Adaptive finite elements with large aspect ratio for evolution problems

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Adaptive finite elements

- A posteriori error estimates for elliptic, parabolic, hyperbolic ? nonlinear ? problems + efficient meshing tools $\longrightarrow$ can be used for industrial problems.
- Adaptive finite element with large aspect ratio: the ultimate tool to reduce the number of vertices given a prescribed level of accuracy.

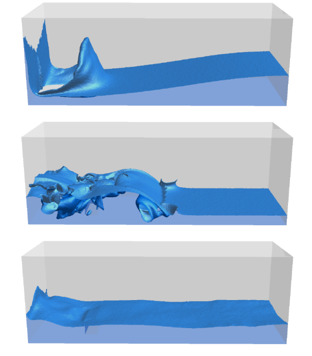
Finite elements with large aspect ratio

- General statement (mathematician): If nothing is known about the solution, then finite elements with large aspect ratio should not be used.
- A priori error estimates: $\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq Ch|u|_{H^2(\Omega)}$ and C is large when the aspect ratio is large.
- But engineers use finite elements with large aspect ratio. Example: viscous compressible flows around aircrafts, aspect ratio 10^3 in the boundary layer.
- Finite elements with large aspect ratio can be used provided the mesh fits the solution.
- The theory of finite elements has to be revisited to handle meshes with large aspect ratio.

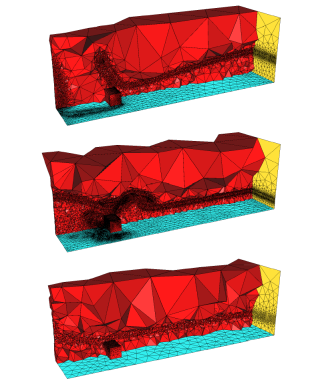
Examples

- Fluid flows with complex free surfaces (no theory, allows very fast computations).
- Microfluidics (parabolic problem, theory and practice).

Example 1: Fluid flows with complex free surfaces



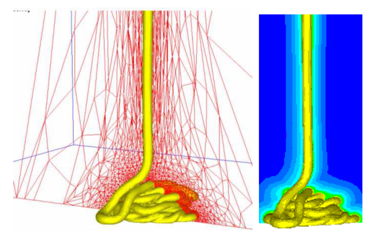
Free surface



Mesh

- Navier-Stokes with level set, finite volumes.
- Anisotropic adaptive remeshing, criterion: distance to interface.
- Alain Guégan Alauzet 2009.

Example 1: Fluid flows with complex free surfaces



Jet buckling of a viscoelastic fluid.

- Navier-Stokes with level set, finite elements.
- Anisotropic adaptive remeshing, criterion: distance to interface.
- Coupez 2010.

Example 2: Microfluidics

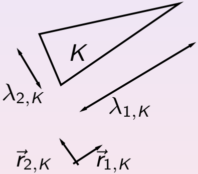
- Heat equation, Lozinski Picasso Prachittham SISC 2009.
- Unsteady convection-diffusion, Picasso Prachittham JCAM 2009.
- Microfluidics, Picasso Prachittham Gijs IJNMF 2009.
- Adaptive time steps and finite elements with large aspect ratio.
- Optimal a posteriori error estimates in the $L^2(H^1)$ norm for the Crank Nicolson scheme.
- Animation.
- Animation.

A priori error estimates for the Laplace equation

- Isotropic meshes: $\exists C > 0$ (dep. aspect ratio, indep. u, h)

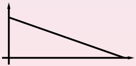
$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \leq Ch|u|_{H^2(\Omega)}.$$

- Anisotropic meshes: Hessian matrix $H(u)$.
  - Dompierre Vallet Bourgault Fortin Habashi 2002, Frey George 2008, Chen Sun Xu 2007, Mirebeau Cohen 2009.
  - Formaggia Perotto 2001, $\exists C > 0$ (indep. aspect ratio, u, h)



$$\int_K |\nabla(u-r_hu)|^2 \leq C\left(\frac{\lambda_{1,K}^4}{\lambda_{2,K}^2}\int_K (\vec{r}_{1,K}^{\,T}H(u)\vec{r}_{1,K})^2 + 2\lambda_{1,K}^2\int_K (\vec{r}_{1,K}^{\,T}H(u)\vec{r}_{2,K})^2 + \lambda_{2,K}^2\int_K (\vec{r}_{2,K}^{\,T}H(u)\vec{r}_{2,K})^2\right).$$

  - Ex: the mesh is aligned with u, $u = u(x_2)$, $\vec{r}_{1,K} = (1\ 0)^T$



$$\int_K |\nabla(u-r_hu)|^2 \leq C\lambda_{2,K}^2\int_K \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2.$$

  - Most anisotropic adaptive algorithms use an estimate of $H(u)$, Vallet Manole Dompierre Dufour Guibault 2007, Picasso Alauzet Borouchaki George 2010.

A posteriori error estimates for the Laplace equation

- Isotropic meshes: $\exists C_1, C_2 > 0$ (dep. aspect ratio, indep. u, h)

$$C_1 \eta \leq \|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C_2 \eta + h.o.t.,$$

  where the error estimator η is a computable quantity dep. on the mesh size, the data and on u_h.
- Anisotropic meshes:
  - Kunert, Kunert Verfürth 2000: C_2 depends on the alignement of the mesh with the (unknown) solution u.
  - Formaggia Perotto 2001 2003, Picasso 2003 2006: C_1 and C_2 indep. aspect ratio, u, h, provided the error estimator is equidistributed in the directions of min. and max. stretching.

Outline

- A posteriori error estimates (iso and aniso) for the Laplace equation in the natural $H^1(\Omega)$ norm.
- The heat equation in the natural $L^2(0, T; H^1(\Omega))$ norm.
- The wave equation.
- The transport equation.

A posteriori error estimates for the Laplace equation

- Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u = f & \qquad \text{in } \Omega, \\ u = 0 & \qquad \text{on } \partial\Omega. \end{aligned}$$

- Let $\mathcal{T}_h$ be a mesh of Ω into triangles K with diameter h_K less than h.
- Find $u_h \in V_h$ (continuous, piecewise linears) such that, for all $v_h \in V_h$

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h.$$

A posteriori error estimates for the Laplace equation

- The classical procedure to obtain an explicit, residual based error estimator is:

$$\begin{aligned}
\int_\Omega |\nabla(u-u_h)|^2 &= \int_\Omega f(u-u_h) - \int_\Omega \nabla u_h \cdot \nabla(u-u_h), \\
&= \int_\Omega f(u-u_h-v_h) - \int_\Omega \nabla u_h \cdot \nabla(u-u_h-v_h) \qquad \forall v_h \in V_h, \\
&= \sum_{K\in\mathcal{T}_h} \left(\int_K (f+\Delta u_h)(u-u_h-v_h) \right. \\
&\qquad\qquad \left. + \frac{1}{2}\int_{\partial K} [\nabla u_h \cdot n](u-u_h-v_h) \right).
\end{aligned}$$

- Use Cauchy-Schwarz inequality, take $v_h = R_h(u-u_h)$ Clément interpolant, use interpolation estimates to obtain ...

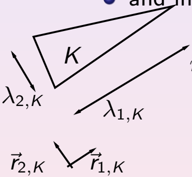
A posteriori error estimates for the Laplace equation

$$\int_\Omega |\nabla(u-u_h)|^2 \le C \sum_{K\in\mathcal{T}_h} \eta_K^2,$$

- where, in the isotropic case (C depends on the aspect ratio)

$$\eta_K^2 = h_K^2 \|f+\Delta u_h\|^2_{L^2(K)} + \frac{1}{2}|\partial K| \ \|[\nabla u_h\cdot n]\|^2_{L^2(\partial K)},$$

- and in the anisotropic case (C does not depend on the aspect ratio)



$$\eta_K^2 = \left(\|f+\Delta u_h\|_{L^2(K)} + \frac{1}{2}\left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}}\right)^{1/2} \|[\nabla u_h\cdot n]\|_{L^2(\partial K)} \right)$$

$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^{\,T} G_K(u-u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^{\,T} G_K(u-u_h)\vec{r}_{2,K} \right) \right)^{1/2}.$$

- ZZ (Zienkiewicz-Zhu) post-processing to guess

$$G_K(u-u_h) = \begin{pmatrix} \displaystyle\int_{\Delta_K} \left(\frac{\partial(u-u_h)}{\partial x_1}\right)^2 & \displaystyle\int_{\Delta_K} \frac{\partial(u-u_h)}{\partial x_1}\frac{\partial(u-u_h)}{\partial x_2} \\ \displaystyle\int_{\Delta_K} \frac{\partial(u-u_h)}{\partial x_1}\frac{\partial(u-u_h)}{\partial x_2} & \displaystyle\int_{\Delta_K} \left(\frac{\partial(u-u_h)}{\partial x_2}\right)^2 \end{pmatrix}.$$

A posteriori error estimates for the Laplace equation

$$\int_\Omega |\nabla(u - u_h)|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K^2,$$

- Can we prove a lower bound ?
- Yes in the isotropic case, but the constant depends on the aspect ratio (Verfürth 1989).
- Yes in the anisotropic case, the constant does not depend on the aspect ratio provided, $\forall K \in \mathcal{T}_h$:

$$\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u - u_h) \vec{r}_{1,K} \right) \simeq \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u - u_h) \vec{r}_{2,K} \right).$$

Adaptive finite elements

- Goal: find $\mathcal{T}_h$ such that:

$$0.75 \ TOL \leq \frac{\left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}}{\left(\int_\Omega |\nabla u_h|^2 \right)^{1/2}} \leq 1.25 \ TOL.$$

- Sufficient condition (N_K is the number of triangles):

$$\frac{0.75^2 \, TOL^2 \int_\Omega |\nabla u_h|^2}{N_K} \leq \eta_K^2 \leq \frac{1.25^2 \, TOL^2 \int_\Omega |\nabla u_h|^2}{N_K}$$

- Isotropic case: if η_K is too large, refine, too small, coarsen.
- Anisotropic case: refine or coarsen in the directions of stretching, align the triangles with the eigenvectors of $G_K(u - u_h)$.
- Use the INRIA remeshing tools: `BL2D` (Laug Borouchaki) `GHS` (George Hecht Saltel) `MMG3D` (Dobrzynski Frey).
- Alternative remeshing tools: Gruau Coupez 2005, Compère Marchandise Remacle 2008.

- 2D: $TOL = 0.25$, 30 mesh generations, animation, zoom.
- The effectivity index is aspect ratio independent on adapted meshes

TOL	vertices	error	ei	ei^{ZZ}	asp. ratio
0.125	854	0.25	2.70	1.00	262
0.0625	2793	0.13	2.75	0.99	288
0.03125	10812	0.062	2.79	0.95	425
0.015625	42562	0.031	2.79	0.98	1199

- 3D: $TOL = 0.25$, 30 mesh generations, animation, zoom.

A posteriori error estimates for the heat equation with space discretization only

- $\dfrac{\partial u}{\partial t} - \Delta u = f$ in $\Omega \times (0, T)$.
- Find $u_h : t \to u_h(\cdot, t) \in V_h$ such that, for all $t \in (0, T)$, for all $v_h \in V_h$

$$\int_\Omega \frac{\partial u_h}{\partial t} v_h dx + \int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$$

- $e = u - u_h$

$$\left\langle \frac{\partial e}{\partial t}, e \right\rangle + \int_\Omega |\nabla e|^2 dx = \int_\Omega \left(f - \frac{\partial u_h}{\partial t} \right) e dx - \int_\Omega \nabla u_h \cdot \nabla e dx$$

$$= \int_\Omega \left(f - \frac{\partial u_h}{\partial t} \right) (e - v_h) dx - \int_\Omega \nabla u_h \cdot \nabla (e - v_h) dx \qquad \forall v_h \in V_h.$$

- Use Cauchy-Schwarz inequality, take $v_h = R_h e$ Clément interpolant, use interpolation estimates to obtain ...

A posteriori error estimates for the heat equation with space discretization only

$$\frac{1}{2}\int_\Omega (u-u_h)^2(T)dx + \int_0^T \int_\Omega |\nabla(u-u_h)(t)|^2 dxdt \leq \frac{1}{2}\int_\Omega (u-u_h)^2(0)dx + C\int_0^T \sum_{K\in\mathcal{T}_h} \eta_K^2,$$

- where, in the isotropic case (C depends on the aspect ratio)

$$\eta_K^2 = h_K^2 \|f - \frac{\partial u_h}{\partial t} + \Delta u_h\|^2_{L^2(K)} + \frac{1}{2}|\partial K| \; \|[\nabla u_h \cdot n]\|^2_{L^2(\partial K)},$$

- and in the anisotropic case (C does not depend on the aspect ratio)

$$\eta_K^2 = \left(\|f - \frac{\partial u_h}{\partial t} + \Delta u_h\|_{L^2(K)} + \frac{1}{2}\left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}}\right)^{1/2} \|[\nabla u_h \cdot n]\|_{L^2(\partial K)} \right) \left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u-u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u-u_h) \vec{r}_{2,K} \right) \right)^{1/2}.$$

A posteriori error estimates for the heat equation with Euler backward scheme

- $\dfrac{\partial u}{\partial t} - \Delta u = f$ in $\Omega \times (0, T)$.
- For $n = 1, \ldots, N$, find $u_h^n \in V_h$ such that, for all $v_h \in V_h$

$$\frac{1}{\tau}\int_\Omega (u_h^n - u_h^{n-1}) v_h dx + \int_\Omega \nabla u_h^n \cdot \nabla v_h dx = \int_\Omega f^n v_h v dx.$$

- Isotropic meshes: Picasso 1998, Verfürth 2003, Bergam Bernardi Mghazli 2004.
- $u_{h\tau}(x,t) = \dfrac{t - t^{n-1}}{\tau} u_h^n(x) + \dfrac{t^n - t}{\tau} u_h^{n-1}(x)$.
- $\displaystyle\int_\Omega \frac{\partial u_{h\tau}}{\partial t} v_h dx + \int_\Omega \nabla u_{h\tau} \cdot \nabla v_h dx$
  $\displaystyle= \int_\Omega f v_h dx + \int_\Omega (f^n - f) v_h dx + (t^n - t)\int_\Omega \nabla \frac{\partial u_{h\tau}}{\partial t} \cdot \nabla v_h dx.$

A posteriori error estimates for the heat equation with Euler backward scheme

There exists C independent of u, the mesh size, time step, aspect ratio, such that

$$\frac{1}{2}\int_\Omega (u-u_{h\tau})^2(T)dx + \int_0^T \int_\Omega |\nabla(u-u_{h\tau})(t)|^2 dxdt$$
$$\leq \frac{1}{2}\int_\Omega (u-u_{h\tau})^2(0)dx + C\sum_{n=1}^N \sum_{K\in\mathcal{T}_h} \eta_{K,n}^2,$$

with $\eta_{K,n}^2$

$$= \int_{t^{n-1}}^{t^n} \Bigg\{ \left(\left\| f - \frac{\partial u_{h\tau}}{\partial t} + \Delta u_{h\tau} \right\|_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \left\| [\nabla u_{h\tau}\cdot \vec{n}] \right\|_{L^2(\partial K)} \right)$$
$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^{\,T} G_K(u-u_{h\tau}) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^{\,T} G_K(u-u_{h\tau}) \vec{r}_{2,K} \right) \right)^{1/2}$$
$$+ \|f - f^n\|_{L^2(K)}^2 + \tau^2 \left\| \nabla \frac{\partial u_{h\tau}}{\partial t} \right\|_{L^2(K)}^2 \Bigg\} dt.$$

A posteriori error estimates for the heat equation with Crank-Nicolson scheme

Optimal $L^2(H^1)$ a posteriori error estimates for Crank-Nicolson time discretization: Akrivis Makridadkis Nochetto 2006, Lozinski Prachittham Picasso 2009.

$$\eta_{K,n}^2 = \int_{t^{n-1}}^{t^n} \Bigg\{ \left(\left\| f - \frac{\partial u_{h\tau}}{\partial t} + \Delta u_{h\tau} \right\|_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \left\| [\nabla u_{h\tau} \cdot \vec{n}] \right\|_{L^2(\partial K)} \right)$$

$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u - \widetilde{u}_{h\tau}) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u - \widetilde{u}_{h\tau}) \vec{r}_{2,K} \right) \right)^{1/2}$$

$$+ \left\| f - \left(f^{n-1/2} + (t - t^{n-1/2}) \frac{f^n - f^{n-2}}{2\tau} \right) \right\|_{L^2(K)}^2 + \tau^4 \left\| \nabla \partial_n^2 u_h \right\|_{L^2(K)}^2 \Bigg\} dt,$$

$$\text{where } \widetilde{u}_{h\tau}(x,t) = u_{h\tau}(x,t) + \frac{1}{2}(t - t^{n-1})(t - t^n) \partial_n^2 u_h,$$

$$\partial_n^2 u_h = \frac{\dfrac{u_h^n - u_h^{n-1}}{\tau} - \dfrac{u_h^{n-1} - u_h^{n-2}}{\tau}}{\tau}.$$

Adaptive space-time algorithm

- Choose the time step and the mesh size so that

$$0.875\,TOL \leq \frac{\eta^{space} + \eta^{time}}{\left(\int_0^T \int_\Omega |\nabla u_{h\tau}|^2 \; dx \; dt\right)^{1/2}} \leq 1.125\,TOL.$$

- Test case 1 Animation
- Test case 2 Animation

TOL	*error*	ei^{ZZ}	ei^{space}	ei^{time}	*Vert*	*N*	*Gen.*	*AR*
0.125	0.03	0.99	2.87	2.68	155	142	84	63
0.0625	0.015	0.99	2.89	2.91	348	201	52	108
0.03125	0.0078	0.99	2.96	2.99	892	285	52	165
0.015625	0.0040	1.00	2.88	2.71	4408	401	40	118

- Conclusion: $Vert = O(TOL^{-2})$ and $N = O(TOL^{-1/2})$.

A posteriori error estimates for the wave equation with space discretization only

- $\frac{\partial^2 u}{\partial t^2} - \Delta u = f$ in $\Omega \times (0, T)$.
- Find $u_h : t \rightarrow u_h(\cdot, t) \in V_h$ such that, for all $t \in (0, T)$, for all $v_h \in V_h$

$$\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v_h dx + \int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$$

- A posteriori error estimates in the natural norm $\rightarrow$ fails.
- A posteriori error estimates in a weaker norm $\rightarrow$ Bernardi Süli 2005.
- Goal oriented isotropic adaptive finite elements $\rightarrow$ Bangerth Geiger Rannacher 2010.
- A posteriori error estimates in the $L^2(0, T; H^1(\Omega))$ norm using the elliptic reconstruction Makridakis Nochetto 2004 $\rightarrow$ Picasso 2010.

A posteriori error estimates for the wave equation: the natural norm

- $\frac{\partial^2 u}{\partial t^2} - \Delta u = f$ in $\Omega \times (0, T)$.
- Find $u_h : t \rightarrow u_h(\cdot, t) \in V_h$ such that, for all $t \in (0, T)$, for all $v_h \in V_h$

$$\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v_h dx + \int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$$

- $e = u - u_h$

$$\left\langle \frac{\partial^2 e}{\partial t^2}, \frac{\partial e}{\partial t} \right\rangle + \int_\Omega \nabla e \cdot \nabla \frac{\partial e}{\partial t} dx = \int_\Omega \left(f - \frac{\partial^2 u_h}{\partial t^2} \right) \frac{\partial e}{\partial t} dx - \int_\Omega \nabla u_h \cdot \nabla \frac{\partial e}{\partial t} dx$$

$$= \int_\Omega \left(f - \frac{\partial^2 u_h}{\partial t^2} \right) \left(\frac{\partial e}{\partial t} - v_h \right) dx - \int_\Omega \nabla u_h \cdot \nabla \left(\frac{\partial e}{\partial t} - v_h \right) dx \quad \forall v_h \in V_h.$$

$$\leq C \sum_{K \in \mathcal{T}_h} h_K \| f - \frac{\partial^2 u_h}{\partial t^2} + \Delta u_h \|_{L^2(K)} \left\| \nabla \frac{\partial e}{\partial t} \right\|_{L^2(K)} + \ldots$$

- Not the correct norm in the rhs...

A posteriori error estimates for the wave equation: Bernardi Süli 2005

- $\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f$ in $\Omega \times (0, T)$.
- Introduce $z = (-\Delta)^{-1} \partial u / \partial t$ that is:

$$\int_\Omega \nabla z \cdot \nabla v dx = \int_\Omega \frac{\partial u}{\partial t} v dx \qquad \forall v \in H_0^1(\Omega).$$

- Then, choosing z as test function in the wave equation:

$$\left\langle \frac{\partial^2 u}{\partial t^2}, z \right\rangle + \int_\Omega \nabla u \cdot \nabla z dx = \int_\Omega f z dx,$$

$$\int_\Omega \nabla \frac{\partial z}{\partial t} \cdot \nabla z dx + \int_\Omega \frac{\partial u}{\partial t} u dx = \int_\Omega f z dx,$$

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \left(|\nabla z|^2 + u^2 \right) dx = \int_\Omega f z dx.$$

A posteriori error estimates for the wave equation: Bernardi Süli 2005

- $e_u = u - u_h$, $e_z = (-\Delta)^{-1}\partial/\partial t(u - u_h)$

$$\left\langle \frac{\partial^2 e_u}{\partial t^2}, e_z \right\rangle + \int_\Omega \nabla e_u \cdot \nabla e_z dx$$
$$= \int_\Omega \left(f - \frac{\partial^2 u_h}{\partial t^2} \right) e_z dx - \int_\Omega \nabla u_h \cdot \nabla e_z dx,$$
$$\frac{1}{2}\frac{d}{dt} \int_\Omega \left(|\nabla e_z|^2 + e_u^2 \right) dx$$
$$= \int_\Omega \left(f - \frac{\partial^2 u_h}{\partial t^2} \right) (e_z - v_h) dx$$
$$- \int_\Omega \nabla u_h \cdot \nabla (e_z - v_h) dx \qquad \forall v_h \in V_h.$$

- The norms are not those corresponding to parabolic problems $\rightarrow$ too many changes in the software to switch from the heat equation to the wave equation.

A posteriori error estimates for the wave equation in the $L^2(0, T; H_0^1(\Omega))$ norm

- $\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f$ in $\Omega \times (0, T)$.
- Find $u_h : t \to u_h(\cdot, t) \in V_h$ such that, for all $t \in (0, T)$, for all $v_h \in V_h$

$$\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v_h dx + \int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$$

- Elliptic reconstruction Makridakis Nochetto 2004: let $U \in L^2(0, T; H_0^1(\Omega))$ be such that

$$\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v dx + \int_\Omega \nabla U \cdot \nabla v dx = \int_\Omega f v dx \qquad \forall v \in H_0^1(\Omega).$$

A posteriori error estimates for the wave equation in the $L^2(0,T;H_0^1(\Omega))$ norm

- $\displaystyle\int_\Omega \frac{\partial^2 u}{\partial t^2} v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$
- $\displaystyle\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v_h dx + \int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$
- $\displaystyle\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v dx + \int_\Omega \nabla U \cdot \nabla v dx = \int_\Omega f v dx.$
- Easy task:

$$\int_\Omega |\nabla(U-u_h)|^2 dx = \int_\Omega \left(f - \frac{\partial^2 u_h}{\partial t^2}\right)(U-u_h)dx - \int_\Omega \nabla u_h \cdot \nabla(U-u_h)dx$$

- We finally obtain ...

A posteriori error estimates for the wave equation in the $L^2(0,T;H^1_0(\Omega))$ norm

$$\int_0^T \int_\Omega |\nabla(U-u_h)|^2 \leq C \int_0^T \sum_{K\in\mathcal{T}_h} \eta_K^2,$$

- where, in the isotropic case (C depends on the aspect ratio)

$$\eta_K^2 = h_K^2 \|f - \frac{\partial^2 u_h}{\partial t^2} + \Delta u_h\|^2_{L^2(K)} + \frac{1}{2}|\partial K| \; \|[\nabla u_h \cdot n]\|^2_{L^2(\partial K)},$$

- and in the anisotropic case (C does not depend on the aspect ratio)

$$\eta_K^2 = \left(\|f - \frac{\partial^2 u_h}{\partial t^2} + \Delta u_h\|_{L^2(K)} + \frac{1}{2}\left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}}\right)^{1/2} \|[\nabla u_h \cdot n]\|^2_{L^2(\partial K)} \right)$$

$$\left(\lambda_{1,K}^2\left(\mathbf{r}_{1,K}^T G_K(U-u_h)\mathbf{r}_{1,K}\right) + \lambda_{2,K}^2\left(\mathbf{r}_{2,K}^T G_K(U-u_h)\mathbf{r}_{2,K}\right)\right)^{1/2}.$$

A posteriori error estimates for the wave equation in the $L^2(0,T;H_0^1(\Omega))$ norm

- $\displaystyle\int_\Omega \frac{\partial^2 u}{\partial t^2} v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$
- $\displaystyle\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v_h dx + \int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$
- $\displaystyle\int_\Omega \frac{\partial^2 u_h}{\partial t^2} v dx + \int_\Omega \nabla U \cdot \nabla v dx = \int_\Omega f v dx.$
- We have an error estimator for $\|\nabla(U - u_h)\|_{L^2(\Omega)}$.
- What about $\|\nabla(u - U)\|_{L^2(\Omega)}$???
- $\displaystyle\int_\Omega \frac{\partial^2}{\partial t^2}(u - U) v dx + \int_\Omega \nabla(u - U) \cdot \nabla v dx = \int_\Omega \frac{\partial^2}{\partial t^2}(u_h - U) v dx.$
- $$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\left| \frac{\partial}{\partial t}(u - U) \right|^2 + |\nabla(u - U)|^2 \right) dx \leq \left\| \frac{\partial^2}{\partial t^2}(u_h - U) \right\|_{L^2(\Omega)} \left\| \frac{\partial}{\partial t}(u - U) \right\|_{L^2(\Omega)}.$$
- The term $\displaystyle\left\| \frac{\partial^2}{\partial t^2}(u_h - U) \right\|_{L^2(\Omega)}$ leads to high order terms, $O(h^2)$, provided Ω is a convex polygon and f is smooth.

Numerical results for unstructured, non adapted, anisotropic meshes

- 1D solution Animation.
- Unstructured, non adapted, anisotropic meshes.
- Implicit Newmark scheme, $O(h + \tau^2)$.
- When $\tau = O(h)$, the error due to time discretization becomes negligible

h_x	h_y	τ	e	ei^{ZZ}	ei^{A}	N_{vert}
0.01	0.1	0.001	0.83	0.51	1.34	1292
0.005	0.05	0.0005	0.27	0.71	2.00	5070
0.0025	0.025	0.00025	0.10	0.87	2.44	20001
0.00125	0.0125	0.000125	0.046	0.96	2.63	80391

Numerical results for adapted, anisotropic meshes

TOL	τ	e	ei^{ZZ}	ei^{A}	N_{vert}	N_{mesh}	ar
0.125	0.001	0.27	0.45	1.23	2382	28	145
0.0625	0.0005	0.10	0.64	1.73	9781	27	245
0.03125	0.00025	0.051	0.64	1.77	36865	39	226
0.015625	0.000125	0.023	0.71	1.96	182914	45	285

- Discrepancy between error and estimator.
- Conjecture: due to the interpolation error between meshes.
- Discrepancy is smaller when conservative interpolation is used (F. Alauzet INRIA).
- Check conjecture: $\varepsilon \frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\partial u}{\partial t} = f.$

Numerical results for adapted, anisotropic meshes

- $\varepsilon \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \dfrac{\partial u}{\partial t} = f.$

ε	e	ei^{ZZ}	ei^{A}
1.	0.47	0.23	0.62
0.1	0.18	0.35	0.92
0.01	0.033	0.64	1.74
0.001	0.010	0.90	2.43
0.0001	0.0068	0.96	2.63

- Conclusion: interpolation error between meshes can be neglected for parabolic problems, not for the wave equation.
- Probably the same for the transport equation.
- Alauzet Frey George Mohammadi 2007: transient fixed point mesh adaptation for CFD.
- PhD Thesis G. Olivier, supervisor F. Alauzet, INRIA Gamma: anisotropic adaptation for Euler equations with ALE. Animation.

Conclusions

- Adaptive finite elements with large aspect ratio $\longrightarrow$ reduce the number of vertices.
- Anisotropic interpolation estimates (Formaggia-Perotto, Kunert) + ZZ postprocessing $\longrightarrow$ residual based, explicit anisotropic error estimator for the H^1 norm.
- Effectivity index aspect ratio independent whenever the estimator is equidistributed in the direction of maximum and minimum stretching.
- Applied to a wide range of elliptic and parabolic problems, very well suited for problems with boundary or internal layers.

Perspectives

- Residual based, implicit error estimators?
- Hyperbolic problems? Compressible Navier-Stokes?
- Interpolation error induced by remeshing?
- Bernardi Süli: elliptic projection between two consecutive meshes.
  - Intersection between meshes.
  - Solve a Laplace problem (more expensive than solving the wave equation when using iterative methods).
  - Stability for an order one time discretization. Order two Newmark scheme ?