Goal-oriented a posteriori error estimation for conforming and nonconforming approximations with inexact solvers

Gouranga Mallik, Soleiman Yousef, and **Martin Vohralík**

Inria Paris & Ecole des Ponts

ADMOS 2021

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega,\\
-\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}},\\
u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}
$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega, \\ -\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}, \\ u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \end{aligned}$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega,\\
-\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}},\\
u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}
$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega,\\
-\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}},\\
u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}
$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

a posteriori estimate

$$|Q(u) - Q(u_h^i)| \leq \frac{\eta_h^i \tilde{\eta}_h^i}{2}$$

adaptive algorithm

- inexactly solve both primal and dual discrete problems (stop on i)
- estimate spatial error distribution
- adaptively refine mesh

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega,\\
-\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}},\\
u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}
$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

a posteriori estimate

$$
|Q(u) - Q(u_h^i)| \le \frac{\eta_h^i\,\tilde{\eta}_h^i}{2} \le \frac{(\eta_{h,\mathrm{disc}}^i + \eta_{h,\mathrm{alg}}^i)(\tilde{\eta}_{h,\mathrm{disc}}^i + \tilde{\eta}_{h,\mathrm{alg}}^i)}{2}
$$

adaptive algorithm

- **inexactly solve** both primal and dual discrete problems (stop. on it. $\bar{i}$)
- estimate spatial **error distribution**
- **adaptively refine mesh**

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega, \\ -\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}, \\ u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \end{aligned}$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

a posteriori estimate

$$|Q(u) - Q(u_h^i)| \leq \frac{\eta_h^i\, \tilde{\eta}_h^i}{2}$$

$$\leq \frac{(\eta_{h,\mathrm{disc}}^i + \eta_{h,\mathrm{alg}}^i)\,(\tilde{\eta}_{h,\mathrm{disc}}^i + \tilde{\eta}_{h,\mathrm{alg}}^i)}{2}$$

adaptive algorithm

- **inexactly solve** both primal and dual discrete problems (stop. on it. $\bar{i}$)
- estimate spatial **error distribution**
- **adaptively refine mesh**

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega,\\
-\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}},\\
u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}
$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

a posteriori estimate

$$
|Q(u) - Q(u_h^i)| \le \frac{\eta_h^i\,\tilde{\eta}_h^i}{2} \le \frac{(\eta_{h,\mathrm{disc}}^i + \eta_{h,\mathrm{alg}}^i)\,(\tilde{\eta}_{h,\mathrm{disc}}^i + \tilde{\eta}_{h,\mathrm{alg}}^i)}{2}
$$

adaptive algorithm

- **inexactly solve** both primal and dual discrete problems (stop. on it. $\bar{i}$)
- estimate spatial **error distribution**
- **adaptively refine mesh**

Setting and goals

Model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla\cdot(\underline{\boldsymbol{K}}\nabla u) &= f && \text{in } \Omega,\\
-\underline{\boldsymbol{K}}\nabla u\cdot\boldsymbol{n} &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}},\\
u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}
$$

Numerical approximation

$u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ **arbitrary**: nonconforming, high-order, on **iteration** i of an **algebraic iterative solver** (corresponding dual approximation $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ arbitrary as well)

Goals For goal functional $Q(v) := (\tilde{f}, v) - (\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}})_{\Gamma_{\mathrm{D}}}$, $v \in H^1(\mathcal{T}_h)$, design

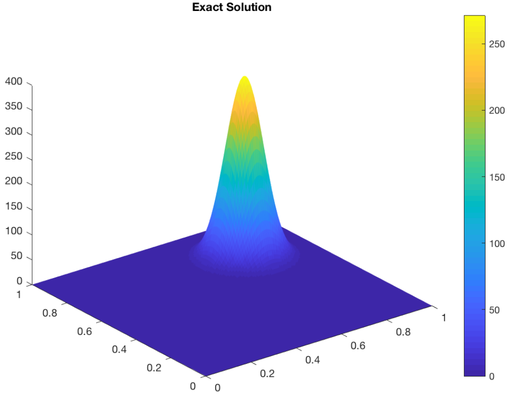
a posteriori estimate

$$
|Q(u) - Q(u_h^i)| \le \frac{\eta_h^i\,\tilde{\eta}_h^i}{2} \le \frac{(\eta_{h,\mathrm{disc}}^i + \eta_{h,\mathrm{alg}}^i)\,(\tilde{\eta}_{h,\mathrm{disc}}^i + \tilde{\eta}_{h,\mathrm{alg}}^i)}{2}
$$

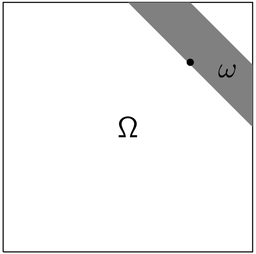
adaptive algorithm

- **inexactly solve** both primal and dual discrete problems (stop. on it. $\bar{i}$)
- estimate spatial **error distribution**
- **adaptively refine mesh**

2D regular solution and uniform mesh refinement

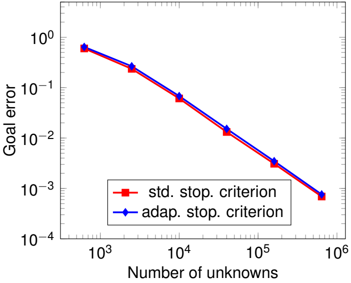


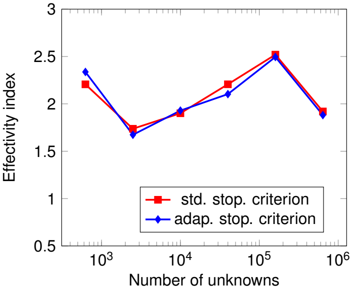
Exact solution

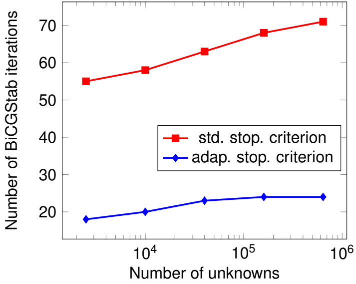


Goal functional $Q(v) = \frac{1}{|\omega|}(1, v)_\omega$

2D regular solution and uniform mesh refinement





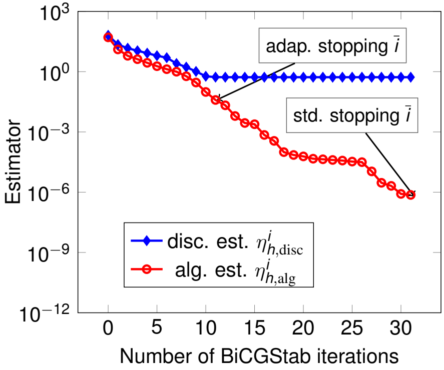


Goal error $|Q(u) - Q(u_h^{\bar{i}})|$

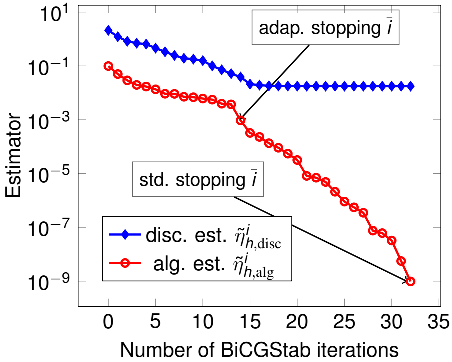
Eff. indices $\frac{\eta_h^{\bar{i}} \; \tilde{\eta}_h^{\bar{i}}}{2|Q(u)-Q(u_h^{\bar{i}})|}$

Algebraic solver iterations $\bar{i}$

2D regular solution and uniform mesh refinement

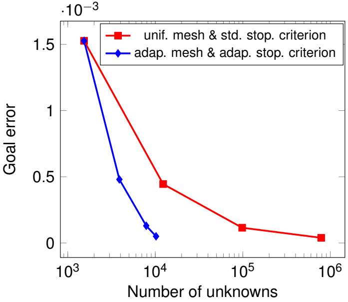


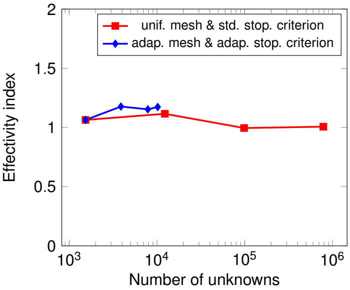
Primal problem (3rd mesh)

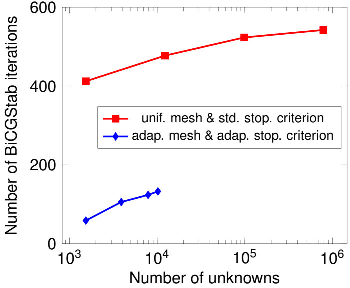


Dual problem (3rd mesh)

3D singular solution and adaptive mesh refinement





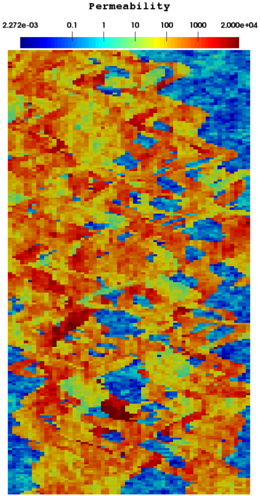


Goal error $\left|Q(u) - Q(u_h^{\bar{i}})\right|$

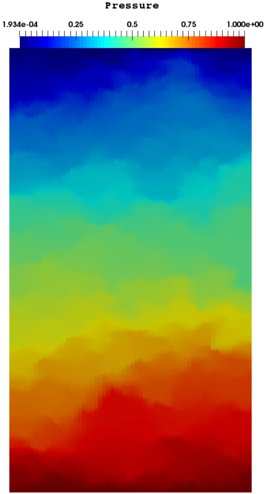
Eff. indices $\frac{\eta_h^{\bar{i}}\ \tilde{\eta}_h^{\bar{i}}}{2|Q(u)-Q(u_h^{\bar{i}})|}$

Algebraic solver iterations $\bar{i}$

2D heterogeneous media and uniform mesh refinement



SPE10 permeability $\underline{\boldsymbol{K}}$



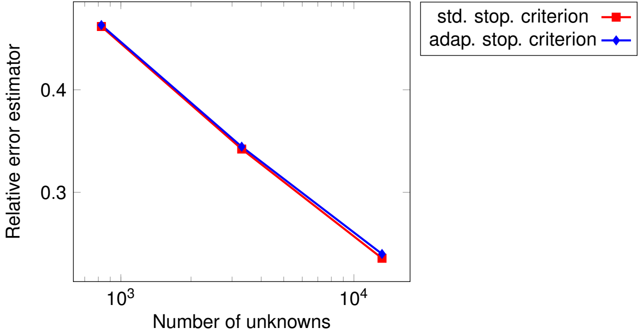
Pressure approx. $u_h^{\bar{i}}$

Outflow goal functional

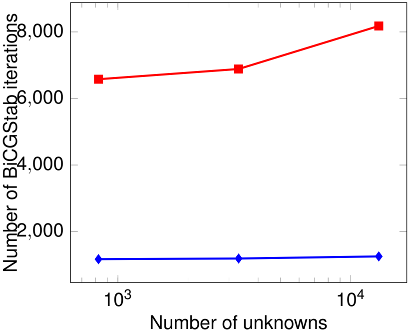
$$Q(v) = -\left(\underline{\boldsymbol{K}}\nabla v\cdot\boldsymbol{n}, \tilde{u}_{\mathrm{D}}\right)_{\Gamma_{\mathrm{D}}}$$

$$\tilde{u}_{\mathrm{D}}|_{\{y=0\}} = 0, \ \tilde{u}_{\mathrm{D}}|_{\{y=2200\}} = 1$$

2D heterogeneous media and uniform mesh refinement



Relative error estimator $\frac{\eta_h^{\bar{i}}\,\tilde{\eta}_h^{\bar{i}}}{2Q(u_h^{\bar{i}})}$



Algebraic solver iterations $\bar{i}$