A posteriori error estimator competition for 2nd-order PDEs

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Contents & Overview

C. Carstensen, C. Merdon, *Estimator competition for Poisson problems*, J. Comp. Math., 28 (2010), pp. 309-330.

C. Carstensen, C. Merdon, *A posteriori error estimator competition for conforming obstacle problems, Part I: Theoretical findings (in preparation)*

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Introduction: Goal 1

Goal 1. Compare methodologies from literature.
Unified approach [C (2005)] leads to estimation of

$$\mathrm{Res} \in V^\star, \quad \mathrm{Res}(v) = \int_\Omega f \cdot v \, dx - \int_\Omega \sigma_\ell : Dv \, dx \, .$$

Five classes of a posteriori error control compete: explicit, equilibration, localisation, averaging, least-square error estimators η with

$$|||\mathrm{Res}|||_\star := \sup_{\substack{v \in V \\ |||v||| \equiv 1}} \mathrm{Res}(v) \leq \eta$$

with smaller or larger

efficiency index $\quad |||\mathrm{Res}|||_\star / \eta$

Introduction: Goal 2

Goal 2. Error Estimators for nonstandard problems.
Nonstandard and nonlinear problems also lead to

$$\text{error} \leq |||\mathrm{Res}|||_\star + \text{pertubations} =: \text{GUB}$$

Pertubations or computable terms but only control of the fraction

$$|||\mathrm{Res}|||_\star/\text{GUB} \leq C_{\text{rel}}\eta/\text{GUB}$$

depends on quality of estimator η

Is higher accuracy in η rewarding in presence of overhead?

Introduction: Benchmarks

No	Domain	Features
1	slit domain	slit singularity for Laplace
2	square domain	RHS oscillations for Laplace
3	L-shaped domain	corner singularity, NC-FEM
4	octagon domain	jumping diffusion, NC-FEM
5	square domain	affine obstacle
6	L-shaped domain	affine obstacle & corner sing.

Introduction: Error Estimators

No	Class error estimators	Examples
1	explicit residual-based	η_{R}
2	averaging	η_{A1}, η_{A2}, η_{MP1}, η_{RT}, η_{MRT}
3	equilibration	η_{B}, η_{MFEM}, η_{LW}, η_{EQL}, η_{EQB}
4	least-square	$\eta_{\mathrm{LS}}, \eta_{\mathrm{REPIN}}$
5	localisation	η_{CF}
6	interpolation (NC-FEM)	$\eta_{\mathrm{A}}, \eta_{\mathrm{MARED}}, \eta_{\mathrm{PWRED}}$

What is best? What to do in practice?
Why new estimators?

Introduction: Literature

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- W. Han, *A posteriori error analysis via duality theory. With applications in modeling and numerical approximations.*, Springer-Verlag, New York, 2005.
- C. Carstensen, *A unifying theory of a posteriori finite element error control*, Numer. Math., 100 (2005), pp. 617-637.
- P. Neittaanmäki, S. Repin, *Reliable methods for computer simulation. Error control and a posteriori estimates*, Elsevier Science B.V., Amsterdam, 2004.
- I. Babuška, T. Strouboulis, *The finite element method and its reliability*, The Clarendon Press Oxford University Press, New York, 2001.
- M. Ainsworth, J. T. Oden, *A posteriori error estimation in finite element analysis*, John Wiley & Sons, New York, 2000.
- R. Verfürth, *A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Amsterdam, 1996.
- K. Eriksson, C. Johnson et al, *Computational differential equations*, Cambridge University Press, Cambridge, 1996.

Model Poisson Problem

Seek $u \in V := H^1_0(\Omega)$ s.t., for all $v \in V$,

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx =: F(v)$$

P_1-FEM seeks $u_h \in V(\mathcal{T}) := P_1(\mathcal{T}) \cap C_0(\Omega)$ with

$$a(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V(\mathcal{T})$$

Energy error $|||e||| := a(e,e)^{1/2}$ for $e := u - u_h \in V$?

Error Residual Identity

Theorem. Given Hilbert space (V, a) and residual

$$\mathrm{Res} := F - a(u_h, \cdot) \in V^\star.$$

For all $v \in V$ with $|||v||| = 1$,

$$\frac{|||e||| - \mathrm{Res}(v)}{|||e|||} = \frac{1}{2}\left|\left|\left| v - \frac{e}{|||e|||} \right|\right|\right|^2$$

In particular,

$$|||e||| = |||\mathrm{Res}|||_\star := \sup_{\substack{v \in V \\ |||v|||=1}} \mathrm{Res}(v) \qquad \text{(dual norm)}$$

Evaluation of minimiser $e/|||e|||$ as costly as the calculation of exact error e resp. of exact solution u

A Posteriori Error Estimator Collection

Alternative approaches for bounds of $|||\mathrm{Res}|||_\star$?

1. explicit residual-based
2. averaging error
3. equilibration error
4. least-square
5. localisation

. . . error estimators

1. Explicit Residual-Based Error Estimator

$$\eta_{\mathrm{R}} := \|h_{\mathcal{T}} f\|_{L^2(\Omega)} + \left(\sum_{E \in \mathcal{E}(\Omega)} h_E \, \|[\nabla u_h \cdot \nu_E]_E\|^2_{L^2(E)} \right)^{1/2}$$

First-order approximation operator $J : V \to V(\mathcal{T})$ yields

$$\begin{aligned} \mathrm{Res}(v) &= \int_\Omega f(v - Jv)\, dx + \int_\Omega \nabla u_h \cdot \nabla(v - Jv)\, dx \\ &\lesssim \eta_R \, |||v||| \end{aligned}$$

If $\mathcal{T}$ consists of right-isosceles triangles,

$$|||\mathrm{Res}|||_\star \le \eta_{\mathrm{R}} \qquad \text{[C-Funken SISC (1999)]}$$

2. Averaging A Posteriori Error Estimators

$$\eta_{\mathrm{MP1}} := \min_{q_h \in P_1(\mathcal{T};\mathbb{R}^2)\cap C(\Omega;\mathbb{R}^2)} \|\nabla u_h - q_h\|_{L^2(\Omega)}$$
$$\leq \|\nabla u_h - \mathcal{A}(\nabla u_h)\|_{L^2(\Omega)} =: \eta_{\mathrm{A}}$$

with $\mathcal{A}(\nabla u_h)(z) := \fint_{\omega_z} \nabla u_h \, dx$ for $z \in \mathcal{N}$.

Efficiency follows from triangle inequality

$$\eta_{\mathrm{MP1}} \leq \|\nabla(u - u_h)\|_{L^2(\Omega)} + \underbrace{\min_{q_h \in Q_1(\mathcal{T})} \|\nabla u - q_h\|_{L^2(\Omega)}}_{= \text{HOT?}}$$

Reliability from [C-Bartels MathComp (2002)] reads

$$|||u - u_h||| \lesssim \eta_{\mathrm{MP1}} + \mathrm{osc}(f, \mathcal{N})$$

3. Equilibration A Post. Error Estimators

Design some $q \in H(\mathrm{div}, \Omega)$ s.t., for all $v \in V$,

$$\int_\Omega \nabla(u - u_h) \cdot \nabla v \, dx =$$
$$\int_\Omega (f - f_\mathcal{T}) v \, dx + \int_\Omega (f_\mathcal{T} + \mathrm{div}\, q) v \, dx + \int_\Omega (\nabla u_h - q) \cdot \nabla v \, dx$$

Hence

$$\begin{aligned} |||\mathrm{Res}|||_\star \le & C_F \, \|f_\mathcal{T} + \mathrm{div}\, q\|_{L^2(\Omega)} \\ & + \|\nabla u_h - q\|_{L^2(\Omega)} + \mathrm{osc}(f, \mathcal{T})/\pi \end{aligned}$$

Here: Discussion of examples $\eta_{\mathsf{LS}}, \eta_{\mathsf{REPIN}}, \eta_{\mathsf{MFEM}}, \eta_{\mathsf{CF}}, \eta_{\mathsf{EQ}}$

How to Design $q \in H(\mathrm{div}, \Omega)$?

1. least-square FEM (Repin) $\eta_{\mathrm{LS}},\ \eta_{\mathrm{REPIN}}$
2. mixed FEM (hypercircle) η_{MFEM}
3. equilibration $\eta_{\mathrm{B}},\ \eta_{\mathrm{LW}},\ \eta_{\mathrm{EQ}}$
4. localisation η_{CF}

Raviart-Thomas MFEM

$$RT_k(\mathcal{T}) := \Big\{ q = a_{\mathcal{T}} x + \begin{pmatrix} b_{\mathcal{T}} \\ c_{\mathcal{T}} \end{pmatrix} \in H(\mathrm{div}, \Omega) \mid a_{\mathcal{T}}, b_{\mathcal{T}}, c_{\mathcal{T}} \in P_k(\mathcal{T}) \Big\}$$

4. LSFEM and MFEM

$$\eta_{\mathsf{LS}} := \min_{q\in RT_0(\mathcal{T})} \Big(C_F \, \|f_{\mathcal{T}} + \operatorname{div} q\|_{L^2(\Omega)} + \|\nabla u_h - q\|_{L^2(\Omega)} \Big) + \operatorname{osc}(f,\mathcal{T})/\pi$$

all discrete functions

$$\eta_{\mathsf{MFEM}} := \min_{\substack{q\in RT_0(\mathcal{T})\\ \operatorname{div} q = -f_{\mathcal{T}}}} \|\nabla u_h - q\|_{L^2(\Omega)} + \operatorname{osc}(f,\mathcal{T})/\pi$$

discrete functions in equilibrium

minimiser q_{MFEM} solves $RT_0(\mathcal{T})$-MFEM

Remark: From [Brandts etal (2006)]

$$\eta_{\mathsf{LS}} \le \eta_{\mathsf{MFEM}} \le \eta_{\mathsf{LS}} + \text{supercloseness terms}$$

5. Repin Majorants

Without previous oscillation split,

$$\eta_{\mathsf{REPIN}} := \min_{q \in RT_0(\mathcal{T})} C_F \, \|f + \operatorname{div} q\|_{L^2(\Omega)} + \|\nabla u_h - q\|_{L^2(\Omega)}$$

N.B. $$\eta_{\mathsf{LS}} \le \sqrt{1 + h^2/(\pi^2 C_F^2)} \; \eta_{\mathsf{REPIN}}$$

Benchmarks with oscillating RHS show that indeed

$\eta_{\mathsf{LS}} \ll \eta_{\mathsf{REPIN}}$ is possible

6. Braess Equilibration A Post. Error Est.

For each nodal patch ω_z [Braess Math.Comp. 2008] designs r_z with

$$\operatorname{div} r_z|_T = -1/|T| \int_T f\varphi_z\, dx \quad \text{for all } T \in \mathcal{T}|_{\omega_z}$$

$$[r_z \cdot \nu_E]_E = -[\nabla u_h \cdot \nu_E]_E/2 \quad \text{for all } E \in \mathcal{E}(z)$$

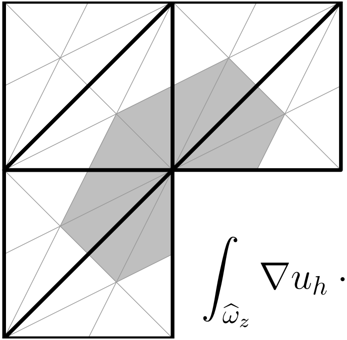
and $q_{\mathrm{B}} := \nabla u_h + \sum_{z\in\mathcal{N}} r_z \in RT_0(\mathcal{T})$ & $f_{\mathcal{T}} + \operatorname{div} q_{\mathrm{B}} = 0$

Hence

$$\eta_{\mathrm{MFEM}} \leq \eta_{\mathrm{B}} := \|\nabla u_h - q_{\mathrm{B}}\|_{L^2(\Omega)} + \operatorname{osc}(f,\mathcal{T})/\pi$$

7. Luce-Wohlmuth Equilibration Error Est.

[Luce-Wohlm. SINUM (2004)] consider finer triangulation $\widehat{\mathcal{T}}$ and design $q_{\mathsf{LW}} \in RT_0(\widehat{\mathcal{T}})$ separately on node patches $\widehat{\omega}_z, z \in \mathcal{N}$,



$$\operatorname{div} q_{\mathsf{LW}}|_{\widehat{\omega}_z} = -3 \int_{\omega_z} f \varphi_z \, dx$$

$$\int_{\widehat{\omega}_z} \nabla u_h \cdot \operatorname{Curl} \widehat{\varphi}_z \, dx = \int_{\widehat{\omega}_z} q_{\mathsf{LW}} \cdot \operatorname{Curl} \widehat{\varphi}_z \, dx$$

$$\eta_{\mathsf{LW}} := \left\| \nabla u_h - q_{\mathsf{LW}} \right\|_{L^2(\Omega)} + C_{\mathsf{LW}} \left(\sum_{z \in \mathcal{N}} h_z^2 \, \| f + \operatorname{div} q_{\mathsf{LW}} \|^2_{L^2(\widehat{\omega}_z)} + \operatorname{osc}(f, \widehat{\mathcal{T}}) \right)$$

8. Ladeveze-Leguillon Equilibration

Given $q \in H(\mathrm{div}, \Omega)$ with $\int_T (f + \mathrm{div}\, q)\, dx = 0$, seek $w_T \in W_T \subseteq H^1(T)$ s.t., for all $T \in \mathcal{T}$ and $v \in W_T$,

$$\int_T \nabla w_T \cdot \nabla v \, dx = \mathrm{Res}(v) + \int_{\partial T} q \cdot \nu_T \, v \, ds$$

Then (with approximation $w_T \in P_4(T)$ with extra errors)

$$|||\mathrm{Res}|||_\star \leq \eta_{\mathsf{EQ}} := \left(\sum_{T \in \mathcal{T}} \|\nabla w_T\|^2_{L^2(T)} \right)^{1/2}$$

This works with $q = q_{\mathsf{B}}$, q_{MFEM}, q_{L} and $I_{\mathsf{RT}}\, q_{\mathsf{L}}$.

[LL SINUM (1983), AO Book (2000)]

9. Carstensen-Funken Localisation

Given nodal basis functions φ_z on patch ω_z, seek $w_z \in W_z \subseteq H^1_{\text{loc}}(\omega_z)$ s.t., for all $z \in \mathcal{N}$ & $v \in W_z$,

$$\int_{\omega_z} \varphi_z \nabla w_z \cdot \nabla v \, dx = \mathrm{Res}(\varphi_z v)$$

Then (with approximation $w_T \in P_4(\mathcal{T}|_{\omega_z}) \cap C_0(\omega_z)$)

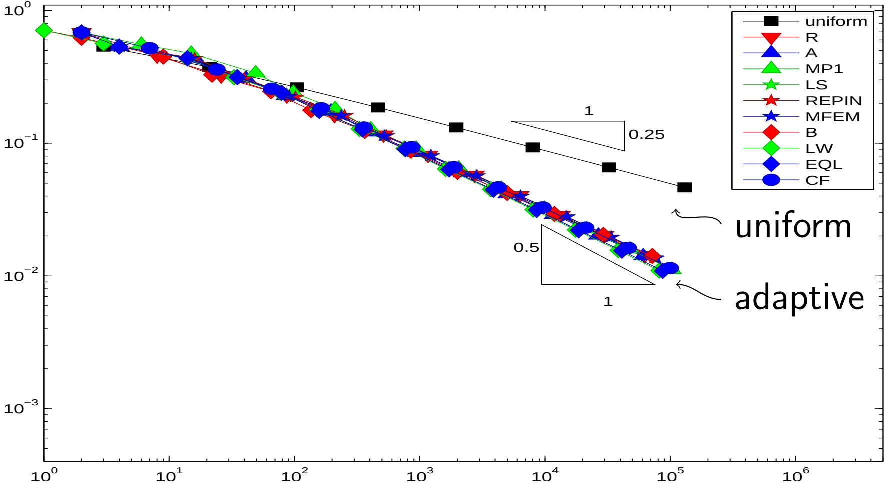
$$|||u - u_h||| \leq \eta_{\text{CF}} := \left(\sum_{z \in \mathcal{N}} \left\| \varphi_z^{1/2} \nabla w_z \right\|^2_{L^2(\omega_z)} \right)^{1/2}$$

[CF SISC (1999), Nochetto etal MathComp (2003)]

Ex. 1 $\Delta u + 1 = 0$ on Slit Domain

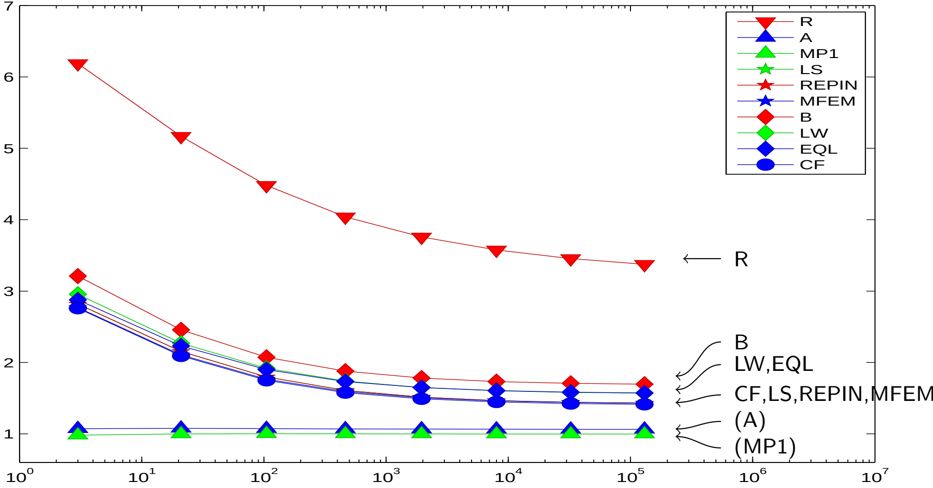
$$u(r,\varphi) = r^{1/2}\sin(\varphi/2) - 1/2\,\sin^2\varphi \text{ with } u_D \neq 0$$

Error vs. ndof for uniform & adaptive mesh refinement



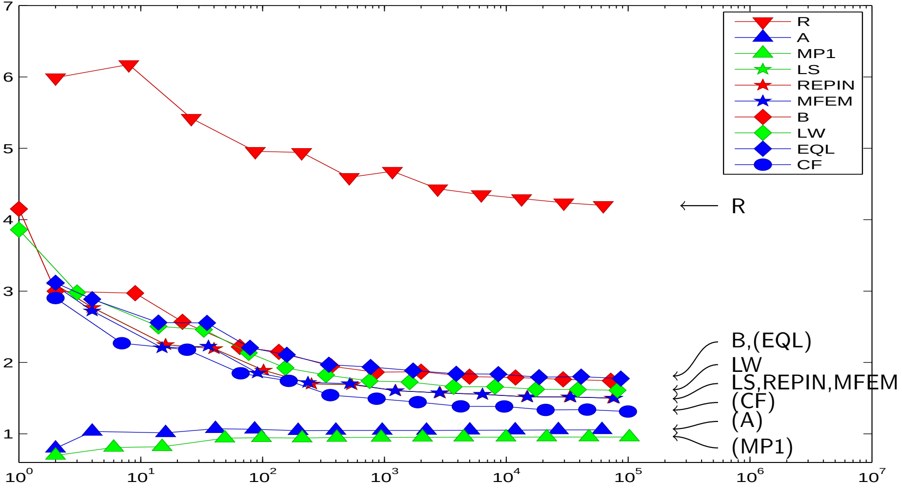
Ex. 1 $\Delta u + 1 = 0$ on Slit Domain

Efficiency index $|||e|||/\eta_{\text{xyz}}$ vs. ndof for uniform meshes



Ex. 1 $\Delta u + 1 = 0$ on Slit Domain

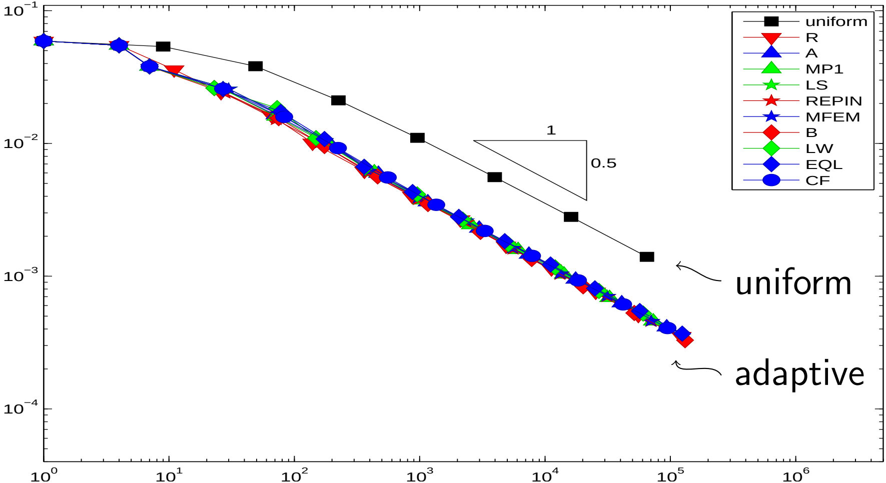
Efficiency index $|||e|||/\eta_{\text{xyz}}$ vs. ndof for adaptive meshes



Ex. 2 $\operatorname{osc}(f, \mathcal{T}) > 0$ on Unit Square

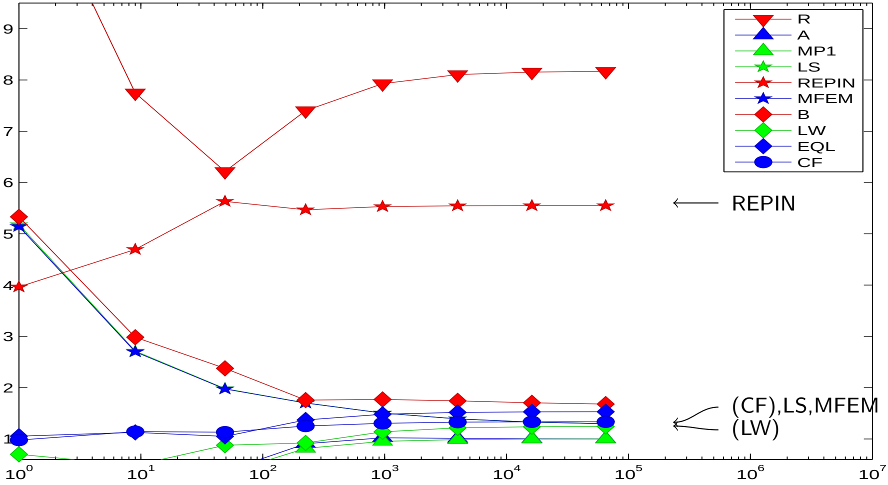
$$u(x,y) = x(x-1)y(y-1)\exp(-100(x-1/2)^2 - 100(y-117/1000)^2)$$

Error vs. ndof for uniform & adaptive mesh refinement



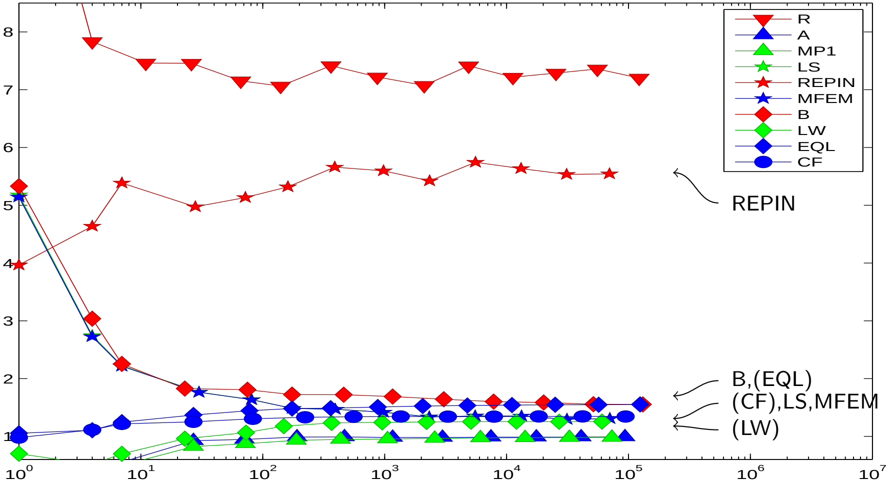
Ex. 2 $\operatorname{osc}(f, \mathcal{T}) > 0$ on Unit Square

Efficiency index $|||e|||/\eta_{xyz}$ vs. ndof for uniform meshes



Ex. 2 $\operatorname{osc}(f, \mathcal{T}) > 0$ on Unit Square

Efficiency index $|||e|||/\eta_{\text{xyz}}$ vs. ndof for adaptive meshes



Crouzeix-Raviart NC-FEM

Crouzeix-Raviart finite element space

$$CR(\mathcal{T}) := \left\{ v \in P_1(\mathcal{T}) \cap C\,(\mathrm{mid}(\mathcal{E}(\Omega))) \;\middle|\; v|_{\mathrm{mid}(\mathcal{E}(\partial\Omega))} \equiv 0 \right\}$$

CR-FEM seeks $u_{\mathrm{CR}} \in CR(\mathcal{T})$ s.t., for all $v_{\mathrm{CR}} \in CR(\mathcal{T})$,

$$a_{\mathrm{NC}}(u_{\mathrm{CR}}, v_{\mathrm{CR}}) := \int_\Omega \nabla_{\mathrm{NC}} u_{\mathrm{CR}} \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\, dx = F(v_{\mathrm{CR}})$$

Energy error $|||e|||_{\mathrm{NC}} := a_{\mathrm{NC}}(e,e)^{1/2}$ for $e := u - u_{\mathrm{CR}}$?

Inconsistency Residual

Unified analysis from [C (2005)] leads to

$$\mathrm{Res}_{\mathrm{NC}}(v) := -\int_\Omega \nabla_{\mathrm{NC}} u_{\mathrm{CR}} \cdot \mathrm{Curl}\, v \, dx$$

for $v \in H^1(\Omega)$ with $\mathrm{Curl}\, v := (-\partial v/\partial x_2, \partial v/\partial x_1)$.

Then with constants $= 1$ as displayed,

$$|||e|||_{\mathrm{NC}}^2 \le \underbrace{|||\mathrm{Res}_{\mathrm{NC}}|||_\star^2}_{\le \eta} + \underbrace{\left(\|f_{\mathcal{T}}/2\;(\bullet - \mathrm{mid}(\mathcal{T}))\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathcal{T})/\pi\right)^2}_{\text{overhead}}$$

Approximation of $|||\mathrm{Res}_{\mathrm{NC}}|||_\star$

1. by conforming interpolations of u_{CR}

$$|||\mathrm{Res}_{\mathrm{NC}}|||_\star = \min_{v \in H_0^1(\Omega)} \|\nabla_{\mathrm{NC}} u_{\mathrm{CR}} - \nabla v\|_{L^2(\Omega)}$$

2. by any previous technique with substitution of ∇u_h by $\mathrm{Curl}\, u_{\mathrm{CR}}$ in previous analysis! Modify $f \equiv g \equiv 0$, $\Gamma_N := \partial\Omega$ etc. and notice

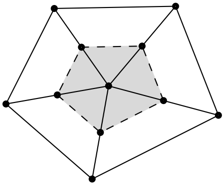
$$\mathrm{Res}_{\mathrm{NC}}(\varphi_z) = 0 \text{ for all } z \in \mathcal{N}$$

Conforming Approximations v_{xyz} of u_{CR}

Ainsworth (2005) chooses $v_{\mathsf{A}} \in P_1(\mathcal{T}) \cap C_0(\Omega)$

$$v_{\mathsf{A}}(z) = \int_{\omega_z} u_{\mathrm{CR}}\, dx \quad \text{for all } z \in \mathcal{N}(\Omega)$$

C-Merdon (2010) $v_{\mathsf{MARED}}, v_{\mathsf{PWRED}} \in P_1(\mathrm{red}(\mathcal{T})) \cap C_0(\Omega)$



$$v_{\mathsf{MARED}} = v_{\mathsf{PWRED}} = u_{\mathrm{CR}} \quad \text{in } \mathrm{mid}(\mathcal{E}(\Omega))$$

$$v_{\mathsf{MARED}} = v_{\mathsf{A}} \quad \text{in } \mathcal{N}(\Omega)$$

$$v_{\mathsf{PWRED}} = \text{patchwise optimal} \quad \text{in } \mathcal{N}(\Omega)$$

Optimal Conforming Interpolations of u_{CR}

Optimal choice v_{MPk} in $P_k(\mathcal{T}) \cap C_0(\Omega)$

Optimal choice v_{MP1RED} in $P_1(\mathrm{red}(\mathcal{T})) \cap C_0(\Omega)$

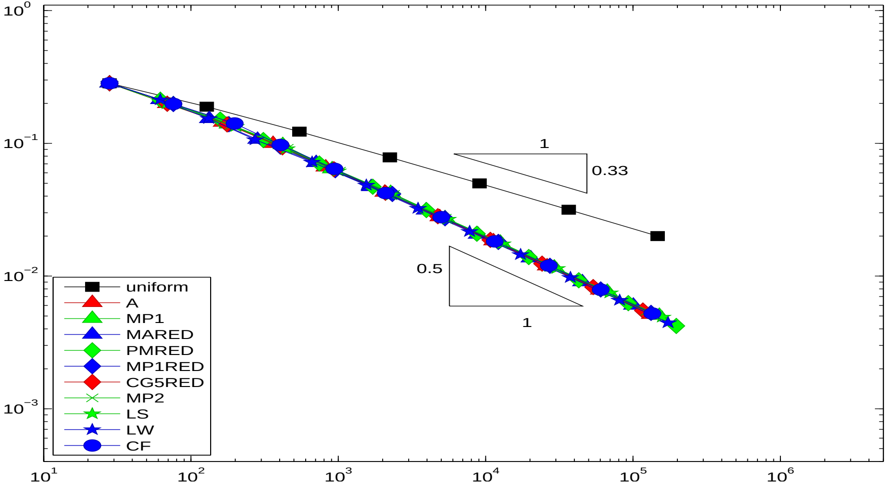
CG-Approximation v_{CG5RED} of v_{MP1RED} with

- diagonal preconditioner
- initial value v_{MARED}
- maximal 5 iterations

Ex. 3 $\Delta u + 1 = 0$ on L-shaped Domain

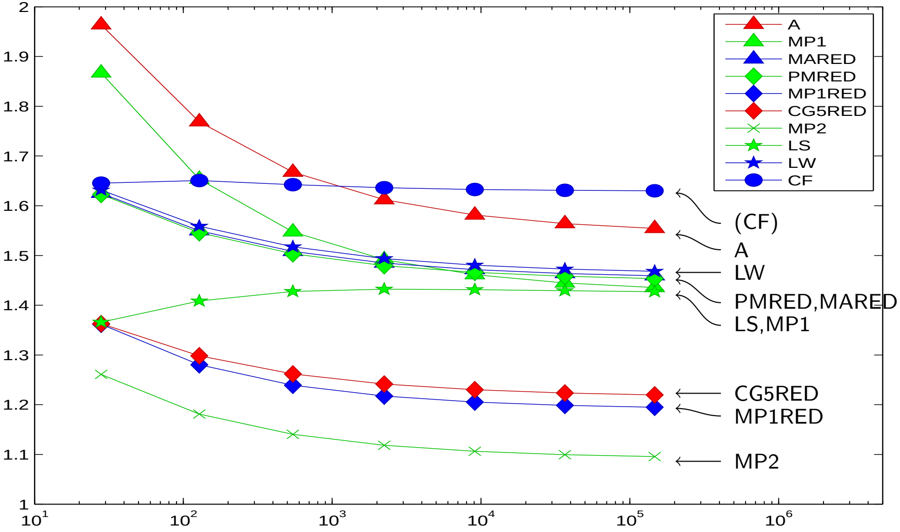
exact solution $u(r,\varphi) = r^{2/3}\sin(2\varphi/3)$ with $u_D \neq 0$

Error vs. ndof for CR-NCFEM, uniform & adaptive meshes



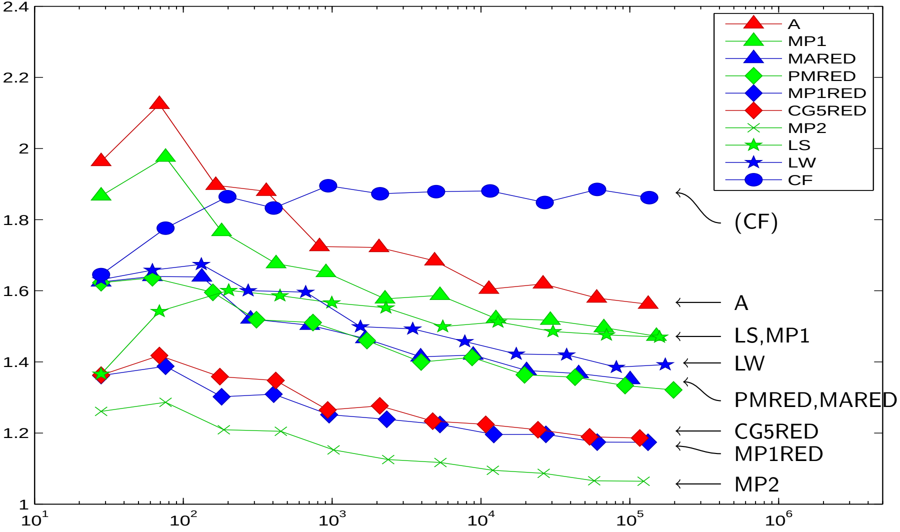
Ex. 3 $\Delta u + 1 = 0$ on L-shaped Domain

Efficiency index $|||e|||_{\mathrm{NC}}/\eta_{\mathrm{xyz}}$ for uniform meshes

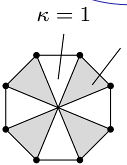


Ex. 3 $\Delta u + 1 = 0$ on L-shaped Domain

Efficiency index $|||e|||_{\mathrm{NC}}/\eta_{\mathrm{xyz}}$ for adaptive meshes

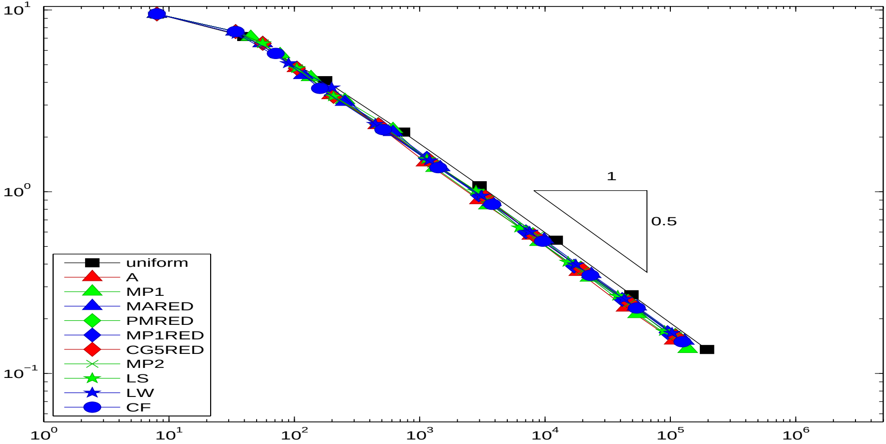


Ex. 4 $f \equiv 0$ & Jumps on Oktagon



$u(x,y) = ((ax^2 - y^2)(ay^2 - x^2))/\varkappa$ with $a = \tan(3\pi/8)^2$

Error vs. ndof for CR-NCFEM, uniform & adaptive meshes



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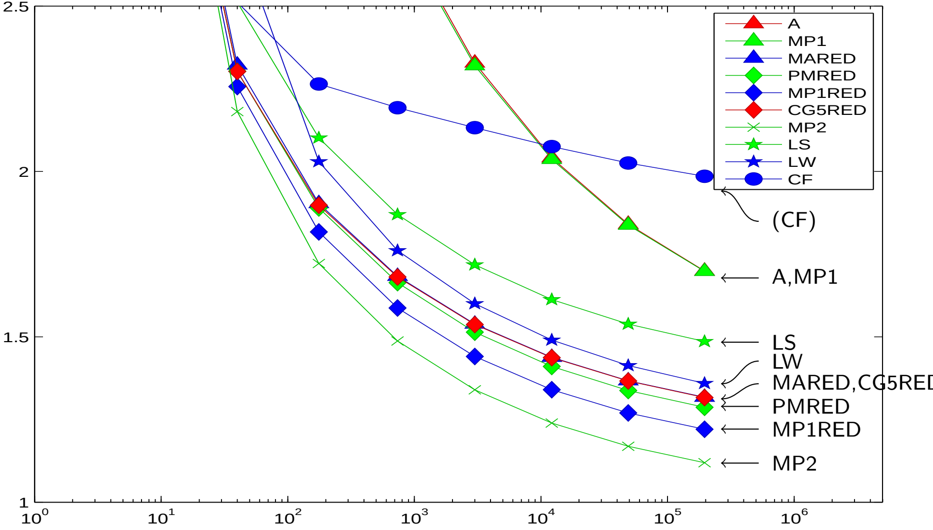
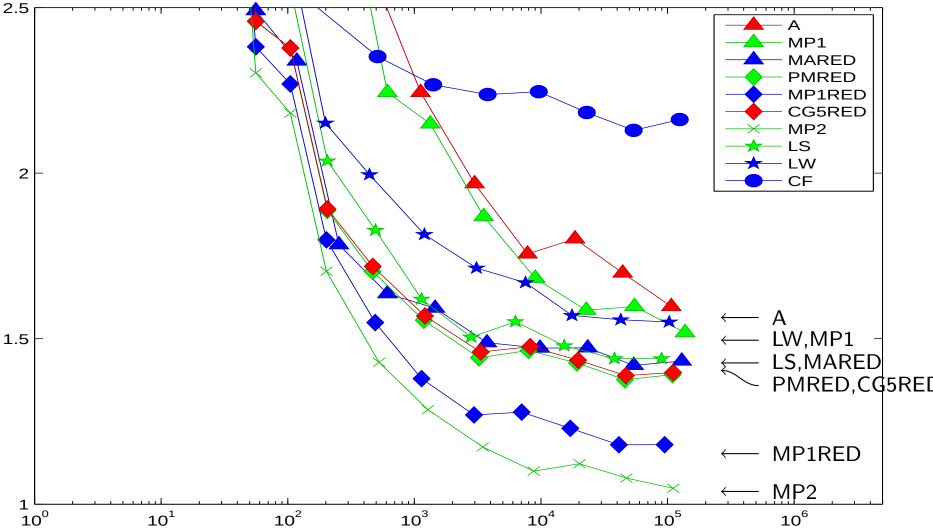
Efficiency index $|||e|||_{\mathrm{NC}}/\eta_{\mathrm{xyz}}$ for uniform meshes



Ex. 4 $f \equiv 0$ & Jumps on Oktagon

Efficiency index $|||e|||_{\mathrm{NC}}/\eta_{\mathrm{xyz}}$ for adaptive meshes



Model Obstacle Problem

Seek $u \in K := \{v \in V \mid v \geq \chi\}$ s.t.

$$a(u, u - v) \leq F(v) \quad \text{for all } v \in K$$

FEM seeks $u_h \in K(\mathcal{T}) := \{v \in V(\mathcal{T}) \mid \mathcal{I}\chi \leq v\}$ s.t.

$$a(u_h, u_h - v_h) \leq F(v_h) \quad \text{for all } v_h \in K(\mathcal{T})$$

Energy error $|||e||| := a(e, e)^{1/2}$ for $e := u - u_h$?

Auxiliary Equation $-\Delta w = f - \Lambda_h$

Define $\Lambda_h \in P_1(\mathcal{T}) \cap C(\Omega)$ s.t., for all $z \in \mathcal{N}$

$$\int_\Omega \Lambda_h \varphi_z \, dx = \int_\Omega f \varphi_{\zeta(z)} \, dx - \int_\Omega \nabla u_h \cdot \nabla \varphi_{\zeta(z)} \, dx$$

for some map $\zeta : \mathcal{N} \to \mathcal{N}(\Omega)$ & $\zeta(z) = z$ for $z \in \mathcal{N}(\Omega)$
Auxiliary Residual [Braess NumMath 2005], for $v \in V$,

$$\mathrm{Res}_{\mathrm{AUX}}(v) := \int_\Omega (f - \Lambda_h) v \, dx - \int_\Omega \nabla u_h \cdot \nabla v \, dx$$

Since $P_1(\mathcal{T}) \cap C_0(\Omega) \subseteq \ker(\mathrm{Res}_{\mathrm{AUX}})$,

$$|||\mathrm{Res}_{\mathrm{AUX}}|||_\star \le \eta \quad \text{for all error estimators}$$

Global Upper Bound for $\chi \leq \mathcal{I}\chi$

$$|||u - u_h||| \leq \mathrm{GUB} := \frac{a}{2} + \Big(\frac{a^2}{4} + \underbrace{\int_\Omega (\chi - u_h)(J\Lambda_h)\, dx}_{\text{overhead}}\Big)^{1/2}$$

with
$$a := \underbrace{|||\mathrm{Res}_{\mathrm{AUX}}|||_\star}_{\leq \eta_{\mathrm{xyz}}} + \underbrace{|||\Lambda_h - J\Lambda_h|||_\star}_{\text{overhead}}$$

for some positive interpolation operator J

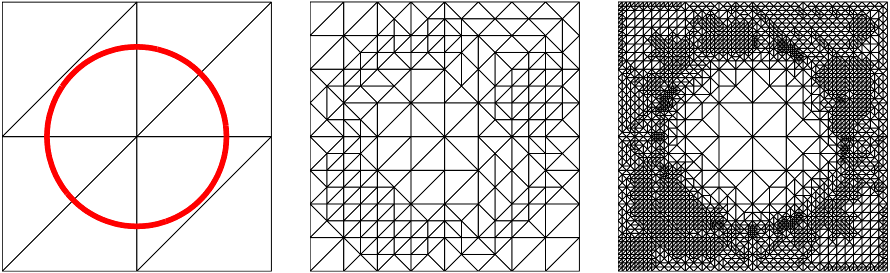
$$Jv = \sum_{z\in\mathcal{N}} \varphi_z \int_\Omega v\varphi_z\, dx \Big/ \int_\Omega \varphi_z\, dx$$

Efficiency $\mathrm{GUB} \lesssim |||u - u_h||| + |||\Lambda - \Lambda_h|||_*$ holds for model scenarios [C-Merdon 2010]

Ex. 5 $f \equiv -2, \chi \equiv 0$ on $\Omega := (-1.5, 1.5)^2$

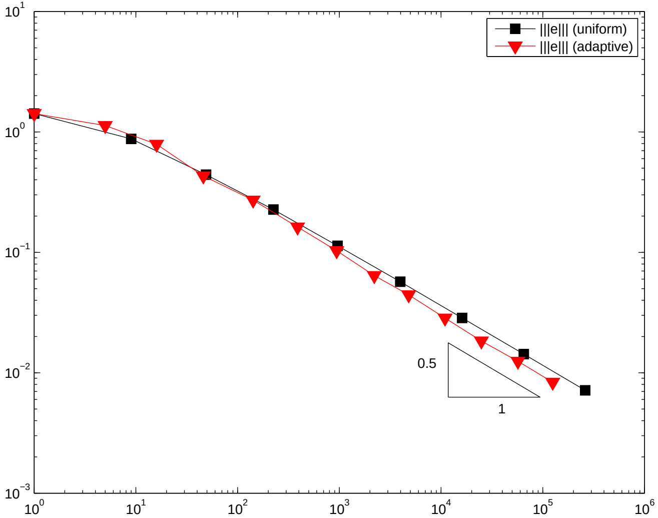
$$u(r,\varphi) = \begin{cases} r^2/2 - \ln(r) - 1/2 & \text{if } r \geq 1, \\ 0 & \text{otherwise} \end{cases} \qquad u_D \not\equiv 0$$

Adaptive meshes for levels $\ell = 0, 4, 7$



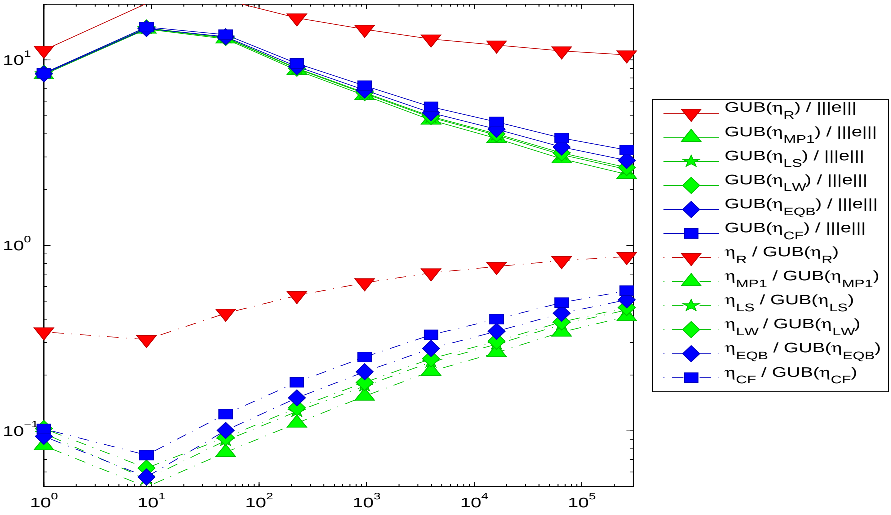
Ex. 5 $f \equiv -2, \chi \equiv 0$ on $\Omega := (-1.5, 1.5)^2$

Error vs. ndof for uniform & adaptive mesh refinement



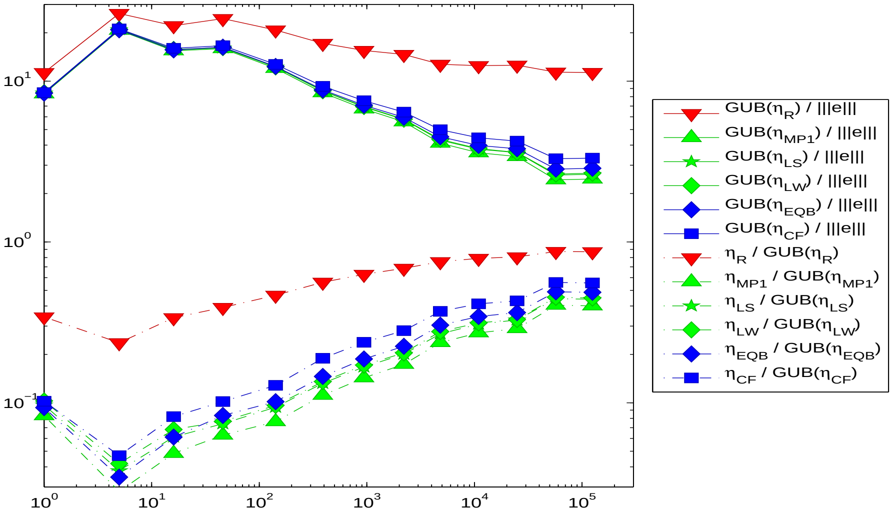
Ex. 5 $f \equiv -2, \chi \equiv 0$ on $\Omega := (-1.5, 1.5)^2$

$\mathrm{GUB}/|||e|||$ and $\eta_{\mathrm{xyz}}/\mathrm{GUB}$ for uniform meshes



Ex. 5 $f \equiv -2, \chi \equiv 0$ on $\Omega := (-1.5, 1.5)^2$

$\mathrm{GUB}/|||e|||$ and $\eta_{\mathrm{xyz}}/\mathrm{GUB}$ for adaptive meshes

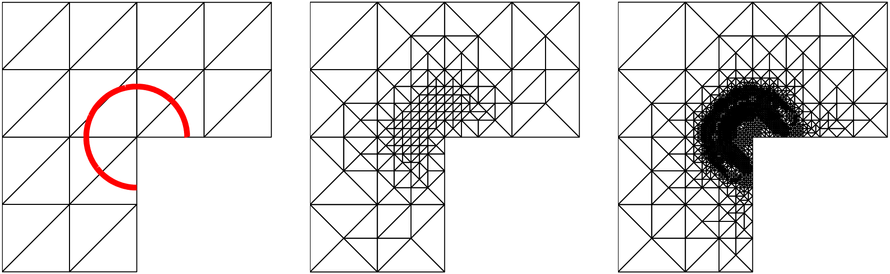


Ex. 6 $\operatorname{osc}(f, \mathcal{T}) > 0, \chi \equiv 0$ on L-shape

$u(r, \varphi) := r^{2/3} g(r) \sin(2\varphi/3)$ with $s := 2(r - 1/4)$ and

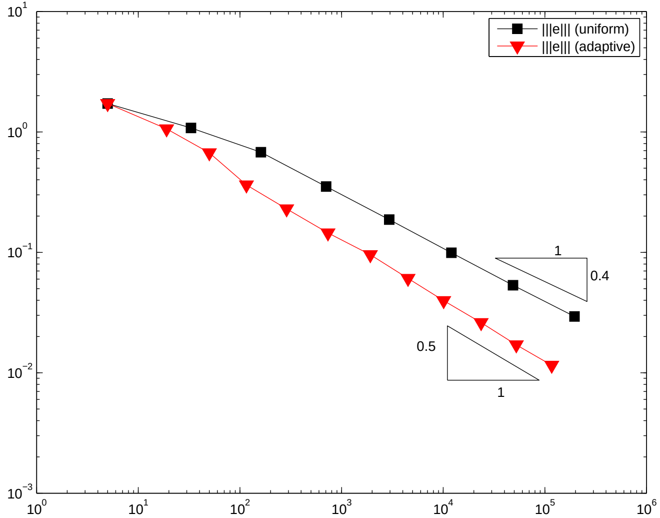
$$g(r) := \max\{0, \min\{1, -6s^5 + 15s^4 - 10s^3 + 1\}\}$$

Adaptive meshes for levels $\ell = 0, 3, 6$



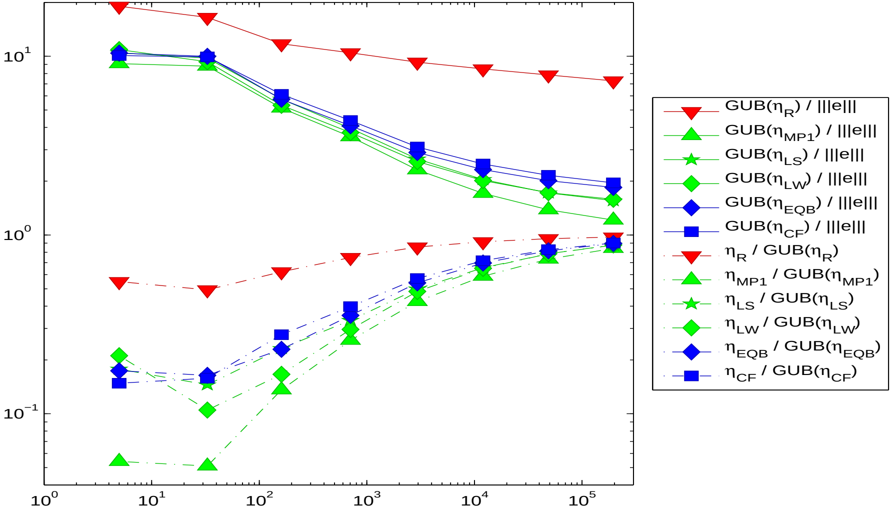
Ex. 6 $\operatorname{osc}(f, \mathcal{T}) > 0, \chi \equiv 0$ on L-shape

Error vs. ndof for uniform & adaptive mesh refinement



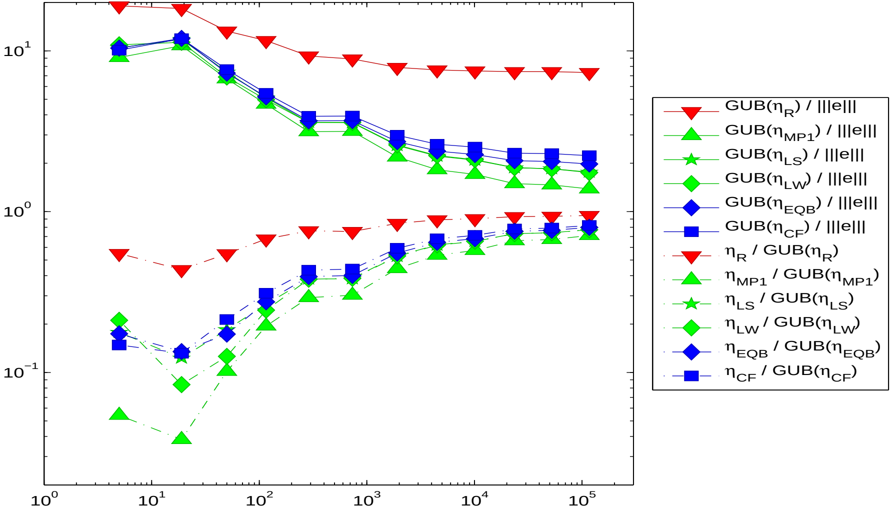
Ex. 6 $\operatorname{osc}(f, \mathcal{T}) > 0, \chi \equiv 0$ on L-shape

$\mathrm{GUB}/|||e|||$ and $\eta_{\mathrm{xyz}}/\mathrm{GUB}$ for uniform meshes



Ex. 6 $\operatorname{osc}(f,\mathcal{T}) > 0, \chi \equiv 0$ on L-shape

$\mathrm{GUB}/|||e|||$ and $\eta_{\mathrm{xyz}}/\mathrm{GUB}$ for adaptive meshes



Conclusions

- suitable adaptive mesh refinement with η_{R}, no new research on marking necessary!
- guaranteed accurate error control is possible with efficiency indices between 1 and 3
- overhead does not dominate the GUB, hence more elaborate error estimators pay off
- $(\eta_{\mathrm{LW}}), \eta_{\mathrm{LS}}$ or η_{MFEM} recommended for P1-FEM
- (η_{LW}) or η_{PWRED} recommended for CR-FEM

Is there need and possibility of even higher accuracy?

Thank you four your attention!

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- C. Carstensen, C. Merdon, *Estimator competition for Poisson problems*, J. Comp. Math., 28 (2010), pp. 309-330.
- C. Carstensen, C. Merdon, *Computational survey on a posteriori error estimators for nonconforming finite element methods, Part I: Poisson problems (in preparation)*
- C. Carstensen, C. Merdon, *A posteriori error estimator competition for conforming obstacle problems, Part I: Theoretical findings (in preparation)*
- C. Carstensen, C. Merdon, *A posteriori error estimator competition for conforming obstacle problems, Part II: Numerical results (in preparation)*

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