose that there is a subspace Z_1 of dimension less than the dimension of Z. By this assumption, there is a nonzero vector $w \in (Z \cap Z_1)^{\perp_Z}$, where

$(Z \cap Z_1)^{\perp_Z}$ is the orthogonal complementary of $(Z \cap Z_1)$ in Z, such that $w \perp Z_1$. By construction, we have

$$w^\top B w > \tau w^\top A w.$$

This contradicts the relation (2.8). ◻

The previous lemmas are general and algebraic and not directly related to the preconditioning. In the following section we will review the one- and two-level additive Schwarz preconditioner.

2.2. One- and two-level additive Schwarz preconditioner. In this section we review the definition and general properties of one- and two-level additive Schwarz preconditioners, ASM, ASM_2, respectively. We review, without proving, several lemmas introduced in [2, 3]. These lemmas show how the elements of ASM_2 without any specific property of the coarse space $\mathcal{S}$ verify the conditions 1 and 2 of the fictitious subspace Lemma 2.2.

The two-level preconditioner ASM_2 with coarse space $\mathcal{S}$ is defined as

$$M_{ASM,2}^{-1} = \sum_{i=0}^{N} R_i^\top \left(R_i A R_i^\top\right)^{-1} R_i. \tag{2.9}$$

If $n_0 = 0$, i.e., the subspace $\mathcal{S}$ is trivial, the term

$$R_0^\top \left(R_0 A R_0^\top\right)^{-1} R_0 = 0$$

by convention. The following lemma gives the additive Schwarz method a matrix representation as in [3].

Lemma 2.5. *The additive Schwarz operator can be represented as*

$$M_{ASM,2}^{-1} = \mathscr{R}_2 \mathcal{B}^{-1} \mathscr{R}_2^\top, \tag{2.10}$$

where $\mathscr{R}_2^\top$ is the operator adjoint of $\mathscr{R}_2$ and $\mathcal{B}$ is a block diagonal operator defined as

$$\begin{aligned} \mathcal{B} \;:\; \prod_{i=0}^{N} \mathbb{R}^{n_i} &\to \prod_{i=0}^{N} \mathbb{R}^{n_i}, \\ (u_i)_{0\le i\le N} &\mapsto \left(\left(R_i A R_i^\top\right) u_i\right)_{0\le i\le N}, \end{aligned} \tag{2.11}$$

where $R_i A R_i^\top$ for $0 \le i \le N$ is the ith diagonal block.

Proof. The proof follows directly from the definition of $\mathcal{B}$ and $\mathscr{R}_2$. ◻

We note that the dimension of the matrix representation of $\mathcal{B}$ is larger than the dimension of A. More precisely, $\mathcal{B}$ has the following dimension:

$$n_{\mathcal{B}} = \sum_{i=0}^{N} n_i = n + n_0 + \sum_{i=1}^{N} \delta_i.$$

The one-level additive Schwarz preconditioner can be defined in the same manner. It corresponds to the case where the subspace $\mathcal{S}$ is trivial. The following Lemma 2.6, [3, Lemma 7.10, p. 173] states that the operator $\mathscr{R}_2$ is surjective without any specific assumption about the coarse space $\mathcal{S}$.

Lemma 2.6. *The operator $\mathscr{R}_2$ as defined in* (1.6) *is surjective.*

Proof. The proof follows from the definition of $\mathscr{R}_2$ (1.6) and the definition of the partition of unity (1.4). ∎

Lemma 2.6 shows that the interpolation operator $\mathscr{R}_2$ seen as a matrix verifies the condition 1 in Lemma 2.2. Lemma 2.7 guarantees that the matrix representation of the additive Schwarz verifies condition 2 in Lemma 2.2.

LEMMA 2.7. *Let k_c be the minimum number of distinct colors so that $\left(span\{R_i^\top\}\right)_{1\leq i\leq N}$ of the same color are mutually A-orthogonal. Then, we have*

$$(\mathscr{R}_2 u_{\mathcal{B}})^\top A\,(\mathscr{R}_2 u_{\mathcal{B}}) \leq (k_c+1)\ \sum_{i=0}^{N} u_i^\top \left(R_i A R_i^\top\right) u_i \quad \forall u_{\mathcal{B}} = (u_i)_{0\leq i\leq N} \in \prod_{i=0}^{N} \mathbb{R}^{n_i}. \tag{2.12}$$

Proof. We refer the reader to [2, Theorem 12, p. 93] for a detailed proof. ∎

We note that Lemma 2.7 is true for any coarse space $\mathcal{S}$, especially when this subspace is trivial. This makes the lemma applicable also for the one-level additive Schwarz preconditioner. (The constant on the right-hand side in Lemma 2.7 becomes k_c.) Lemma 2.8 is the first step to obtain a reasonable constant c_l that verifies the third condition in Lemma 2.2

LEMMA 2.8. *Let $u_A \in \mathbb{R}^{n_A}$ and $u_{\mathcal{B}} = (u_i)_{0\leq i\leq N} \in \prod_{i=0}^{N} \mathbb{R}^{n_i}$ such that $u_A = \mathscr{R}_2 u_{\mathcal{B}}$. The additive Schwarz operator without any other restriction on the coarse space $\mathcal{S}$ verifies the following inequality:*

$$\sum_{i=0}^{N} u_i^\top \left(R_i A R_i^\top\right) u_i \leq 2\ u_A^\top A u_A + (2k_c+1) \sum_{i=1}^{N} u_i^\top R_i A R_i^\top u_i, \tag{2.13}$$

where k_c is defined in Lemma 2.7.

Proof. We refer the reader to [3, Lemma 7.12, p. 175] to view the proof in detail. ∎

In order to apply the fictitious subspace Lemma 2.2, the term $\sum_{i=1}^{N} u_i^\top R_i A R_i^\top u_i$ in the right-hand side of (2.13) must be bounded by a factor of $u_A^\top A u_A$. For this aim, the next section presents an algebraic local decomposition of the matrix A. Combining this decomposition with Lemma 2.3 or Lemma 2.4 (depending on the definiteness) defines a class of local generalized eigenvalue problems. By solving them, we can define a coarse space $\mathcal{S}$. The additive Schwarz preconditioner combined with $\mathcal{S}$ satisfies the three conditions of the fictitious subspace Lemma 2.2. Hence, we can control the condition number of the preconditioned system.

3. Algebraic local SPSD splitting of an SPD matrix. In this section we present our main contribution. We introduce the *algebraic local SPSD splitting of an SPD matrix* related to a subdomain. Then, we characterize all the algebraic local SPSD splittings of A that are related to each subdomain. We give a nontrivial bound from below for the energy norm of a vector by a locally determined quantity.

We start by defining the algebraic local SPSD splitting of a matrix related to a subdomain.

DEFINITION 3.1 (algebraic local SPSD splitting of A related to a subdomain). *Following the previous notations, let $\tilde{A}_i$ be the matrix defined as*

$$\mathcal{P}_i \tilde{A}_i \mathcal{P}_i^\top = \begin{pmatrix} R_{i,0} A R_{i,0}^\top & R_{i,0} A R_{i,\delta}^\top & \\ R_{i,\delta} A R_{i,0}^\top & \tilde{A}_\delta^i & \\ & & 0 \end{pmatrix}, \tag{3.1}$$

where $\tilde{A}^i_\delta \in \mathbb{R}^{\delta_i \times \delta_i}$. *We say that* $\tilde{A}_i$ *is an algebraic local SPSD splitting of* A *related to the subdomain* i *if the following condition holds:*

$$0 \leq u^\top \tilde{A}_i u \leq u^\top A u \quad \forall u \in \mathbb{R}^n. \tag{3.2}$$

For $i \in \{1, \ldots, N\}$, the matrix $\mathcal{P}_i A \mathcal{P}_i^\top$ has the form of a block tridiagonal matrix. (The permutation matrix $\mathcal{P}_i$ is defined in the section on notation.) The first diagonal block corresponds to the interior DOF of the subdomain i, the second diagonal block corresponds to the overlapping DOF in the subdomain i, and the third block diagonal is associated to the rest of the DOF.

LEMMA 3.2. *Let* m_1, m_2, m_3 *be strictly positive integers and* $m = m_1 + m_2 + m_3$, *and let* $B \in \mathbb{R}^{m \times m}$ *be a* 3×3 *block tridiagonal SPD matrix*

$$B = \begin{pmatrix} B_{11} & B_{12} & \\ B_{21} & B_{22} & B_{23} \\ & B_{32} & B_{33} \end{pmatrix}, \tag{3.3}$$

where $B_{ii} \in \mathbb{R}^{m_i \times m_i}$ *for* $i \in \{1, 2, 3\}$. *Let* $\tilde{B}_1 \in \mathbb{R}^{m \times m}$ *be*

$$\tilde{B}_1 = \begin{pmatrix} B_{11} & B_{12} & \\ B_{21} & \tilde{B}_{22} & \\ & & 0 \end{pmatrix}, \tag{3.4}$$

where $\tilde{B}_{22} \in \mathbb{R}^{m_2 \times m_2}$ *is a symmetric matrix verifying the inequalities*

$$u^\top B_{21} B_{11}^{-1} B_{12} u \leq u^\top \tilde{B}_{22} u \leq u^\top \left(B_{22} - B_{23} B_{33}^{-1} B_{32} \right) u \quad \forall u \in \mathbb{R}^{m_2}, \tag{3.5}$$

and then the following inequality holds:

$$0 \leq u^\top \tilde{B}_1 u \leq u^\top B u \quad \forall u \in \mathbb{R}^m. \tag{3.6}$$

Proof. Consider the difference matrix $F = B - \tilde{B}_1$. Let $F_2 \in \mathbb{R}^{(m_2+m_3)\times(m_2+m_3)}$ be the lowest 2×2 subblock diagonal matrix of F, i.e.,

$$F_2 = \begin{pmatrix} B_{22} - \tilde{B}_{22} & B_{23} \\ B_{32} & B_{33} \end{pmatrix}.$$

F_2 admits the following decomposition:

$$F_2 = \begin{pmatrix} I & B_{23} B_{33}^{-1} \\ & I \end{pmatrix} \begin{pmatrix} B_{22} - \tilde{B}_{22} - B_{23} B_{33}^{-1} B_{32} & \\ & B_{33} \end{pmatrix} \begin{pmatrix} I & \\ B_{33}^{-1} B_{32} & I \end{pmatrix}. \tag{3.7}$$

Since $\tilde{B}_{22}$ satisfies, by assumption, the inequality (3.5), F_2 satisfies the following inequality:

$$0 \leq u^\top F_2 u \quad \forall u \in \mathbb{R}^{(m_2+m_3)}.$$

This proves the right inequality in (3.6).

Let $E \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)}$ be the upper 2×2 subblock diagonal of $\tilde{B}_1$. E admits the following decomposition:

$$E = \begin{pmatrix} I & \\ B_{21} B_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} B_{11} & \\ & \tilde{B}_{22} - B_{21} B_{11}^{-1} B_{12} \end{pmatrix} \begin{pmatrix} I & B_{11}^{-1} B_{12} \\ & I \end{pmatrix}. \tag{3.8}$$

The positivity of $\tilde{B}_1$ follows directly from (3.5). ∎

LEMMA 3.3. *Using the notation from Lemma* 3.2, *the following holds:*

- *The condition* (3.5) *in Lemma* 3.2 *is not trivial, i.e., the set of matrices* $\tilde{B}_1$ *that verify the condition* (3.5) *is not empty*
- *There exist matrices,* $\tilde{B}_{22}$, *that verify the condition* (3.5) *with strict inequalities*
- *The left inequality in condition* (3.5) *is optimal, i.e., if there exists a nonzero vector* $u_2 \in \mathbb{R}^{m_2}$ *that verifies*

$$u_2^\top B_{21} B_{11}^{-1} B_{12} u_2 > u_2^\top \tilde{B}_{22} u_2.$$

Then, there exists a nonzero vector $u \in \mathbb{R}^m$ *such that*

$$u^\top \tilde{B}_1 u < 0.$$

- *The right inequality in condition* (3.5) *is optimal, i.e., if there exists a nonzero vector* $u_2 \in \mathbb{R}^{m_2}$ *that verifies*

$$u_2^\top \tilde{B}_{22} u_2 > u_2^\top \left(B_{22} - B_{23} B_{33}^{-1} B_{32} \right) u_2.$$

Then, there exists a nonzero vector $u \in \mathbb{R}^m$ *such that*

$$u^\top \tilde{B}_1 u > u^\top B u.$$

Proof. First we prove the nontriviality of the set of matrices verifying (3.5). Indeed, let $S(B_{22})$ be the Schur complement of B_{22} in B, namely,

$$S(B_{22}) = B_{22} - B_{21} B_{11}^{-1} B_{12} - B_{23} B_{33}^{-1} B_{32}.$$

Set $\tilde{B}_{22} := \frac{1}{2} S(B_{22}) + B_{21} B_{11}^{-1} B_{12}$. Then we have

$$\tilde{B}_{22} - B_{21} B_{11}^{-1} B_{12} = \left(B_{22} - B_{23} B_{33}^{-1} B_{32} \right) - \tilde{B}_{22} = \frac{1}{2} S(B_{22}),$$

which is an SPD matrix. Hence, the strict inequalities in (3.5) follow.

Let $u_2 \in \mathbb{R}^{m_2}$ be a vector such that

$$u_2^\top B_{21} B_{11}^{-1} B_{12} u_2 > u_2^\top \tilde{B}_{22} u_2.$$

The block-LDLT factorization (3.8) shows that

$$u^\top \tilde{B}_1 u = u_2^\top \left(\tilde{B}_{22} - B_{21} B_{11}^{-1} B_{12} \right) u_2 < 0,$$

where u is defined as

$$u = \begin{pmatrix} I & B_{11}^{-1} B_{12} \\ & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ u_2 \end{pmatrix}.$$

In the same manner we verify the optimality mentioned in the last point. ∎

Remark 3.4. We note that the matrix $\left(\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & \tilde{B}_{22} \end{smallmatrix}\right)$ defines a seminorm in $\mathbb{R}^{m_1+m_2}$. Furthermore, if $\tilde{B}_{22}$ is set such that the left inequality in (3.5) is strict, then the seminorm becomes a norm.

Now, we can apply Lemma 3.2 on $\mathcal{P}_i A \mathcal{P}_i^\top$ for each subdomain i by considering its interior DOF, overlapping DOF, and the rest of the DOF.

Proposition 3.5. *For each subdomain $i \in \{1, \ldots, N\}$, let $\tilde{A}_i \in \mathbb{R}^{n\times n}$ be defined as*

$$\mathcal{P}_i \tilde{A}_i \mathcal{P}_i^\top = \begin{pmatrix} R_{i,0}AR_{i,0}^\top & R_{i,0}AR_{i,\delta}^\top & \\ R_{i,\delta}AR_{i,0}^\top & \tilde{A}_\delta^i & \\ & & 0 \end{pmatrix}, \tag{3.9}$$

where $\tilde{A}_\delta^i \in \mathbb{R}^{\delta_i\times\delta_i}$ satisfies the following conditions:

$\forall u \in \mathbb{R}^{\delta_i}$,

- $u^\top \left(R_{i,\delta}AR_{i,0}^\top\right)\left(R_{i,0}AR_{i,0}^\top\right)^{-1}\left(R_{i,0}AR_{i,\delta}^\top\right)u \le u^\top \tilde{A}_\delta^i u$,
- $u^\top \tilde{A}_\delta^i u \le u^\top\left(\left(R_{i,\delta}AR_{i,\delta}^\top\right) - \left(R_{i,\delta}AR_{i,c}^\top\right)\left(R_{i,c}AR_{i,c}^\top\right)^{-1}\left(R_{i,c}AR_{i,\delta}^\top\right)\right)u.$

Then, $\forall i \in \{1, \ldots, N\}$ the matrix $\tilde{A}_i$ is an algebraic local SPSD splitting of A related to the subdomain i. Moreover, the following inequality holds:

$$0 \le \sum_{i=1}^{N} u^\top \tilde{A}_i u \le k_m u^\top A u \quad \forall u \in \mathbb{R}^n, \tag{3.10}$$

where k_m is a number bounded by N.

Proof. Lemma 3.2 shows that $\tilde{A}_i$ is an algebraic local SPSD splitting of A related to the subdomain i. The inequality (3.10) holds with the constant N for all algebraic local SPSD splittings of A. Thus, depending on the SPSD splitting related to each subdomain there exists a number $k_m \le N$ such that the inequality holds. ◻

We note that the matrix $\tilde{A}_i$ is considered local since it has nonzero elements only in the overlapping subdomain i. More precisely,

$$\forall\ j,\ k\ \in \mathcal{N} \mid j \notin \mathcal{N}_i\ \vee\ k \notin \mathcal{N}_i,\ \tilde{A}_i(j,k) = 0.$$

Proposition 3.5 shows that the A-norm of a vector $v \in \mathbb{R}^n$ can be bounded from below by a sum of local seminorms (Remark 3.4).

4. Algebraic stable decomposition with $\mathscr{R}_2$. In the previous section we introduced the algebraic local SPSD splitting of A. In this section we present the *τ-filtering* subspace that is associated with each SPSD splitting. In each subdomain a τ-filtering subspace will contribute to the coarse space. We show how this leads to a class of stable decomposition with $\mathscr{R}_2$. We note that the previous results of section 2 hold for any coarse space $\mathcal{S}$. Those results are sufficient to determine the constant c_u in the second condition of the fictitious subspace lemma (Lemma 2.2). However, they do not allow one to control the constant c_l of the third condition of the same lemma.

As we will see, the GenEO coarse space [17, 3] corresponds to a special SPSD splitting of A. Therefore, we follow the presentation in [3] in the construction of the coarse space. We note that the proof of Theorem 4.4 is similar to the proof of [3, Theorem 7.17, p. 177]. We present it for the sake of completeness.

Definition 4.1. *Let $\tilde{A}_i$ be an algebraic local SPSD splitting of A related to the subdomain i for $i = 1, \ldots, N$. Let $\tau > 0$. Let $\tilde{Z}_i \subset \mathbb{R}^{n_i}$ be a subspace and let $\tilde{P}_i$ be an orthogonal projection on $\tilde{Z}_i$. We say that $\tilde{Z}_i$ is a τ-filtering subspace if*

$$u_i^\top \left(R_i A R_i^\top\right) u_i \le \tau\ \left(R_i u\right)^\top \left(R_i \tilde{A}_i R_i^\top\right)\left(R_i u\right) \quad \forall u \in \mathbb{R}^n,$$

where $u_i = \left(D_i\left(I_{n_i} - \tilde{P}_i\right)R_i u\right)$ and D_i is the partition of unity for $i = 1, \ldots, N$.

After the characterization of the local SPSD splitting of A related to each subdomain, we characterize the associated smallest τ-filtering subspace.

LEMMA 4.2. *Let $\tilde{A}_i$ be an algebraic local SPSD splitting of A related to the subdomain i for $i = 1, \ldots, N$. Let $\tau > 0$. For all subdomains $1 \le i \le N$, let*

$$\tilde{G}_i = D_i \left(R_i A R_i^\top \right) D_i,$$

where D_i is the partition of unity. Let $\tilde{P}_{0,i}$ be the projection on range$(R_i \tilde{A}_i R_i^\top)$ parallel to $\ker(R_i \tilde{A}_i R_i^\top)$. Let $K = \ker(R_i \tilde{A}_i R_i^\top)$, $L = \ker(\tilde{G}_i) \cap K$, and $L^{\perp_K}$ the orthogonal complementary of L in K.

- *If $\tilde{G}_i$ is indefinite, consider the following generalized eigenvalue problem:*

$$\begin{aligned} &\textit{Find } (u_{i,k},\ \lambda_{i,k}) \ \in \textit{range}(R_i \tilde{A}_i R_i^\top) \times \mathbb{R} \\ &\textit{such that } \tilde{P}_{0,i} \tilde{G}_i \tilde{P}_{0,i} u_{i,k} = \lambda_{i,k} R_i \tilde{A}_i R_i^\top u_{i,k}. \end{aligned}$$

Set

$$\tilde{Z}_{\tau,i} = L^{\perp_K} \oplus \textit{span}\left\{ u_{i,k} \mid \lambda_{i,k} > \tau \right\}. \tag{4.1}$$

- *If $\tilde{G}_i$ is definite, consider the following generalized eigenvalue problem:*

$$\begin{aligned} &\textit{Find } (u_{i,k},\ \lambda_{i,k}) \ \in \mathbb{R}^{n_i} \times \mathbb{R} \\ &\textit{such that } R_i \tilde{A}_i R_i^\top u_{i,k} = \lambda_{i,k} \tilde{G}_i u_{i,k}. \end{aligned}$$

Set

$$\tilde{Z}_{\tau,i} = \textit{span}\left\{ u_{i,k} \mid \lambda_{i,k} < \frac{1}{\tau} \right\}. \tag{4.2}$$

Then, $\tilde{Z}_{\tau,i}$ is the smallest dimension τ-filtering subspace and the following inequality holds:

$$u_i^\top \left(R_i A R_i^\top \right) u_i \le \tau \ \left(R_i u \right)^\top \left(R_i \tilde{A}_i R_i^\top \right) \left(R_i u \right),$$

where $u_i = \big(D_i \big(I_{n_i} - \tilde{P}_{\tau,i} \big) R_i u \big)$, and $\tilde{P}_{\tau,i}$ is the orhtogonal projection on $\tilde{Z}_{\tau,i}$.

Proof. The proof follows from direct application of Lemmas 2.3 and 2.4. ∎

We will refer to the smallest dimension τ-filtering subspace as $\tilde{Z}_{\tau,i}$ and to the projection on it as $\tilde{P}_{\tau,i}$. Note that for each algebraic local SPSD splitting of A related to a subdomain i, the τ-filtering subspace $\tilde{Z}_{\tau,i}$ defined in Definition 4.1 changes. Thus, the projection $\tilde{P}_{\tau,i}$ depends on the algebraic local SPSD splitting of A related to the subdomain i.

In the rest of the paper, the notation $\tilde{Z}_{\tau,i}$ and $\tilde{P}_{\tau,i}$ will be used according to the algebraic local SPSD splitting of A that we deal with and following Lemma 4.2.

Definition 4.1 leads us to bound the sum in (2.13) by a sum of scalar products associated to algebraic SPSD splittings of A. Therefore, a factor, which depends on the value of τ, of the scalar product associated to A will bound the inequality in (2.13).

DEFINITION 4.3 (coarse space based on algebraic local SPSD splitting of A (ALS)). *Let $\tilde{A}_i$ be an algebraic local SPSD splitting of A related to the subdomain i for $i = 1, \ldots, N$. Let $\tilde{Z}_{\tau,i}$ be the subspace associated to $\tilde{A}_i$ as defined in Lemma 4.2.*

We define $\mathcal{S}$, the coarse space based on the algebraic local splitting of A related to each subdomain, as the sum of expanded weighted τ-filtering subspaces associated to the algebraic local splitting of A related to each subdomain,

$$\mathcal{S} = \bigoplus_{i=1}^{N} R_i^\top D_i \tilde{Z}_{\tau,i}. \tag{4.3}$$

Let $\tilde{Z}_0$ be a matrix whose columns form a basis of $\mathcal{S}$. We denote its transpose by $R_0 = \tilde{Z}_0^\top$.

As mentioned previously, the key point to apply the fictitious subspace lemma (Lemma 2.2) is to find a coarse space that induces a relatively large c_l in the third condition of the lemma. The following theorem proves that ALS satisfies this.

THEOREM 4.4. *Let $\tilde{A}_i$ be an algebraic local SPSD splitting of A related to the subdomain i for $i = 1, \dots, N$. Let $\tilde{Z}_{\tau,i}$ be the τ-filtering subspace associated to $\tilde{A}_i$, and $\tilde{P}_{\tau,i}$ be the projection on $\tilde{Z}_{\tau,i}$ as defined in Lemma 4.2. Let $u \in \mathbb{R}^n$ and let $u_i = \big(D_i\big(I_{n_i} - \tilde{P}_{\tau,i}\big)R_i u\big)$ for $i = 1, \dots, N$. Let u_0 be defined as*

$$u_0 = \left(R_0 R_0^\top\right)^{-1} R_0 \left(\sum_{i=1}^{N} R_i^\top D_i \tilde{P}_{\tau,i} R_i u\right).$$

Let $c_l = (2 + (2k_c + 1)k_m\tau)^{-1}$. Then,

$$u = \sum_{i=0}^{N} R_i^\top u_i,$$

and

$$c_l \sum_{i=0}^{N} u_i^\top R_i A R_i^\top u_i \le u^\top A u.$$

Proof. Since $\forall y \in \mathcal{S},\ y = R_0^\top \left(R_0 R_0^\top\right)^{-1} R_0 y$, the relation

$$u = \sum_{i=0}^{N} R_i^\top u_i = \mathscr{R}_2 \left(u_i\right)_{0 \le i \le N}$$

follows directly. Lemma 2.8 shows that

$$\sum_{i=0}^{N} u_i^\top R_i A R_i^\top u_i \le 2u^\top A u + (2k_c + 1) \sum_{i=1}^{N} u_i^\top \left(R_i A R_i^\top\right) u_i.$$

By using Lemma 4.2 we can write

$$\sum_{i=0}^{N} u_i^\top R_i A R_i^\top u_i \le 2u^\top A u + (2k_c + 1)\tau \sum_{i=1}^{N} (R_i u)^\top \left(R_i \tilde{A}_i R_i^\top\right) (R_i u).$$

Since $\tilde{A}_i$ is local, we can write

$$\sum_{i=0}^{N} u_i^\top R_i A R_i^\top u_i \le 2u^\top A u + (2k_c + 1)\tau \sum_{i=1}^{N} u^\top \tilde{A}_i u.$$

Then, by applying Proposition 3.5, we can write

$$\sum_{i=0}^{N} u_i^\top R_i A R_i^\top u_i \leq 2u^\top Au + (2k_c+1)k_m\tau \; u^\top Au,$$

$$\sum_{i=0}^{N} u_i^\top R_i A R_i^\top u_i \leq (2 + (2k_c+1)k_m\tau)\, u^\top Au.$$

□

THEOREM 4.5. *Let M_{ALS} be the two-level ASM preconditioner combined with ALS. The following inequality holds:*

$$\kappa\left(M_{ALS}^{-1}A\right) \leq (k_c+1)\left(2 + (2k_c+1)k_m\tau\right).$$

Proof. Lemmas 2.6 and 2.7 and Theorem 4.4 show that the two-level preconditioner associated with ALS verifies the conditions of the fictitious subspace lemma (Lemma 2.2). Hence, the eigenvalues of $M_{ALS}^{-1}A$ verify the following inequality,

$$\frac{1}{2 + (2k_c+1)k_m\tau} \leq \lambda\left(M_{ALS}^{-1}A\right) \leq (k_c+1),$$

and the result follows. □

Remark 4.6. Since any τ-filtering subspace $\tilde{Z}_i$ can replace $\tilde{Z}_{\tau,i}$ in Theorem 4.4, the Theorem 4.5 applies for coarse spaces of the form $\mathcal{S} = \bigoplus_{i=1}^{N} R_i^\top D_i \tilde{Z}_i$. The difference is that the dimension of the coarse space is minimal by choosing $\tilde{Z}_{\tau,i}$; see Lemma 4.2.

We note that the previous theorem (Theorem 4.5) shows that the spectral condition number of the preconditioned system does not depend on the number of subdomains. It depends only on k_c, k_m, and τ. k_c is bounded by the maximum number of neighbors of a subdomain. k_m is a number bounded by the number of subdomains. It depends on the algebraic local SPSD splitting of each subdomain. Partitioned graphs of sparse matrices have structures such that k_c is small. The parameter τ can be chosen small enough such that ALS has a relatively small dimension.

4.1. GenEO coarse space. In [3], the authors present the theory of one- and two-level additive Schwarz preconditioners. To bound the largest eigenvalue of the preconditioned system they use the algebraic properties of the additive Schwarz preconditioner. However, to bound the smallest eigenvalue, they benefit from the discretization of the underlying PDE. In the environment of the finite element method, they construct local matrices corresponding to the integral of the operator in the overlapping subdomain. For each subdomain, the expanded matrix has the form

$$\mathcal{P}_i \tilde{A}_i \mathcal{P}_i^\top = \begin{pmatrix} R_{i,0}AR_{i,0}^\top & R_{i,0}AR_{i,\delta}^\top & \\ R_{i,\delta}AR_{i,0}^\top & \tilde{A}_\delta^i & \\ & & 0 \end{pmatrix},$$

where $\tilde{A}_\delta^i$ corresponds to the integral of the operator in the overlapping region with neighbors of the subdomains i. This matrix is SPSD since the global operator is SPD. Since the integral over the subdomain is always smaller than the integral over the global domain (positive integrals), the following inequality holds:

$$0 \leq u^\top \tilde{A}_i u \leq u^\top Au \;\forall u \in \mathbb{R}^n.$$

Hence, Lemma 3.3 confirms that the matrix $\tilde{A}_i$ corresponds to an algebraic local SPSD splitting of A related to the subdomain i. Thus, GenEO is a member of the class of preconditioners that are based on the algebraic local SPSD splitting of A. We note that the parameter k_m, defined in (3.10), with the algebraic local SPSD splitting of A corresponding to GenEO can be shown to be equal to the maximum number of subdomains sharing a DOF.

4.2. Extremum efficient coarse space. In this section we discuss the two obvious choices to have algebraic local SPSD splitting of A. We show how in practice these two choices are costly. However, they have two advantages. The first is that one of these choices gives an answer to the following question that appears in domain decomposition. How many local vectors must be added to the coarse space in order to bound the spectral condition number by a number defined a priori? We are able to answer this question in the case where the additive Schwarz preconditioner is to be used. We note that the answer is given without any analytic information. Only the coefficients of the matrix A have to be known. The second advantage is that both choices give an idea of constructing a noncostly algebraic approximation of an ALS.

In the following discussion we disregard the impact of the parameter k_m. Numerical experiments in section 5 demonstrate that the impact of this parameter can be negligible. We note that this parameter depends only on the algebraic local SPSD splitting and it is bounded by N.

Suppose that we have two SPSD splittings of A related to a subdomain i, $\tilde{A}_i^{(1)}, \tilde{A}_i^{(2)}$, such that

$$u^\top \tilde{A}_i^{(1)} u \leq u^\top \tilde{A}_i^{(2)} u \quad \forall u \in \mathbb{R}^n.$$

We want to compare the number of vectors that contribute to the coarse space for each SPSD splitting. It is clear that a τ-filtering subspace associated to $\tilde{A}_i^{(1)}$ is a τ-filtering subspace associated to $\tilde{A}_i^{(2)}$. Thus, the following inequality holds:

$$dim(\tilde{Z}_{\tau,i}^{(1)}) \geq dim(\tilde{Z}_{\tau,i}^{(2)}),$$

where $\tilde{Z}_{\tau,i}^{(1)}, \tilde{Z}_{\tau,i}^{(2)}$ are the smallest τ-filtering subspaces associated to $\tilde{A}_i^{(1)}, \tilde{A}_i^{(2)}$, respectively. Therefore, Lemma 3.3 shows that the closer we are to the upper bound in (3.5) the fewer vectors that will contribute to ALS. Moreover, the closer we are to the lower bound in (3.5) the more vectors that will contribute to ALS. Indeed, the set of algebraic local SPSD splitting of A related to a subdomain i admits a relation of partial ordering.

$$M_1 \leq M_2 \iff u^\top M_1 u \leq u^\top M_2 u \quad \forall u.$$

This set admits obviously a smallest and a largest element defined by the left and the right bounds in (3.5), respectively.

Hence, the best ALS corresponds to the following algebraic local SPSD splitting of A for $i = 1, \ldots, N$:

(4.4)
$$\mathcal{P}_i \tilde{A}_i \mathcal{P}_i^\top = \begin{pmatrix} R_{i,0} A R_{i,0}^\top & R_{i,0} A R_{i,\delta}^\top & \\ R_{i,\delta} A R_{i,0}^\top & R_{i,\delta} A R_{i,\delta}^\top - \left(R_{i,\delta} A R_{i,c}^\top\right)\left(R_{i,c} A R_{i,c}^\top\right)^{-1}\left(R_{i,c} A R_{i,\delta}^\top\right) & \\ & & 0 \end{pmatrix}.$$

The dimension of the subspace $\tilde{Z}_{\tau,i}$ associated to $\tilde{A}_i$ (4.4) is minimal over all possible algebraic local SPSD splittings of A related to the subdomain i. We remark that this splitting is not a choice in practice since it includes inverting the matrix $\left(R_{i,c}AR_{i,c}^\top\right)$, which is of large size (approximately corresponding to $N-1$ subdomains). We will refer to (4.4) as *the upper bound SPSD splitting*, and the associated coarse space will be referred to as *the upper ALS*. In the same manner, we can find the worst ALS. The corresponding algebraic local SPSD splitting of A related to the subdomain i is the following:

$$\mathcal{P}_i\tilde{A}_i\mathcal{P}_i^\top = \begin{pmatrix} R_{i,0}AR_{i,0}^\top & R_{i,0}AR_{i,\delta}^\top & \\ R_{i,\delta}AR_{i,0}^\top & \left(R_{i,\delta}AR_{i,0}^\top\right)\left(R_{i,0}AR_{i,0}^\top\right)^{-1}\left(R_{i,0}AR_{i,\delta}^\top\right) & \\ & & 0 \end{pmatrix}. \tag{4.5}$$

On the contrary of the best splitting (4.4), this splitting is not costly. It includes inverting the matrix $\left(R_{i,0}AR_{i,0}^\top\right)$, which is considered small. However, the dimension of $\tilde{Z}_{\tau,i}$ associated to $\tilde{A}_i$ (4.5) is maximal. It is of dimension δ_i at least. Indeed, a block-LDLT factorization of $R_i\tilde{A}_iR_i^\top$ shows that its null space is of dimension δ_i. We will refer to (4.5) as *the lower bound SPSD splitting* the associated coarse space will be referred to as *the lower ALS*.

Remark 4.7. A convex linear combination of the lower bound and the upper bound of the SPSD splitting is also an algebraic local SPSD splitting.

$$\alpha \times \text{the upper bound SPSD splitting} + (1-\alpha) \times \text{the lower bound SPSD splitting}.$$

We refer to it as α-convex SPSD splitting. We refer to the corresponding ALS as the α-convex ALS.

In the following section we propose a strategy to compute an approximation of reasonable ALS that is not costly.

4.3. Approximate ALS. As mentioned in subsection 4.2, the extremum cases of ALS are not practical choices. We recall that the restriction matrix $R_{i,c}$ is associated to the DOFs outside the overlapping subdomain i. The bottleneck in computing the upper bound SPSD splitting is the computatation of the term

$$\left(R_{i,\delta}AR_{i,c}^\top\right)\left(R_{i,c}AR_{i,c}^\top\right)^{-1}\left(R_{i,c}AR_{i,\delta}^\top\right)$$

since it induces inverting the matrix $\left(R_{i,c}AR_{i,c}^\top\right)$. To approximate the last term, we look for a restriction matrix $R_{i,\tilde{c}}$ such that

$$\left(R_{i,\delta}AR_{i,c}^\top\right)\left(R_{i,c}AR_{i,c}^\top\right)^{-1}\left(R_{i,c}AR_{i,\delta}^\top\right) \approx \left(R_{i,\delta}AR_{i,\tilde{c}}^\top\right)\left(R_{i,\tilde{c}}AR_{i,\tilde{c}}^\top\right)^{-1}\left(R_{i,\tilde{c}}AR_{i,\delta}^\top\right),$$
$$\left(R_{i,\delta}AR_{i,\tilde{c}}^\top\right)\left(R_{i,\tilde{c}}AR_{i,\tilde{c}}^\top\right)^{-1}\left(R_{i,\tilde{c}}AR_{i,\delta}^\top\right) \text{ is easy to compute.}$$

One choice is to associate $R_{i,\tilde{c}}$ to the DOFs outside the overlapping subdomain i that have the nearest distance from the boundary of the subdomain i through the graph of A. In practice, we fix an integer $d \geq 1$ such that the matrix $R_{i,\tilde{c}}AR_{i,\tilde{c}}^\top$ has a dimension $\dim_i \leq d \times n_i$. Then we can take a convex linear combination of the lower bound SPSD splitting and the approximation of the upper bound SPSD splitting. For instance, the error bound on this approximation is still an open question. Numerical experiments show that d does not need to be large.

TABLE 5.1
Matrices used for tests. n is the size of the matrix, NnZ is the number of nonzero elements. HPD stands for Hermitian positive definite. κ is the condition number related to the second norm.

Matrix name	Type	n	NnZ	κ
SKY3D	Skyscraper	8000	53000	10^5
SKY2D	Skyscraper	10000	49600	10^6
EL3D	Elasticity	15795	510181	3×10^{11}

5. Numerical experiments. In this section we present numerical experiments for ALS. We denote ASM_{ALS} the two-level additive Schwarz combined with ALS. If it is not specified, the number of vectors deflated by subdomain is fixed to 15. We use the preconditioned CG implemented in MATLAB 2017R to compare the preconditioners. The threshold of convergence is fixed to 10^{-6}. Our test matrices arise from the discretization of two types of challenging problems: linear elasticity and diffusion problems [5, 1, 15]. Our set of matrices are given in Table 5.1. The matrices SKY2D and SKY3D arise from the boundary value problem of the diffusion equation on Ω, the two-dimensional (2-D) unit square and the 3-D unit cube, respectively:

$$\begin{aligned} -\mathrm{div}(\kappa(x)\nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_N. \end{aligned} \tag{5.1}$$

They correspond to skyscraper problems. The domain Ω contains several zones of high permeability. These zones are separated from each other. The tensor κ is given by the following relation:

$$\begin{aligned} &\kappa(x) = 10^3([10x_2] + 1) && \text{if } [10x_i] \text{ is odd, } i = 1, 2, \\ &\kappa(x) = 1 && \text{otherwise.} \end{aligned}$$

$\Gamma_D = [0, 1] \times \{0, 1\}$ in the 2-D case. $\Gamma_D = [0, 1] \times \{0, 1\} \times [0, 1]$ in the 3-D case. Γ_N is chosen as $\Gamma_N = \partial\Omega \setminus \Gamma_D$ and n denotes the exterior normal vector to the boundary of Ω. The linear elasticity problem with Dirichlet and Neumann boundary conditions is defined as follows:

$$\begin{aligned} \mathrm{div}(\sigma(u)) + f &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \sigma(u) \cdot n &= 0 && \text{on } \Gamma_N. \end{aligned} \tag{5.2}$$

Ω is a unit cube 3-D. The matrix El3D corresponds to this equation discretized using a triangular mesh with $65 \times 9 \times 9$ vertices. Γ_D is the Dirichlet boundary, Γ_N is the Neumann boundary, f is a force, and u is the unknown displacement field. The Cauchy stress tensor $\sigma(.)$ is given by Hooke's law: it can be expressed in terms of Young's modulus E and Poisson's ration ν. n denotes the exterior normal vector to the boundary of Ω. We consider discontinuous E and ν: $(E_1, \nu_1) = (2 \times 10^{11}, 0.45)$, $(E_2, \nu_2) = (10^7, 0.25)$. Data elements of this problem are obtained by the application FreeFem++ [7]. Table 5.2 presents a comparison between one-level ASM and ASM_2 with the upper bound ALS. As is known, the iteration number of CG preconditioned by ASM increases by increasing the number of subdomains. However, we remark that

TABLE 5.2

Comparison between ASM_2 with the upper ALS and one-level additive Schwarz. n is the dimension of the problem, N is the number of subdomains, n_{uC} is the iteration number of CG preconditioned by ASM_2, and n_{ASM} is the iteration number of CG preconditioned by one-level ASM. The sign $-$ means that the method did not converge in fewer than 100 iteration.

Matrix	n	N	n_{uC}	n_{ASM}
SKY3D	8000	4	23	29
		8	25	35
		16	25	37
		32	22	55
		64	24	79
		128	24	-
SKY2D	10000	4	18	54
		8	19	-
		16	20	-
		32	22	-
		64	26	-
		128	31	-
EL3D	15795	4	38	-
		8	43	-
		16	51	-
		32	51	-
		64	67	-
		128	92	-

TABLE 5.3

Comparison between ALS variants, the upper bound ALS, the α_1-convex ALS, and the α_2-convex CosBALSS. n is the dimension of the problem, N is the number of subdomains, the subscript uC refers to the upper bound ALS, $n_.$ is the iteration number of ASM_2, and α refers to the coefficient in the convex linear combination, $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$.

Matrix	n	N	n_{uC}	n_{α_1}	n_{α_2}
SKY3D	8000	4	23	22	22
		8	25	25	23
		16	25	24	24
		32	22	22	22
		64	24	23	21
		128	24	24	22
SKY2D	10000	4	18	18	17
		8	19	19	19
		16	20	19	19
		32	22	21	18
		64	26	24	20
		128	31	28	20
EL3D	15795	4	38	38	38
		8	43	43	43
		16	51	51	51
		32	51	51	51
		64	67	67	67
		128	92	92	92

the iteration number of the CG preconditioned by ALS is robust when the number of subdomain increases.

In Table 5.3 we compare three ALS, the upper bound, α_1-convex, and α_2-convex, where $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$. Table 5.3 shows the efficiency of three ALS related to different SPSD splittings. The iteration count corresponding to each coarse space increases slightly by increasing the number of subdomains. The main reason behind

TABLE 5.4

Estimation of the spectral condition number of matrix El3D preconditioned by ASM_2 with ALS variants and GenEo coarse space. Results correspond to Table 5.5, N is the number of subdomains, the subscript uC refers to the upper bound ALS, α refers to the coefficient in the convex ALS, $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$, and the subscript Gen stands for the GenEO coarse space.

N	κ_{uC}	κ_{α_1}	κ_{α_2}	κ_{Gen}
4	5	4	4	5
8	8	5	5	7
16	15	10	9	15
32	34	25	15	18
64	100	67	30	31
128	231	178	86	39

TABLE 5.5

Matrix El3D, ALS variants, and GenEo coarse space with the minimum number of deflated vectors disregarding the parameter k_m. N is the number of subdomains, the subscript uC refers to the upper bound ALS. $dim_.$ is the dimension of ALS, $n_.$ is the iteration number of ASM_2, α refers to the coefficient in the convex ALS, $\alpha_1 = 0.75$ and $\alpha_2 = 0.25$, and the subscript Gen stands for the GenEO coarse space. See Table 5.4.

N	dim_{uC}	n_{uC}	dim_{α_1}	n_{α_1}	dim_{α_2}	n_{α_2}	dim_{Gen}	n_{Gen}
4	82	20	92	19	120	18	106	20
8	179	23	209	20	240	20	229	24
16	304	37	394	30	480	28	391	38
32	447	53	583	45	960	36	614	42
64	622	84	769	73	1920	51	850	55
128	969	131	1096	112	3834	77	1326	61

this increasing is that the predifined parameter τ provides an overestimation of the upper bound on the spectral condition number; see Table 5.4.

To illustrate the impact of the parameter k_m, when increasing the number of subdomains, on bounding the spectral condition number, we do the following. We choose τ as

$$\tau = \frac{1}{2}\left(\frac{\tilde{\kappa}}{k_c + 1} - 2\right)(2k_c + 1)^{-1},$$

i.e., we suppose that k_m has no impact on τ. The resulting spectral condition number will be affected only by the parameter k_m; see Table 5.4. Tables 5.4 and 5.5 present results for ALS variants for $\tilde{\kappa} = 100$. We perform this test on the elasticity problem (5.2), where we could also compare against the GenEO coarse space [17, 3]. We note that when GenEO is applied on the elasticity problem (5.2), the domain decomposition performed by freefem++ [7], for all tested values of N, is such that any DOF belongs to at most two subdomains and hence $k_m(GenEO) = 2$. This means that the hyposthesis that k_m has no impact on the selected τ is true for the coarse space GenEO. Nevertheless, this might be false for the other coarse spaces. Therefore, the impact of k_m will be remarked on only for the ALS coarse spaces. Table 5.5 shows the dimension of ALS for each variant as well as the iteration number for preconditioned CG to reach the convergence tolerance. On the other hand, Table 5.4 shows an estimation of the spectral condition number of the preconditioned system. This estimation is performed by computing an approximation of the largest and the smallest eigenvalues of the preconditioned operator by using the Krylov–Schur method [19] in MATLAB. The same tolerance τ is applied for GenEO. In order to avoid a large-dimension coarse space, 30 vectors at max are deflated per subdomain.

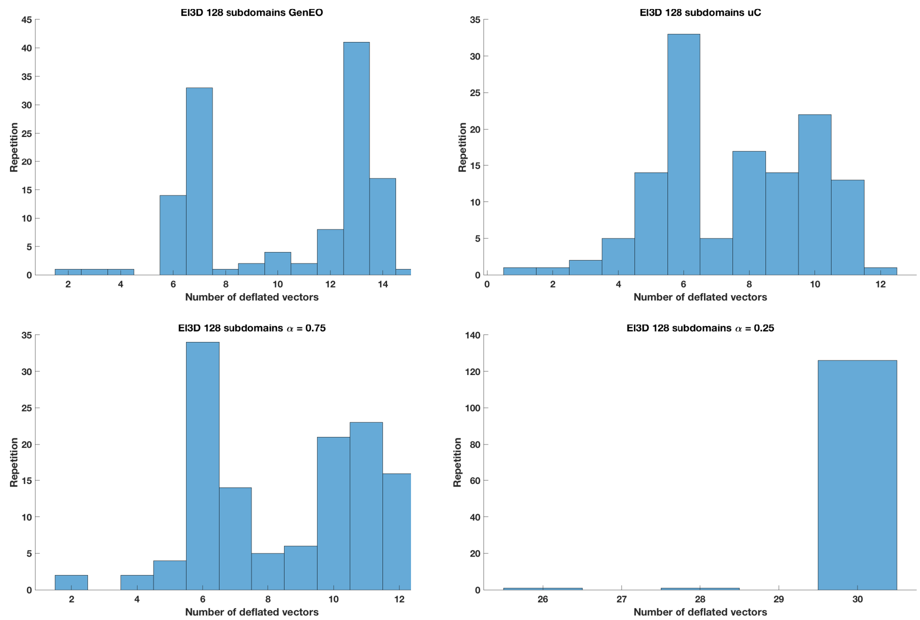

FIG. 5.1. *Histogram of the number of deflated vectors by each subdomain for different ALS, GenEO; uC, the upper bound ALS;* α_1*-convex ALS,* $\alpha_1 = 0.75$*;* α_2*-convex ALS,* $\alpha_2 = 0.25$.

We note that results in Table 5.5 satisfy the discussion in subsection 4.2. Indeed, the upper bound ALS has the minimum dimension; 0.75- and 0.25-convex ALS follow the upper bound ALS, respectively.

Table 5.4 demonstrates the impact of k_m on the bound of the spectral condition number. We notice that its effect increases when α is closer to 1 (the larger α is, the larger k_m becomes). We recall that in the algebraic SPSD splitting $k_m \leq N$. However, when GenEO is applied to the elasticity problem test case (5.2), k_m is independant of N and is equal to 2 as explained previously. The values of the estimated spectral condition number, especially for a small number of subdomains ($N = 4$), show how τ provides an overestimation of the theoretical upper bound on the spectral condition number (estimated(κ) $\ll 100$). For this reason, we consider that this slight augmentaion of the iteration count does not mean that the method is not robust.

In Figure 5.1 we present a histogram of the number of deflated vectors by each subdomain. We remark that the number of vectors that each subdomain contributes to the coarse space is not necessarily equal. In the case of α_2-convex ALS, most subdomains reach the maximum number of deflated vectors, 30, that we fixed. Moreover, Figure 5.2 compares the number of deflated vectors in each subdomain for the GenEO subspace and the upper bound ALS. This figure illustrates the relation of partial ordering between the SPSD splitting as discussed in subsection 4.2.

In Table 5.6 we show the impact of the approximation strategy that we proposed in subsection 4.3. The distance parameter related to the approximation (see subsection 4.3) is fixed for each matrix. It is obtained by tuning. The convex linear combination is chosen as $\alpha = 0.01$. Each subdomain contributes 20 vectors to the coarse space. We remark that the approximation strategy gives interesting results with the conviction-diffusion problem matrices SKY2D and SKY3D. With a small factor of the local dimension $d = 2$ and $d = 3$, respectively, the approximate ALS

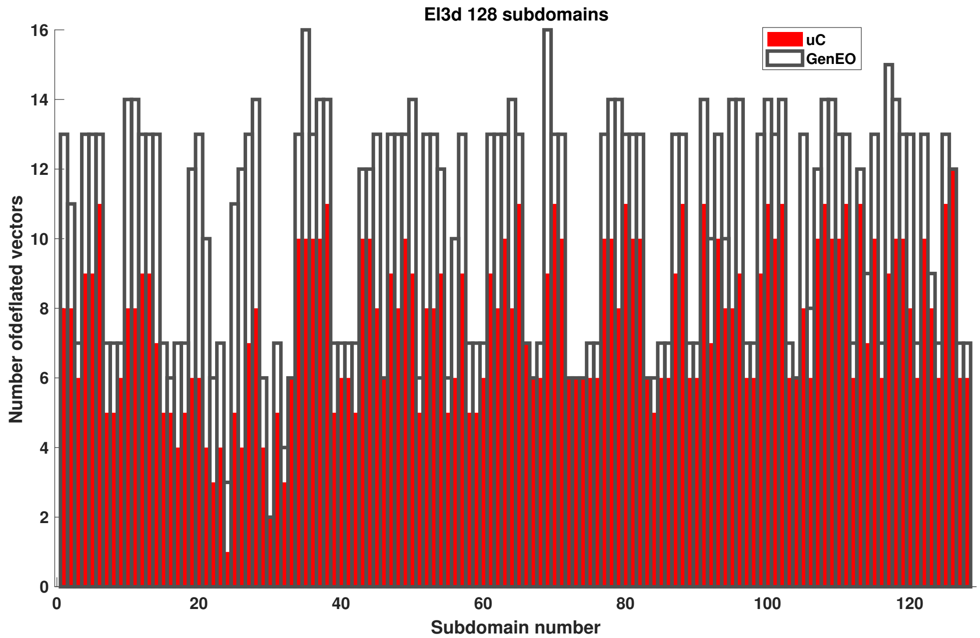

FIG. 5.2. *Comparison between the number of deflated vectors per subdomain GenEO coarse space and the upper bound ALS.*

TABLE 5.6

Comparison between the upper bound ALS and the approximation strategy presented in section 4.3. *n is the dimension of the problem, N is the number of subdomains, n_{uC} is the iteration number of CG preconditioned by ASM_2 with the upper bound ALS, d stands for the factor of local dimension to approximate the upper bound SPSD splitting, as explained in section* 4.3, *and n_{ap} is the iteration number of CG preconditioned by ASM_2 with approximation of ALS; the convex linear combination is chosen as* $(0.01 \times$ *approximation of the upper bound* $+ 0.99 \times$ *lower bound). The sign* $-$ *means that the method did not converge in fewer than* 150 *iterations.*

Matrix	n	N	n_{uC}	d	n_{ap}
		4	22		22
		8	23		23
SKY3D	8000	16	24	2	22
		32	22		22
		64	24		22
		128	22		44
		4	17		17
		8	18		18
SKY2D	10000	16	20	3	19
		32	22		22
		64	26		59
		128	31		90
		4	27		54
		8	36		56
EL3D	15795	16	37	5	77
		32	43		136
		64	61		-
		128	83		-

is able to perform relatively as efficiently as the upper bound ALS. For the elasticity problem with a larger factor $d = 5$, the approximate ALS reduces the iteration number; however, we remark that the latter increases by increasing the number of subdomains.

6. Conclusion. In this paper we reviewed generalities of one- and two-level additive Schwarz preconditioners. We introduced the algebraic local SPSD splitting of an SPD matrix A. We characterized all possible algebraic local SPSD splitting. To study the minimality of the dimension of the coarse space, we introduced the τ-filtering subspaces. Based on the algebraic local SPSD splitting and inspired by the GenEO method [17, 3], we introduced a class of algebraic coarse spaces that are constructed locally (ALS). The characterization of algebraic local SPSD splitting of A and the associated τ-filtering subspaces makes an algebraic framework for studying the coarse spaces related to the additive Schwarz method. We proved that the coarse space of GenEO corresponds to a special case of the SPSD splitting. We discussed different types of ALS and suggested a simple method to approximate a valuable coarse space. For matrices issued from the conviction-diffusion problem, the simple method that we proposed gave very interesting results. The algebraic formulation presented in this paper is particularly important when the theory of GenEO cannot be applied. We also note that in our ongoing work, we develop a theoretical and practical framework that will give rise to a three-level additive Schwarz preconditioner combining GenEO and ALS.

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