# Robust A Posteriori Error Estimates for Non-Stationary Convection-Diffusion Problems

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# Goal

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Establish residual a posteriori error estimates for SUPG-discretizations of non-stationary convection-diffusion problems which yield upper and lower bounds for the energy norm of the error that are uniform with respect to all possible relative sizes of convection to diffusion.

# Outline

Variational Problem

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# Differential Equation

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\partial_t u - \operatorname{div}(d\nabla u) + \mathbf{a} \cdot \nabla u + ru &= f && \text{in } \Omega \times (0,T] \\
u &= 0 && \text{on } \Gamma \times (0,T] \\
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- $\mathbf{a} \in C^1(\Omega\times(0,T])^d$

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- Error norm

$$\|u\|_{X(a,b)} = \left\{ \operatorname*{ess.\,sup}_{t \in (a,b)} \|u(\cdot,t)\|^2 + \int_a^b |||u(\cdot,t)|||^2 dt + \int_a^b |||(\partial_t u + \mathbf{a} \cdot \nabla u)(\cdot,t)|||_*^2 dt \right\}^{\frac{1}{2}}$$

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- Transition condition: There is a common refinement $\widetilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ such that $h_K \le c h_{K'}$ for all $K \in \mathcal{T}_n$ and all $K' \in \widetilde{\mathcal{T}}_n$ with $K' \subset K$.

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- $V_n \subset H_0^1(\Omega)$ finite element space corresponding to $\mathcal{T}_n$.

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$$\int_\Omega \frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} v_{\mathcal{T}_n} + a(\theta \nabla u_{\mathcal{T}_n}^n + (1-\theta)\nabla u_{\mathcal{T}_{n-1}}^{n-1}, v_{\mathcal{T}_n})$$

$$= \int_\Omega f v_{\mathcal{T}_n}$$

with

$$a(u,v) = d(\nabla u, \nabla v) + (\mathbf{a} \cdot \nabla u, v) + r(u,v),$$

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$$+ \sum_{K \in \widetilde{\mathcal{T}}_n} \delta_K \int_K \Big(\frac{u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}}{\tau_n} + L(\theta u^n_{\mathcal{T}_n} + (1-\theta) u^{n-1}_{\mathcal{T}_{n-1}})\Big) \mathbf{a} \cdot \nabla v_{\mathcal{T}_n}$$

$$= \int_\Omega f v_{\mathcal{T}_n} + \sum_{K \in \widetilde{\mathcal{T}}_n} \delta_K \int_K f \mathbf{a} \cdot \nabla v_{\mathcal{T}_n}$$

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$a(u,v) = d(\nabla u, \nabla v) + (\mathbf{a} \cdot \nabla u, v) + r(u,v),$

$Lv = -\operatorname{div}(d \nabla u) + \mathbf{a} \cdot \nabla u + ru$

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- Lower bound:

$$\|R(u_{\mathcal{I}})\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \leq \sqrt{2}\|u - u_{\mathcal{I}}\|_{X(t_{n-1},t_n)}$$

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- Upper bound:

$$\|u - u_{\mathcal{I}}\|_{X(0,t_n)} \leq \left\{4\|u_0 - \pi_0 u_0\|^2 + 6\|R(u_{\mathcal{I}})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}\right\}^{\frac{1}{2}}$$

# Proof of the Equivalence

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- Relation of residual and error:

$$\langle R(u_{\mathcal{I}}), v\rangle = (\partial_t e, v) - (\mathbf{a} \cdot \nabla e, v) - (d\nabla e, \nabla v) - (re, v)$$

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- Lower bound: Definition of primal and dual norm plus Cauchy-Schwarz inequality.
- Upper bound: Parabolic energy estimate with $v = e$ as test-function.

# Decomposition of the Residual

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- Temporal residual:

$$\begin{aligned}\langle R_\tau(u_{\mathcal{I}}), v\rangle &= (d\nabla(u^n_{\mathcal{T}_n} - u_{\mathcal{I}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u_{\mathcal{I}}), v) \\ &\quad + (r(u^n_{\mathcal{T}_n} - u_{\mathcal{I}}), v)\end{aligned}$$

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- Spatial residual:

$$\langle R_h(u_{\mathcal{I}}), v\rangle = (f, v) - (\partial_t u_{\mathcal{I}}, v) - (d\nabla u^n_{\mathcal{T}_n}, \nabla v) - (\mathbf{a}\cdot\nabla u^n_{\mathcal{T}_n}, v) - (r u^n_{\mathcal{T}_n}, v)$$

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- Splitting: $R(u_\mathcal{I}) = R_\tau(u_\mathcal{I}) + R_h(u_\mathcal{I})$
- Estimate for $L^2(t_{n-1}, t_n; H^{-1}(\Omega))$-norms:

$$\frac{1}{5}\left\{\|R_\tau(u_\mathcal{I})\|^2 + \|R_h(u_\mathcal{I})\|^2\right\}^{\frac{1}{2}} \le \|R_\tau(u_\mathcal{I}) + R_h(u_\mathcal{I})\| \le \|R_\tau(u_\mathcal{I})\| + \|R_h(u_\mathcal{I})\|$$

# Motivation of the Lower Bound

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- Strengthened Cauchy-Schwarz inequality for $v = c$ and $w = \frac{b-t}{b-a}$:

$$\int_a^b vw = \frac{1}{2}c(b-a) = \frac{\sqrt{3}}{2}\|v\|_{(a,b)}\|w\|_{(a,b)}$$

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- Hence:

$$\|v+w\|_{(a,b)}^2 \geq \left(1 - \frac{\sqrt{3}}{2}\right)\left\{\|v\|_{(a,b)}^2 + \|w\|_{(a,b)}^2\right\}$$

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$$\langle \rho^n, v\rangle = (d\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) + (r(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v).$$

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- Choose $v, w \in H^1_0(\Omega)$ such that

$$|||v||| = |||R_h(u_{\mathcal{I}})|||_*, \qquad \langle R_h(u_{\mathcal{I}}), v\rangle = |||R_h(u_{\mathcal{I}})|||_*^2,$$
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- Insert $3\left(\frac{t-t_{n-1}}{\tau_n}\right)^2 v + \frac{t_n - t}{\tau_n} w$ as test-function in representation of $R(u_{\mathcal{I}})$.

# Estimation of the Spatial Residual

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- Spatial error indicator $\eta^n_{\mathcal{T}_n}$:

$$\eta^n_{\mathcal{T}_n} = \Big\{ \sum_{K \in \widetilde{\mathcal{T}}_n} \alpha_K^2 \| R_K \|_K^2 + \sum_{E \in \mathcal{E}_{\widetilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}} \alpha_E \| R_E \|_E^2 \Big\}^{\frac{1}{2}}.$$

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- Standard arguments for stationary problems yield:

$$\begin{aligned} |||R_h(u_{\mathcal{I}})|||_* &\le c^\dagger \eta^n_{\mathcal{T}_n}, \\ \eta^n_{\mathcal{T}_n} &\le c_\dagger |||R_h(u_{\mathcal{I}})|||_*. \end{aligned}$$

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- $c^\dagger c_\dagger$ only depend on the polynomial degrees and on the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

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- Upper bound:

$$|||\rho^n|||_* \leq \{ |||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}||| + |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_* \}$$

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- Follows from definition of $\rho^n$ and $|||\cdot|||_*$.
- Lower bound:

$$\frac{1}{3}\{|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}||| + |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*\} \le |||\rho^n|||_*$$

# Proof of the Lower Bound

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- Set $w^n = u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}$ and choose $v \in H^1_0(\Omega)$ with $|||v||| = |||\mathbf{a} \cdot \nabla w^n|||_*$ and $(\mathbf{a} \cdot \nabla w^n, v) = |||\mathbf{a} \cdot \nabla w^n|||_*^2$

# Proof of the Lower Bound

- Set $w^n = u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}$ and choose $v \in H^1_0(\Omega)$ with $|||v||| = |||\mathbf{a} \cdot \nabla w^n|||_*$ and $(\mathbf{a} \cdot \nabla w^n, v) = |||\mathbf{a} \cdot \nabla w^n|||_*^2$
- Insert $\frac{1}{2}w^n + \frac{1}{2}v$ in the definition of $\rho^n$:

$$\begin{aligned}
&\langle \rho^n, \frac{1}{2}w^n + \frac{1}{2}v \rangle \\
&= \underbrace{\frac{1}{2}(d\nabla w^n, \nabla w^n) + \frac{1}{2}(rw^n, w^n)}_{=\frac{1}{2}|||w^n|||^2} + \underbrace{\frac{1}{2}(\mathbf{a} \cdot \nabla w^n, w^n)}_{=0} \\
&\quad + \underbrace{\frac{1}{2}(d\nabla w^n, \nabla v) + \frac{1}{2}(rw^n, v)}_{\geq -\frac{1}{2}|||w^n||| \, |||\mathbf{a}\cdot\nabla w^n|||_*} + \underbrace{\frac{1}{2}(\mathbf{a} \cdot \nabla w^n, v)}_{=\frac{1}{2}|||\mathbf{a}\cdot\nabla w^n|||_*^2}
\end{aligned}$$

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- Friedrichs' inequality implies

$$\begin{aligned}(\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) &\le \|\mathbf{a}\|_\infty \|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\| \|v\| \\ &\le \|\mathbf{a}\|_\infty \|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\| c_\Omega \|\nabla v\|\end{aligned}$$

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&\le \|\mathbf{a}\|_\infty \|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\| c_\Omega \|\nabla v\|
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$$

- Hence $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_* \le c_c c_\Omega |||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}|||$ and $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ is equivalent to $|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}|||$.

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- Consider the auxiliary problem

$$d(\nabla v^n_{\mathcal{T}_n}, \nabla\varphi) + r(v^n_{\mathcal{T}_n}, \varphi) = (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \varphi) \qquad (*)$$

with variational and discrete solutions $\Phi$ and $\Phi_{\mathcal{T}_n}$.

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- Hence $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ is equivalent to $|||\Phi_{\mathcal{T}_n}||| + \eta^n_{\mathcal{T}}$.

# A Posteriori Error Estimate

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- Define the space-time error estimator by:

$$\eta^n = \tau_n^{\frac{1}{2}} \Big[ \underbrace{\left( \eta_{\mathcal{T}_n}^n \right)^2}_{\text{spatial}} + \underbrace{|||u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}|||^2 + \left( \eta_\tau^n \right)^2}_{\text{temporal}} \Big]^{\frac{1}{2}}.$$

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- Then

$$\|e\|_{X(0,T)} \le c^* \Big\{ \|u_0 - \pi_0 u_0\|^2 + \sum_{n=1}^{N_{\mathcal{I}}} \big(\eta^n\big)^2 \Big\}^{\frac{1}{2}},$$

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- $c_* c^*$ only depends on the polynomial degrees and the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

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