Guaranteed and robust a posteriori error estimation based on flux reconstruction for discontinuous Galerkin methods

Alexandre Ern[1], Annette F. Stephansen[2] & Martin Vohralík[3]

[1]CERMICS
Université Paris-Est (Ecole des Ponts)

[2]Centre for Integrated Petroleum Research
Bergen, Norway

[3]Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris 6)

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Outline

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What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE.
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|u - u_h\|_\Omega \le f(u)h^q$. Dependent on u, not computable. Useful in theory.
- A posteriori error estimate: $\|u - u_h\|_\Omega \lesssim f(u_h)$. Only uses u_h, computable. Great in practice.

Usual form

- $f(u_h)^2 = \sum_{T\in\mathcal{T}_h} \eta_T(u_h)^2$, where $\eta_T(u_h)$ is an element indicator.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

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What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|u - u_h\|_\Omega^2 \leq \sum_{T\in\mathcal{T}_h} \eta_T(u_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|u - u_h\|_\Omega^2 \leq C \sum_{T\in\mathcal{T}_h} \eta_T(u_h)^2$

Local efficiency (local error lower bound)

- $\eta_T(u_h)^2 \leq C_{\mathrm{eff},T}^2 \sum_{T' \text{ close to } T} \|u - u_h\|_{T'}^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{T\in\mathcal{T}_h} \eta_T(u_h)^2 / \|u - u_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},T}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

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Previous results on a posteriori error estimation in DG

DG, pure diffusion case

- Karakashian and Pascal (2003), Becker, Hansbo, and Larson (2003), residual-based estimates
- Rivière and Wheeler (2003), L^2-estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (preprint, 2008), Lazarov, Repin, and Tomar (preprint, numerical experiments, 2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

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Problems with discontinuous coefficients

- Bernardi and Verfürth (2000), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Convection–diffusion problems

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Residual estimates for $-\triangle u = f$

Corollary (Classical residual error estimate in FEs)

There holds (cf. Verfürth 96)

$$\|\nabla(u-u_h)\| \leq C_1 \left\{ \sum_{T\in\mathcal{T}_h} h_T^2 \|f+\triangle u_h\|_T^2 \right\}^{1/2} + C_2 \left\{ \sum_{F\in\mathcal{F}_h} h_F \|[\![\nabla u_h\cdot\mathbf{n}]\!]\|_F^2 \right\}^{1/2}.$$

Drawbacks

- What are C_1 and C_2?
- If C_1 and C_2 evaluated: overestimation by a factor of 30 (uniform refinement) and 60 (adaptive refinement).
- $\triangle u_h = 0$: $h_T\|f\|_T$ as estimator gives no good sense.
- Not robust for inhomogeneities.

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FEs residual constants C_1 and C_2

Constants C_1 and C_2, Carstensen & Funken '00

$$C_V := \begin{cases} C_{\mathrm{P},\mathcal{T}_V}^{\frac{1}{2}} h_{\mathcal{T}_V} & V \in \mathcal{V}_h^{\mathrm{int}}, \\ C_{\mathrm{F},\mathcal{T}_V,\partial\Omega}^{\frac{1}{2}} h_{\mathcal{T}_V} & V \in \mathcal{V}_h^{\mathrm{ext}}, \end{cases}$$

$$C_1 := \max_{T\in\mathcal{T}_h} \left\{ \sum_{V\in\mathcal{V}_T} C_V^2 / \min_{T\in\mathcal{T}_V} h_T^2 \right\}^{\frac{1}{2}},$$

$$C_2^2 := 3C_1 \max_{T\in\mathcal{T}_h} \max_{F\in\mathcal{F}_T} \{h_T/h_F h_T^2/|T|\} + \frac{1}{2} 3^{\frac{3}{2}} C_1^2 \max_{T\in\mathcal{T}_h} \max_{F\in\mathcal{F}_T} \{h_T/h_F h_T^2/|T|(3 + h_T^2/|T|)\}.$$

Zienkiewicz–Zhu averaging estimate for $-\triangle u = f$

Corollary (Zienkiewicz–Zhu averaging error estimate in FEs)

There holds (cf. Zienkiewicz–Zhu '87)

$$\|\nabla(u - u_h)\| \lesssim \|\nabla u_h + \mathbf{t}_h\|,$$

where $\mathbf{t}_h$ is an averaged smooth flux (but not $\mathbf{H}(\mathrm{div}, \Omega)$-conforming).

Drawbacks

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Equilibrated residuals estimate for $-\triangle u = f$

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Let $\phi_T \in H^1(T)$, $\phi_T = 0$ on $\partial\Omega$, $T \in \mathcal{T}_h$, be the solutions of the local problems

$$(\nabla \phi_T, \nabla v_T)_T = (f, v_T)_T - (\nabla u_h, \nabla v_T)_T + \langle g_T, v_T\rangle_{\partial T}$$
$$\forall v_T \in H^1(T),\ v_T = 0 \text{ on } \partial\Omega.$$

Then there holds (cf. Ainsworth & Oden '00)

$$\|\nabla(u - u_h)\| \le \left\{ \sum_{T \in \mathcal{T}_h} \|\nabla \phi_T\|_T^2 \right\}^{1/2}.$$

Drawbacks

- Infinite-dimensional local problems would need to be solved to get a guaranteed upper bound.
- Their approximation may be quite expensive.

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Motivations and key points

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- establish an optimal abstract framework for a posteriori error estimation in potential- and flux-nonconforming methods
- derive estimates satisfying as many as possible of the five optimal properties

Key points

- focus on inhomogeneous and anisotropic diffusion
- case of nonmatching meshes
- singular regimes of dominant convection or reaction

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$$\begin{aligned} -\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Bilinear form

$$\mathcal{B}(u,v) := (\mathbf{K}\nabla u, \nabla v) + (\boldsymbol{\beta}\cdot\nabla u, v) + (\mu u, v), \qquad u, v \in H^1(\mathcal{T}_h)$$

Weak solution

Find $u \in H^1_0(\Omega)$ such that $\mathcal{B}(u,v) = (f,v) \quad \forall v \in H^1_0(\Omega)$.

Energy norm

Decompose $\mathcal{B}$ into $\mathcal{B} = \mathcal{B}_{\mathrm{S}} + \mathcal{B}_{\mathrm{A}}$, where

$$\begin{aligned} \mathcal{B}_{\mathrm{S}}(u,v) &:= (\mathbf{K}\nabla u, \nabla v) + \left((\mu - \tfrac{1}{2}\nabla\cdot\boldsymbol{\beta})u, v\right), \\ \mathcal{B}_{\mathrm{A}}(u,v) &:= \left(\boldsymbol{\beta}\cdot\nabla u + \tfrac{1}{2}(\nabla\cdot\boldsymbol{\beta})u, v\right). \end{aligned}$$

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Optimal abstract estimate in the energy norm

Theorem (Optimal abstract framework, energy norm (Vohralík '07, Ern & Stephansen '08))

Let $u \in H_0^1(\Omega)$ *and* $u_h \in H^1(\mathcal{T}_h)$ *be arbitrary. Then*

$$|||u - u_h||| \leq \inf_{s \in H_0^1(\Omega)} \Big\{ |||u_h - s||| + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}(u - u_h, \varphi) + \mathcal{B}_{\mathrm{A}}(u_h - s, \varphi)\} \Big\}$$

$$\leq 2|||u - u_h|||$$

- specific to the nonconforming and nonsymmetric (CDR) case

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$$\begin{aligned}|||u - u_h||| \leq \inf_{s \in H_0^1(\Omega)} \Big\{ |||u_h - s||| + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}(u - u_h, \varphi) \\ + \mathcal{B}_{\mathrm{A}}(u_h - s, \varphi)\} \Big\} \\ \leq 2|||u - u_h|||\end{aligned}$$

- specific to the nonconforming and nonsymmetric (CDR) case

Optimal abstract estimate in the energy norm

Theorem (Optimal abstract estimate, energy norm)

Let u be the weak sol. and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
|||u - u_h||| \leq & \inf_{s \in H_0^1(\Omega)} \Big\{ |||u_h - s||| \\
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& - (\mathbf{K}\nabla u_h + \mathbf{t}, \nabla\varphi) + \big((\mu - \tfrac{1}{2}\nabla\cdot\boldsymbol{\beta})(s - u_h), \varphi \big) \Big| \Big\} \\
\leq & \; 2|||u - u_h|||.
\end{aligned}
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of u_h.
- Not computable (infimum over an infinite-dim. space).

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Outline

A first computable estimate in the energy norm

Theorem (A first computable estimate, energy norm)

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ and any $s_h \in H_0^1(\Omega)$. Then

$$|||u - u_h||| \leq |||u_h - s_h||| + \min\left\{ \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{\min_{T\in\mathcal{T}_h} c_{\mathbf{K},T}^{1/2}}, \frac{1}{\min_{T\in\mathcal{T}_h} c_{\boldsymbol{\beta},\mu,T}^{1/2}} \right\} \times \|f - \nabla\cdot\mathbf{t}_h - \boldsymbol{\beta}\cdot\nabla s_h - \mu s_h\| + \left(\|\mathbf{K}^{\frac12}\nabla u_h + \mathbf{K}^{-\frac12}\mathbf{t}_h\|^2 + \|(\mu - \tfrac12 \nabla\cdot\boldsymbol{\beta})^{\frac12}(u_h - s_h)\|^2 \right)^{\frac12}.$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \leq 1$).
- $\|\mathbf{K}^{\frac12}\nabla u_h + \mathbf{K}^{-\frac12}\mathbf{t}_h\|$ results from $-\mathbf{K}\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $|||u_h - s_h|||$ results from $u_h \notin H_0^1(\Omega)$.
- Advantage: scheme-independent (promoted by Repin).
- Disadvantage: no info. from computation used, **too rough**.

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Outline

A dual norm augmented by the convective derivative

- define

$$\mathcal{B}_{\mathrm{D}}(u, v) := -\sum_{F\in\mathcal{F}_h} (\beta\cdot\mathbf{n}_F[\![u]\!], \{\!\{\Pi_0 v\}\!\})_F.$$

- introduce the **augmented norm**

$$|||v|||_{\oplus} := |||v||| + \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}_{\mathrm{A}}(v,\varphi) + \mathcal{B}_{\mathrm{D}}(v,\varphi)\}$$

- when $\|\nabla\cdot\beta\|_{\infty,T}$ is controlled by $(\mu - \frac{1}{2}\nabla\cdot\beta)$ on T for all T and when $v \in H_0^1(\Omega)$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

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Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let u be the weak sol. and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
&|||u - u_h|||_\oplus \\
&\leq 2 \inf_{s \in H_0^1(\Omega)} \Big\{ |||u_h - s||| + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{ (f - \nabla\cdot\mathbf{t} - \boldsymbol{\beta}\cdot\nabla s - \mu s, \varphi) \\
&\quad - (\mathbf{K}\nabla u_h + \mathbf{t}, \nabla\varphi) + \big((\mu - \tfrac{1}{2}\nabla\cdot\boldsymbol{\beta})(s - u_h), \varphi \big) \} \Big\} + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \\
&\quad \Big\{ \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \{ (f - \nabla\cdot\mathbf{t} - \boldsymbol{\beta}\cdot\nabla u_h - \mu u_h, \varphi) - (\mathbf{K}\nabla u_h + \mathbf{t}, \nabla\varphi) \} - \mathcal{B}_{\mathrm{D}}(u_h, \varphi) \Big\} \\
&\leq 5 |||u - u_h|||_\oplus .
\end{aligned}
$$

Comments

- only the highlighted terms are new
- their form is similar to the energy estimate
- necessary for robustness in the convection-dominated case

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Outline

Discontinuous Galerkin method for $-\nabla\cdot(\mathbf{K}\nabla u) = f$

Discontinuous Galerkin method

Find $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that for all $v_h \in \mathbb{P}_k(\mathcal{T}_h)$

$$(\mathbf{K}\nabla u_h, \nabla v_h) - \sum_{F\in\mathcal{F}_h} \{(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla u_h\}\!\}_\omega, [\![v_h]\!])_F + \theta(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla v_h\}\!\}_\omega, [\![u_h]\!])_F\} + \sum_{F\in\mathcal{F}_h} (\alpha_F \gamma_{\mathbf{K},F} h_F^{-1}[\![u_h]\!], [\![v_h]\!])_F = (f, v_h).$$

- jump operator $[\![v]\!] = v^- - v^+$
- average operator $\{\!\{v\}\!\} = \frac{1}{2}(v^- + v^+)$
- diffusivity-weighted av. operator $\{\!\{v\}\!\}_\omega = (\omega^- v^- + \omega^+ v^+)$
- diffusivity-dependent penalties $\gamma_{\mathbf{K},F}$ (Ern, Stephansen, and Zunino 08)
- θ: different scheme types (SIPG/NIPG/IIPG/OBB)
- $u_h \notin H_0^1(\Omega)$, $-\mathbf{K}\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega)$

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Potential- and flux-conforming reconstructions

Choice of s_h: the Oswald interpolate of u_h

- $\mathcal{I}_{\mathrm{Os}} : \mathbb{P}_k(\mathcal{T}_h) \to \mathbb{P}_k(\mathcal{T}_h) \cap H_0^1(\Omega)$
- prescribed at Lagrange nodes by arithmetic averages

$$\mathcal{I}_{\mathrm{Os}}(v_h)(V) = \frac{1}{\#(\mathcal{T}_V)} \sum_{T \in \mathcal{T}_V} v_h|_T(V)$$

- one can also use diffusivity-weighted averages (Ainsworth '05)

Choice of $\mathbf{t}_h$: a new $\mathbf{H}(\mathrm{div},\Omega)$ flux reconstruction

- Ern, Nicaise & Vohralík '07 (matching meshes)
- the present work (nonmatching meshes)

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Outline

Diffusive flux reconstructions for DG

Previous work by Bastian and Rivière ’03

- cheap (local) construction
- use the neighboring values of $-\mathbf{K}\nabla u_h$ but not the DG scheme
- projection onto the Brezzi–Douglas–Marini space
- L^2-norm a priori estimate

Present work

- cheap (local) construction
- use the DG scheme
- projection onto the Raviart–Thomas–Nédélec space
- $\mathbf{H}(\mathrm{div})$-norm a priori estimate
- can be used for many DG schemes (not only SIP)

Diffusive flux reconstructions for DG

Previous work by Bastian and Rivière '03

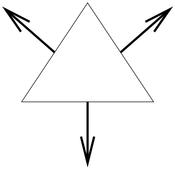
- cheap (local) construction
- use the neighboring values of $-\mathbf{K}\nabla u_h$ but not the DG scheme
- projection onto the Brezzi–Douglas–Marini space
- L^2-norm a priori estimate

Present work

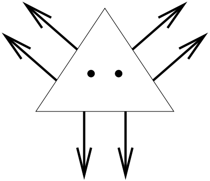
- cheap (local) construction
- use the DG scheme
- projection onto the Raviart–Thomas–Nédélec space
- $\mathbf{H}(\mathrm{div})$-norm a priori estimate
- can be used for many DG schemes (not only SIP)

Diffusive flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}_h)$: Raviart–Thomas–Nédélec spaces of degree l



$l = 0$



$l = 1$

Construction of $\mathbf{t}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(F)$,

$$(\mathbf{t}_h \cdot \mathbf{n}_F, q_h)_F = \left(-\mathbf{n}_F \cdot \{\!\{\mathbf{K}\nabla u_h\}\!\}_\omega + \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} [\![u_h]\!], q_h\right)_F$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(T)$,

$$(\mathbf{t}_h, \mathbf{r}_h)_T = -(\mathbf{K}\nabla u_h, \mathbf{r}_h)_T + \theta \sum_{F \in \mathcal{F}_T} \omega_{T,F} (\mathbf{n}_F \cdot \mathbf{K}\mathbf{r}_h, [\![u_h]\!])_F$$

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Diffusive flux reconstruction property

Crucial diffusive flux reconstruction property

- note that all the terms of the DG scheme are used in the construction of $\mathbf{t}_h$
- denote by Π_l the L^2-orthogonal projection onto $\mathbb{P}_k(\mathcal{T}_h)$
- the above construction yields $\nabla\cdot\mathbf{t}_h = \Pi_l(f)$

  Proof: $(\nabla\cdot\mathbf{t}_h, \xi_h)_T = -(\mathbf{t}_h, \nabla\xi_h)_T + \langle\mathbf{t}_h\cdot\mathbf{n}, \xi_h\rangle_{\partial T} = \mathcal{B}_h(u_h, \xi_h) = (f, \xi_h)_T$
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A post. estimate for $-\nabla\cdot(\mathbf{K}\nabla u) = f$

Towards an a posteriori error estimate

- recall that the energy norm framework gives

$$|||u-u_h||| \le |||u_h-s_h||| + \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \left|(f-\nabla\cdot\mathbf{t}_h,\varphi)-(\mathbf{K}\nabla u_h+\mathbf{t}_h,\nabla\varphi)\right|$$

- note that, by the Cauchy–Schwarz inequality:

$$(\mathbf{K}\nabla u_h + \mathbf{t}_h, \nabla\varphi)_T \le \|\mathbf{K}^{\frac{1}{2}}\nabla u_h + \mathbf{K}^{-\frac{1}{2}}\mathbf{t}_h\|_T |||\varphi|||_T$$

- local conservativity of $\mathbf{t}_h$:

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$$\|\varphi-\Pi_0(\varphi)\|_T \le C_{\mathrm{P}}^{\frac{1}{2}} h_T \|\nabla\varphi\|_T \le \frac{C_{\mathrm{P}}^{1/2} h_T}{c_{\mathbf{K},T}^{1/2}} |||\varphi|||_T$$

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Theorem (A posteriori error estimate, pure diffusion case)

There holds

$$|||u - u_h|||^2 \leq \sum_{T\in\mathcal{T}_h} \left\{\eta_{\mathrm{NC},T}^2 + (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2\right\}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},T} := |||u_h - \mathcal{I}_{\mathrm{Os}}(u_h)|||_T$
- **diffusive flux estimator**
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Local efficiency of the estimates for $-\nabla\cdot(\mathbf{K}\nabla u) = f$

Theorem (Local efficiency, pure diffusion case)

There holds

$$\eta_{\mathrm{NC},T} \leq \tilde{C} \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},\mathfrak{T}_T}^{1/2}} |||u - u_h|||_{*,\mathfrak{F}_T},$$

$$\eta_{\mathrm{DF},T} \leq \tilde{C} \frac{C_{\mathbf{K},T}^{1/2}}{c_{\mathbf{K},T}^{1/2}} \left(|||u - u_h|||_{*,\mathcal{F}_T} + \sum_{T' \in \mathcal{T}_T} \frac{C_{\mathbf{K},T'}^{1/2}}{c_{\mathbf{K},T'}^{1/2}} |||u - u_h|||_{T'} \right),$$

where

$$|||v|||_{*,\mathcal{F}}^2 := \sum_{F \in \mathcal{F}} \|\gamma_F^{1/2} [\![v]\!]\|_{0,F}^2.$$

Local efficiency of the estimates for $-\nabla\cdot(\mathbf{K}\nabla u) = f$

Residual estimator $\eta_{\mathrm{R},T}$

- $\eta_{\mathrm{R},T}$ is a higher-order term (equal to the data oscillation)

Diffusive flux estimator $\eta_{\mathrm{DF},T}$

- robust w.r.t. inhomogeneities
- the weights $\omega_{T,F}$ play a key role
- anisotropy ratios remain local

Nonconformity estimator $\eta_{\mathrm{NC},T}$

- with “monotonicity around vertices” assumption (Bernardi & Verfürth ’00) and using the weighted Oswald interpolate (Ainsworth ’05) can be made robust w.r.t. inhomogeneities

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Outline

Nonmatching grids

Oswald interpolate on nonmatching grids

- consider a matching simplicial submesh $\widehat{\mathcal{T}}_h$ of $\mathcal{T}_h$
- consider $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ as function in $\mathbb{P}_k(\widehat{\mathcal{T}}_h)$
- take $\mathcal{I}_{\mathrm{Os}}(u_h)$ on $\widehat{\mathcal{T}}_h$

Reconstruction of $\mathbf{t}_h$ by direct prescription

- directly prescribe $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ by the values of u_h
- this gives $(\nabla\cdot\mathbf{t}_h, \xi_h)_T = (f, \xi_h)_T$ for all $T \in \mathcal{T}_h$ and all $\xi_h \in \mathbb{P}_l(T)$

Reconstruction of $\mathbf{t}_h$ by solving local linear systems

- consider the simplicial submesh $\mathfrak{R}_T$ of each T
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Outline

Numerical experiments: smooth solution

- consider the pure diffusion equation

$$-\triangle u = f \quad \text{in} \quad \Omega = (0,1) \times (0,1)$$

- analytical solution:

$$u(x,y) = \cos(0.5\pi x)\cos(0.5\pi y)$$

- $k = 1$, unstructured meshes

Estimated and actual errors, smooth solution

			$l = 0$			$l = 1$		
N	$\|\|\|u - u_h\|\|\|$	η_{NC}	η_{R}	η_{DF}	eff.	η_{R}	η_{DF}	eff.
112	3.16e-1	1.25e-1	7.01e-2	3.60e-1	1.2	5.13e-3	3.58e-1	1.2
448	1.58e-1	6.85e-2	1.76e-2	1.82e-1	1.2	6.90e-4	2.22e-1	1.5
1792	7.88e-2	3.53e-2	4.40e-3	9.10e-2	1.2	8.05e-5	9.43e-2	1.3
7168	3.93e-2	1.77e-2	1.10e-3	4.55e-2	1.2	1.01e-5	4.76e-2	1.3
order	1.1	1.1	2.1	1.1	-	3.2	1.1	-

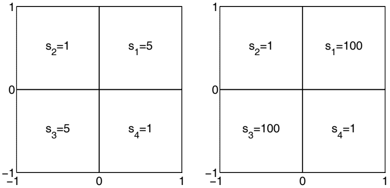
- η_{NC} and η_{DF} optimally convergent
- η_{R} superconvergent
- choice of l:
  - η_{DF} similar in both cases
  - η_{R} by one order sharper for $l = 1$

Discontinuous diffusion tensor

- consider the pure diffusion equation

$$-\nabla\cdot(\mathbf{K}\nabla u) = 0 \quad \text{in} \quad \Omega = (-1,1)\times(-1,1)$$

- discontinuous and inhomogeneous **K**, two cases:

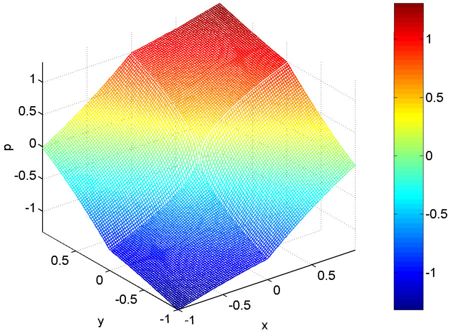


- analytical solution: singularity at the origin

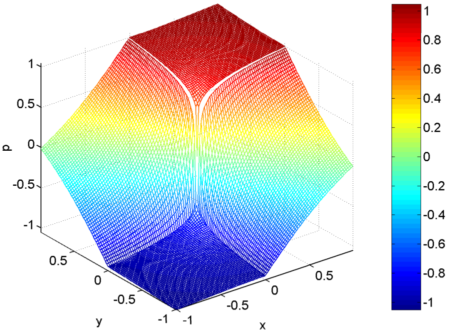
$$u(r,\theta)|_{\Omega_i} = r^\alpha(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution: $u \in H^{1+\alpha}$

Analytical solutions



Case 1



Case 2

Estimated and actual errors, case 1

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N	$\|\|\|u - u_h\|\|\|$	$\eta_{\rm NC}$	$\eta_{\rm DF}$	eff.	$\eta_{\rm DF}$	eff.
112	6.11e-01	8.70e-1	7.43e-1	1.9	6.00e-1	1.7
448	4.28e-01	6.09e-1	5.35e-1	1.9	4.32e-1	1.7
1792	2.97e-01	4.23e-1	3.74e-1	1.9	3.05e-1	1.8
7168	2.01e-01	2.92e-1	2.60e-1	1.9	2.12e-1	1.8
order	0.53	0.53	0.53	-	0.52	-

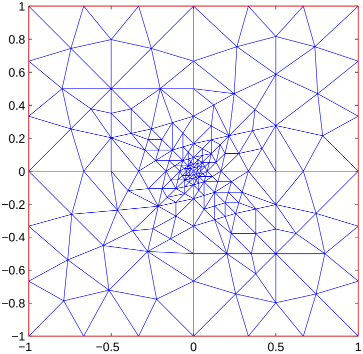
- $\eta_{\rm R} = 0$ in both cases
- $\eta_{\rm DF}$ slightly sharper for $l = 1$

Estimated and actual errors, case 2

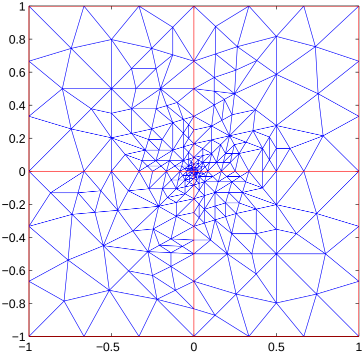
			$l=0$		$l=1$	
N	$\lvert\lvert\lvert u-u_h \rvert\rvert\rvert$	η_{NC}	η_{DF}	eff.	η_{DF}	eff.
112	3.27	11.8	2.39	3.7	1.89	3.7
448	3.11	11.3	2.33	3.7	1.84	3.7
1792	2.93	10.8	2.23	3.8	1.77	3.7
7168	2.75	10.3	2.12	3.8	1.68	3.8
order	0.09	0.08	0.08	-	0.07	-

- $\eta_{\mathrm{R}} = 0$ in both cases
- η_{DF} slightly sharper for $l = 1$
- η_{NC} dominates

Series of refined meshes, case 1



Mesh with 342 elements



Mesh with 494 elements

Outline

Discontinuous Galerkin method

Discontinuous Galerkin method for the CDR case

Find $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that for all $v_h \in \mathbb{P}_k(\mathcal{T}_h)$

$$
\begin{aligned}
&(\mathbf{K}\nabla u_h, \nabla v_h) + ((\mu - \nabla\cdot\boldsymbol{\beta})u_h, v_h) - (u_h, \boldsymbol{\beta}\cdot\nabla v_h) \\
&- \sum_{F\in\mathcal{F}_h} \{(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla u_h\}\!\}_\omega, [\![v_h]\!])_F + \theta(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla v_h\}\!\}_\omega, [\![u_h]\!])_F\} \\
&+ \sum_{F\in\mathcal{F}_h} \left\{((\alpha_F\gamma_{\mathbf{K},F} h_F^{-1} + \gamma_{\boldsymbol{\beta},F})[\![u_h]\!], [\![v_h]\!])_F + (\boldsymbol{\beta}\cdot\mathbf{n}_F\{\!\{u_h\}\!\}, [\![v_h]\!])_F\right\} \\
&= (f, v_h).
\end{aligned}
$$

- $\gamma_{\boldsymbol{\beta},F}$: upwind-weighting stabilization

Convective flux reconstruction

Diffusive flux reconstruction $\mathbf{t}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- as in the pure diffusion case

Convective flux reconstr. $\mathbf{q}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(F)$,

$$(\mathbf{q}_h \cdot \mathbf{n}_F, q_h)_F = (\beta \cdot \mathbf{n}_F \{\!\{u_h\}\!\} + \gamma_{\beta,F} [\![u_h]\!], q_h)_F$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(T)$,

$$(\mathbf{q}_h, \mathbf{r}_h)_T = (u_h, \beta \cdot \mathbf{r}_h)_T$$

Crucial property

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (\mu - \nabla \cdot \beta) u_h, \xi_h)_T = (f, \xi_h)_T \qquad \forall T \in \mathcal{T}_h, \forall \xi_h \in \mathbb{P}_l(T)$$

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Outline

A post. estimate for $-\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f$

Theorem (A posteriori error estimate, energy norm)

There holds

$$|||u - u_h||| \le \eta,$$

$$\eta := \left\{\sum_{T\in\mathcal{T}_h} \eta_{\mathrm{NC},T}^2\right\}^{1/2} + \left\{\sum_{T\in\mathcal{T}_h} (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{C},1,T} + \eta_{\mathrm{C},2,T} + \eta_{\mathrm{U},T})^2\right\}^{\frac{1}{2}},$$

where

- $\eta_{\mathrm{NC},T} = |||u_h - \mathcal{I}_{\mathrm{Os}}(u_h)|||_T$ (*nonconformity*),
- $\eta_{\mathrm{DF},T} = \min\left\{\eta_{\mathrm{DF},T}^{(1)}, \eta_{\mathrm{DF},T}^{(2)}\right\}$ (*diffusive flux*),
- $\eta_{\mathrm{R},T} = m_T\|f - \nabla\cdot\mathbf{t}_h - \nabla\cdot\mathbf{q}_h - (\mu - \nabla\cdot\boldsymbol{\beta})u_h\|_{0,T}$ (*residual*),
- $\eta_{\mathrm{C},1,T} = m_T\|(Id - \Pi_0)(\nabla\cdot(\mathbf{q}_h - \boldsymbol{\beta}s_h))\|_{0,T}$ (*convection*),
- $\eta_{\mathrm{C},2,T} = c_{\boldsymbol{\beta},\mu,T}^{-1/2}\left\|\frac{1}{2}(\nabla\cdot\boldsymbol{\beta})(u_h - s_h)\right\|_{0,T}$ (*convection*),
- $\eta_{\mathrm{U},T} = \sum_{F\in\mathcal{F}_T} m_F\|\Pi_{0,F}((\mathbf{q}_h - \boldsymbol{\beta}s_h)\cdot\mathbf{n}_F)\|_F$ (*upwinding*).

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Individual estimators

Diffusive flux estimator $\eta_{\mathrm{DF},T}$

- $\eta_{\mathrm{DF},T} = \min\left\{\eta^{(1)}_{\mathrm{DF},T}, \eta^{(2)}_{\mathrm{DF},T}\right\}$
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- cutoff fcts of local Péclet and Damköhler numbers in $\eta^{(2)}_{\mathrm{DF},T}$:

$$m_T := \min\{C_{\mathrm{P}}^{1/2} h_T c_{\mathbf{K},T}^{-1/2}, c_{\beta,\mu,T}^{-1/2}\},$$

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- $\eta^{(1)}_{\mathrm{DF},T}$ alone cannot be shown semi-robust (Verfürth '08)
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Individual estimators

Upwinding estimator $\eta_{\mathrm{U},T}$

- $\eta_{\mathrm{U},T} = \sum_{F \in \mathcal{F}_T} m_F \|\Pi_{0,F}((\mathbf{q}_h - \beta s_h)\cdot\mathbf{n}_F)\|_F$
- cutoff function of local Péclet and Damköhler numbers:

$$m_F^2 = \min\left\{\max_{T\in\mathcal{T}_F}\left\{C_{\mathrm{F},T,F}\frac{|F|h_T^2}{|T|c_{\mathbf{K},T}}\right\}, \max_{T\in\mathcal{T}_F}\left\{\frac{|F|}{|T|c_{\beta,\mu,T}}\right\}\right\}$$

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Properties of the estimate

Principal properties

- guaranteed upper bound
- no constants in principal estimators, known constants in the other ones
- cutoff functions of local Péclet ($h_T \|\boldsymbol{\beta}\|_{\infty,T} c_{\mathbf{K},T}^{-1}$) and Damköhler ($h_T^2 c_{\boldsymbol{\beta},\mu,T} c_{\mathbf{K},T}^{-1}$) numbers (here $c_{\boldsymbol{\beta},\mu,T}$ is the (essential) minimum of $(\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta})$)
- explicit dependence on the mesh and data
- valid for arbitrary polynomial degree and data
- nonmatching meshes
- residual estimator $\eta_{\mathrm{R},T}$ is a higher-order term (data oscillation)

Loc. efficiency for $-\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f$

Theorem (Local efficiency, energy norm)

There holds

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{C},1,T} + \eta_{\mathrm{C},2,T} + \eta_{\mathrm{U},T} \leq C_{\mathrm{eff},T} |||u - u_h|||_{*,\widetilde{\mathcal{E}}_T} .$$

Properties

- the estimates are locally efficient
- only semi-robustness: overestimation is a function of local Péclet and Damköhler numbers

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Outline

Augmented norm a posteriori error estimate

Theorem (A posteriori error estimate, augmented norm)

There holds

$$|||u - u_h|||_{\oplus} \le \widetilde{\eta} := 2\eta + \left\{ \sum_{T \in \mathcal{T}_h} (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \widetilde{\eta}_{\mathrm{C},1,T} + \widetilde{\eta}_{\mathrm{U},T})^2 \right\}^{1/2},$$

where η has been defined previously for the energy norm and

$$\widetilde{\eta}_{\mathrm{C},1,T} = m_T \|(Id - \Pi_0)(\nabla \cdot (\mathbf{q}_h - \beta u_h))\|_{0,T},$$

$$\widetilde{\eta}_{\mathrm{U},T} = \sum_{F \in \mathcal{F}_T} m_F \|\Pi_{0,F}(\gamma_{\beta,F} [\![u_h]\!])\|_F.$$

- $\widetilde{\eta}_{\mathrm{C},1,T}$ and $\widetilde{\eta}_{\mathrm{U},T}$ are only slight modifications of $\eta_{\mathrm{C},1,T}$ and $\eta_{\mathrm{U},T}$, respectively
- the term $\mathcal{B}_{\mathrm{D}}(u_h, \varphi)$ added in the definition of $|||\cdot|||_{\oplus}$ is crucial so that $\widetilde{\eta}_{\mathrm{U},T}$ has its present (optimal) form with the cutoff function m_F

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- $\widetilde{\eta}_{\mathrm{C},1,T}$ and $\widetilde{\eta}_{\mathrm{U},T}$ are only slight modifications of $\eta_{\mathrm{C},1,T}$ and $\eta_{\mathrm{U},T}$, respectively
- the term $\mathcal{B}_{\mathrm{D}}(u_h, \varphi)$ added in the definition of $|||\cdot|||_\oplus$ is crucial so that $\widetilde{\eta}_{\mathrm{U},T}$ has its present (optimal) form with the cutoff function m_F

Efficiency of the augmented norm estimate

Global jump seminorm

- define

$$|||v|||^2_{\#,\mathcal{F}_h} = \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathfrak{F}_T}\frac{1}{\#(\mathfrak{T}_F)}\left\{\frac{c_{\mathbf{K},T}}{c_{\mathbf{K},\mathfrak{T}_T}}\alpha_F\gamma_{\mathbf{K},F}h_F^{-1}\|[\![v]\!]\|_F^2 + c_{\beta,\mu,T}h_F\|[\![v]\!]\|_F^2 + m^2_{\mathcal{T}_T}\|\beta\|^2_{\infty,\mathcal{T}_T}h_F^{-1}\|[\![v]\!]\|^2_{0,\mathcal{F}_F\cap\mathfrak{F}_T}\right\},$$

- the first two terms are natural for DG methods
- the third term is not, but at least contains the cutoff factor $m_{\mathcal{T}_T}$
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Efficiency of the augmented norm estimate

Theorem (Fully robust a posteriori estimate)

There holds

$$|||u - u_h|||_{\oplus} + |||u - u_h|||_{\#,\mathcal{F}_h} \le \widetilde{\eta} + |||u_h|||_{\#,\mathcal{F}_h}$$
$$\le \tilde{C}(|||u - u_h|||_{\oplus} + |||u - u_h|||_{\#,\mathcal{F}_h}).$$

- fully robust with respect to convection- or reaction dominance
- sharper than Schötzau & Zhu '08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and cannot be evaluated
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

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Outline

Convection-dominated problem

- consider the convection–diffusion–reaction equation

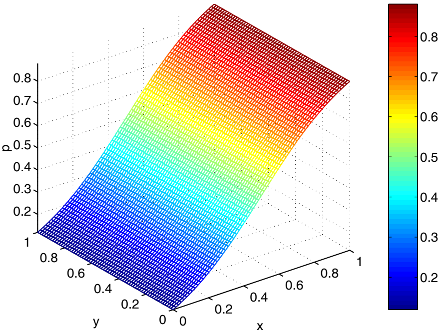
$$-\varepsilon\triangle u + \nabla\cdot(u(0,1)) + u = f \quad \text{in} \quad \Omega = (0,1)\times(0,1)$$

- analytical solution: layer of width a

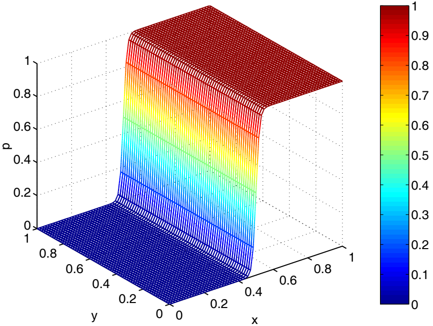
$$u(x,y) = 0.5\left(1 - \tanh\left(\frac{0.5 - x}{a}\right)\right)$$

- consider
  - $\varepsilon = 10^{-4}$, $a = 0.02$
- $k = 1$, uniformly refined structured grid

Analytical solutions



Case $\varepsilon = 1$, $a = 0.5$



Case $\varepsilon = 10^{-4}$, $a = 0.02$

Estimated and actual errors, $\varepsilon = 10^{-2}$

N	$\|\|\|u - u_h\|\|\|$	η_{NC}	eff. $l = 0$	eff. $l = 1$
128	1.72e-3	2.73e-3	80	89
512	5.68e-4	6.74e-4	124	128
2048	2.14e-4	1.66e-4	145	152
8192	1.00e-4	6.78e-5	126	127
order	1.1	1.3	-	-

		$l = 0$			$l = 1$			
N	η_{R}^*	η_{R}	$\eta_{\mathrm{DF}}^{(1)}$	η_{U}	η_{R}	$\eta_{\mathrm{DF}}^{(1)}$	η_{U}	$\eta_{\mathrm{C},1}$
128	7.77e-2	6.84e-2	1.06e-3	6.98e-2	1.92e-2	1.03e-3	6.98e-2	6.55e-2
512	3.90e-2	3.41e-2	6.20e-4	3.60e-2	3.44e-3	5.71e-4	3.60e-2	3.38e-2
2048	1.87e-2	1.63e-2	3.23e-4	1.47e-2	2.01e-3	2.86e-4	1.60e-2	1.46e-2
8192	6.69e-3	5.80e-3	1.60e-4	6.70e-3	3.66e-4	1.45e-4	6.70e-3	5.68e-3
order	1.5	1.5	1.0	1.1	2.5	1.0	1.1	1.5

- $\eta_{\mathrm{DF}}^{(1)}$ takes small values in both cases
- $\eta_{\mathrm{C},2} = 0$ since $\nabla \cdot \beta = 0$
- η_{U} and $\eta_{\mathrm{C},1}$ dominate ($\eta_{\mathrm{C},1} = 0$ for $l = 0$)

Outline

Comments on the estimates and their efficiency

General comments

- $u \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- no shape-regularity and polynomial data needed for the upper bounds (only for the efficiency proofs)
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- based on local conservativity (no global Galerkin orthogonality used)

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Essentials of the estimates

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- nonconformity estimate: compare the approximate solution u_h to a $H^1(\Omega)$-conforming potential s_h
- diffusive flux estimate: compare the flux of the approximate solution $-\mathbf{K}\nabla u_h$ to a $\mathbf{H}(\mathrm{div}, \Omega)$-conforming flux $\mathbf{t}_h$
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Conclusions

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- guaranteed, locally efficient, and robust a posteriori error estimates
- directly and locally computable
- almost asymptotically exact
- optimal framework (exact and robust)
- works for all major numerical schemes (FDs, FVs, FEs, NCFEs, MFEs)
- based on local conservativity

Open questions and future work

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- are the energy/augmented norms optimal?
- can a robust estimate without the jump seminorm be obtained?
- can a robust estimate in the energy norm be obtained?

Future work

- nonlinear (degenerate) cases: in collaboration with Linda El Alaoui in the framework of the MoMaS project
- estimates of quantities of interest

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