Potential and flux reconstructions for optimal a priori and a posteriori error estimates

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Inria Paris & Ecole des Ponts

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Outline

A model partial differential equation

Poisson equation

Find $u : \Omega \to \mathbb{R}$

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Setting

- $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, line segment, Lipschitz polygon, or Lipschitz polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

$$u \in H_0^1(\Omega), \quad -\nabla u \in \mathbf{H}(\mathrm{div}, \Omega), \quad \nabla \cdot (-\nabla u) = f$$

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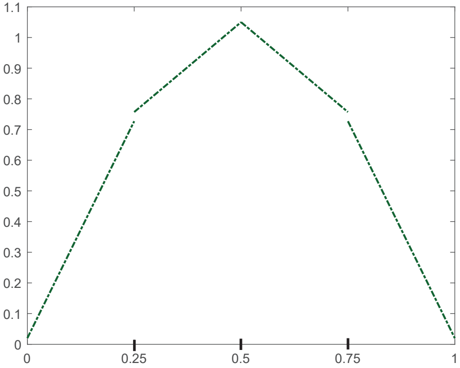
Numerical approximation

- $\mathcal{T}_h$ a simplicial mesh of Ω with characteristic mesh size $h := \max_{K \in \mathcal{T}_h} h_K$
- $\mathcal{P}_p(\mathcal{T}_h)$: piecewise polynomials of total degree $p \geq 0$
- numerical approximation u_h of u

Numerical approximation: potential in 1D



$$u_h \in \mathcal{P}_1(\mathcal{T}_h)$$

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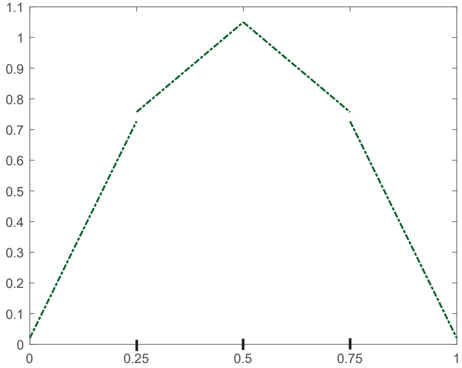

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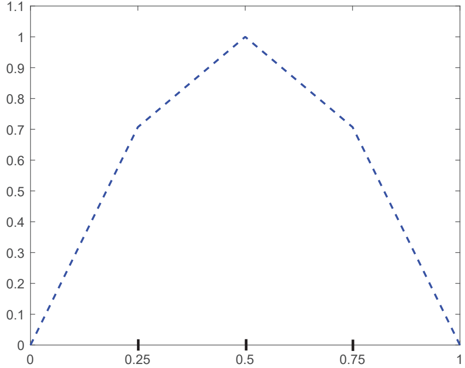
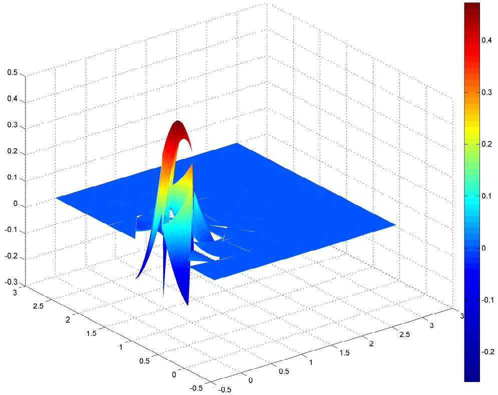
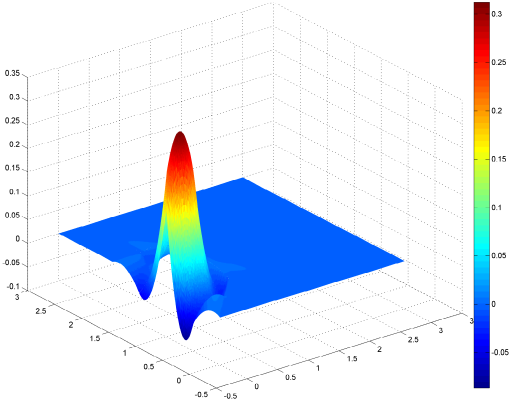


$u_h \in \mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

Numerical approximation: potential in 2D



$u_h \in \mathcal{P}_2(\mathcal{T}_h)$



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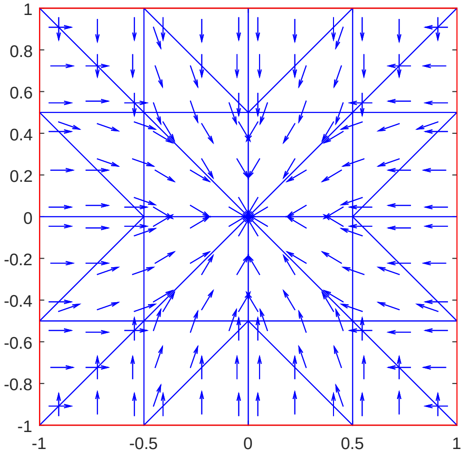
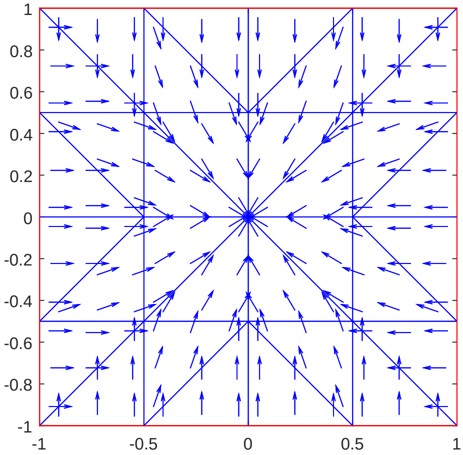
Numerical approximation: flux in 2D



$$-\nabla u_h \in [\mathcal{P}_0(\mathcal{T}_h)]^2$$

Numerical approximation: flux in 2D



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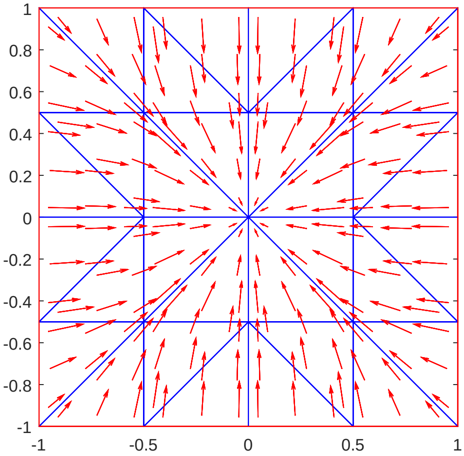

$-\nabla u_h \in \mathcal{RT}_1(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$

Error characterization

Theorem (Error equality)

Let $u \in H_0^1(\Omega)$ *be the weak solution and let* $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 0$, *be arbitrary. Then*

$$\|\nabla_h(u - u_h)\|^2 \qquad .$$

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$$\|\nabla_h(u - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla_h(u_h - v)\|^2}_{} + \underbrace{\min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div},\Omega) \\ \nabla\cdot\mathbf{v} = f}} \|\nabla_h u_h + \mathbf{v}\|^2}_{}.$$

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a posteriori error estimate, reliability

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- **local** construction of **piecewise polynomial** s_h and $\boldsymbol{\sigma}_h$ from u_h

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Theorem (Optimal a posteriori error estimate)

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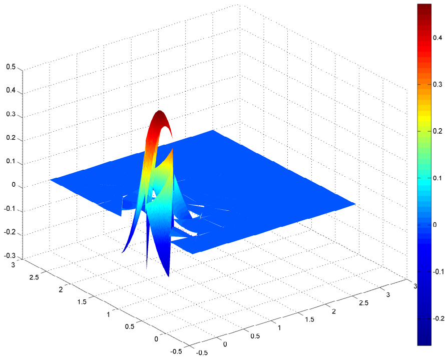
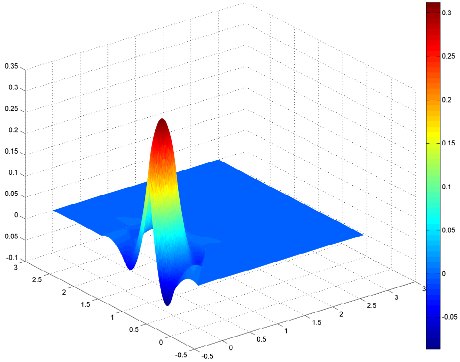
Potential reconstruction



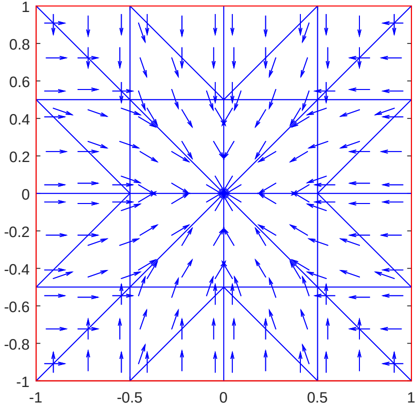
Potential ξ_h



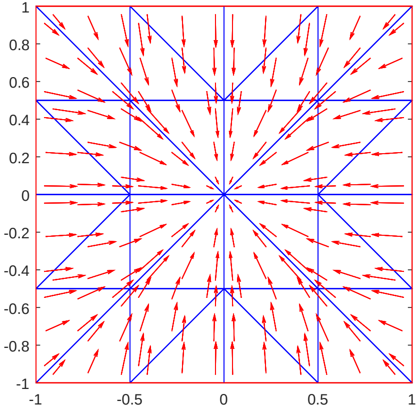
Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

flux reconstruction



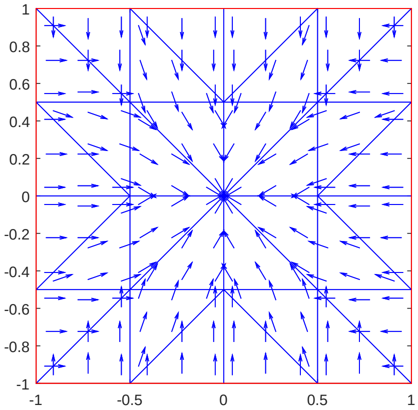
Flux ξ_h



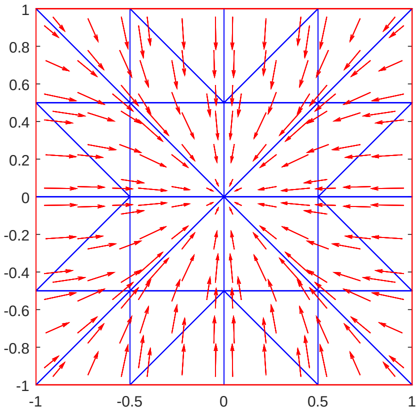
flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)} \to \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div}, \Omega)$$

Equilibrated flux reconstruction



Flux ξ_h



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}+(\xi_h,\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}=0\ \forall \mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$$

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Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

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Theorem (Optimal a priori error estimate)

There holds

$$\underbrace{\|\nabla(u - u_h)\|}_{\substack{\min\limits_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \\ \text{global-best}}}$$

a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$$

Theorem (Optimal a priori error estimate)

There holds

$$\underbrace{\|\nabla(u - u_h)\|}_{\substack{\min\limits_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \\ \text{global-best}}}$$

- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$

Optimal a priori error estimate elementwise, both h and p

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

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global-best

- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H^1_0(\Omega)$

a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

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- s_h: **potential reconstruction** of ξ_h: $s_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$

a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$$

Theorem (Optimal a priori error estimate)

There holds

$$\underbrace{\|\nabla(u - u_h)\|}_{\substack{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\| \\ \text{global-best}}} \leq \|\nabla(u - s_h)\|$$

- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H_0^1(\Omega)$
- s_h: **potential reconstruction** of ξ_h: $s_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$

Optimal a priori error estimate, elementwise, both h and p

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$$

Theorem (Optimal a priori error estimate)

There holds

$$\underbrace{\|\nabla(u - u_h)\|}_{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|} \leq \|\nabla(u - s_h)\| \leq \|\nabla_h(u - \xi_h)\| + \|\nabla_h(\xi_h - s_h)\|$$

global-best

- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H_0^1(\Omega)$
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Optimal a priori error estimate, elementwise, both h and p

Conforming finite element approximation

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global-best

- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H_0^1(\Omega)$
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- s_h so good that **no** $v \in H_0^1(\Omega)$ **can do better** (up to a constant)

Optimal a priori error estimate, **elementwise**, both h and p

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$$

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There holds

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$$= \Bigg\{ \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K^2}_{\substack{\text{local-best approximation of } u \text{ on each } K \\ \text{no interface constraints} \\ \text{regularity only in } K \text{ counts}}} \Bigg\}^{1/2}$$

- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H_0^1(\Omega)$
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Optimal a priori error estimate, elementwise, both h and p

Conforming finite element approximation

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$$= \Bigg\{ \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K^2}_{\substack{\text{local-best approximation of } u \text{ on each } K \\ \text{no interface constraints} \\ \text{regularity only in } K \text{ counts}}} \Bigg\}^{1/2} \lesssim \left(\frac{h}{p}\right)^p.$$

- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H_0^1(\Omega)$
- s_h: **potential reconstruction** of ξ_h: $s_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$
- s_h so good that **no** $v \in H_0^1(\Omega)$ **can do better** (up to a constant)

Optimal a priori error estimate, **elementwise**, both h and p

Theorem (Optimal a priori error estimate)

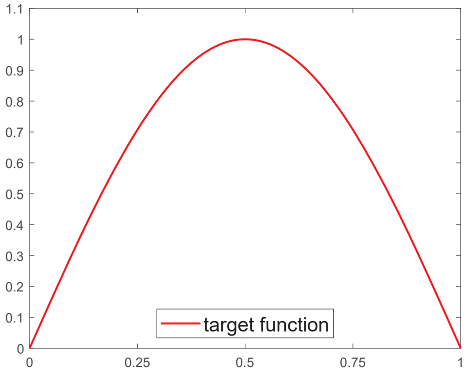
There holds

$$\underbrace{\|\nabla(u-u_h)\|}_{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h)\cap H_0^1(\Omega)} \|\nabla(u-v_h)\|} \lesssim \left\{ \sum_{K\in\mathcal{T}_h} \underbrace{\min_{v_h\in\mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K^2}_{} \right\}^{1/2}$$

global-best

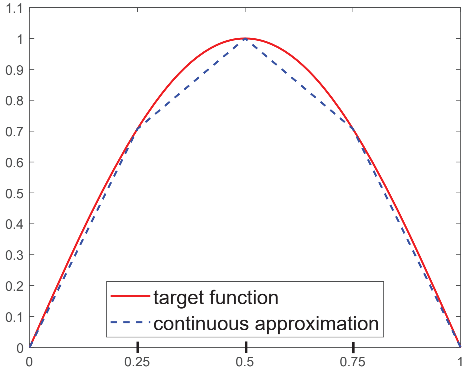
local-best approximation of u on each K
no interface constraints
regularity only in K counts

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



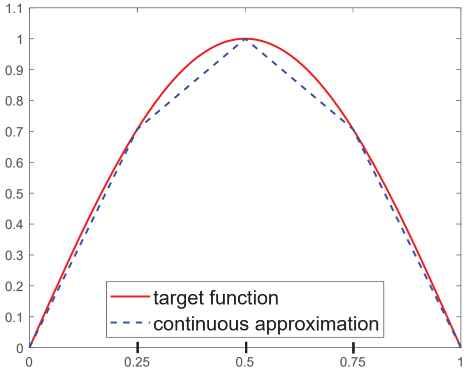
Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

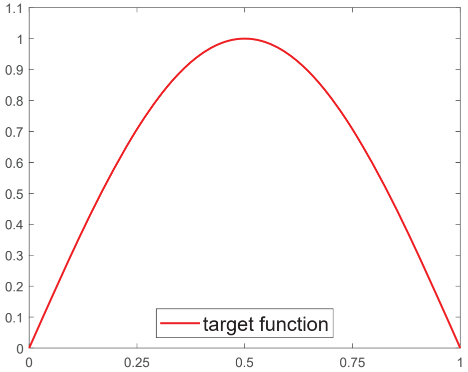


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global**-best

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

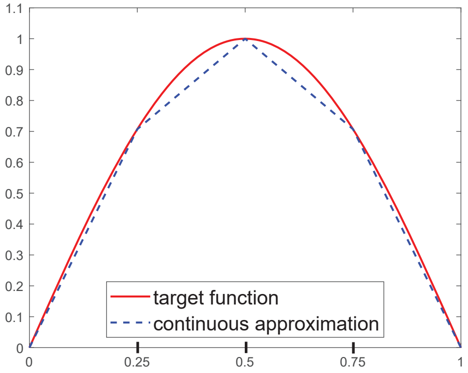


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global**-best

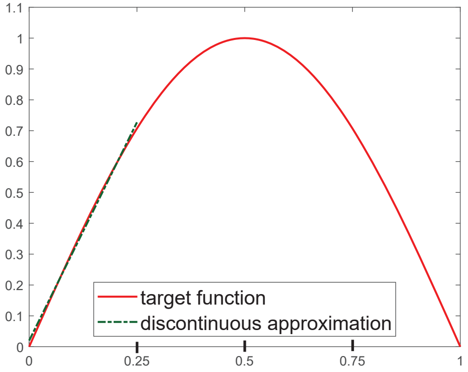


Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$: **1D**

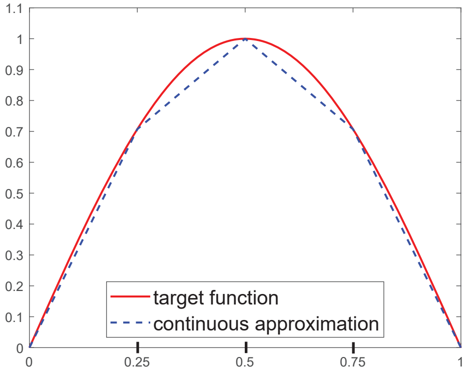


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H^1_0(\Omega)$, **global**-best

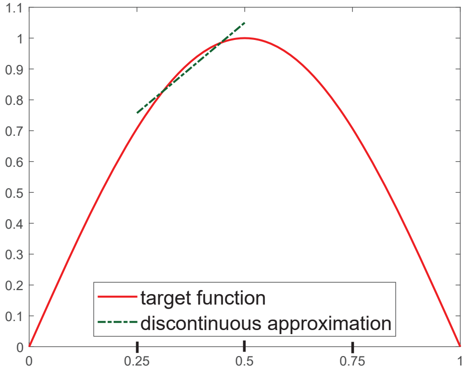


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local**-best

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

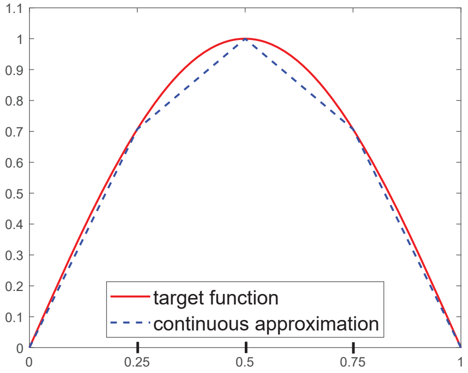


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global**-best

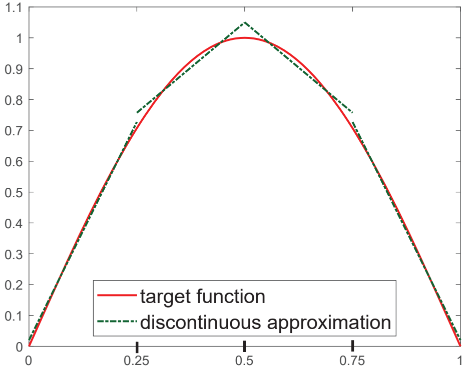


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local**-best

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$: **1D**

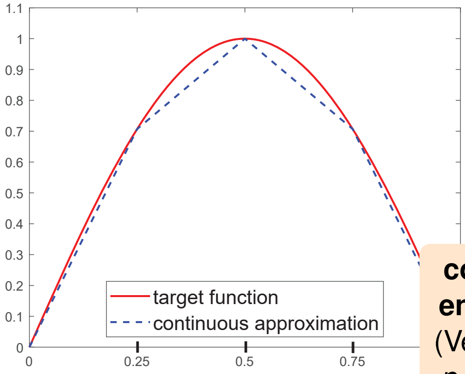


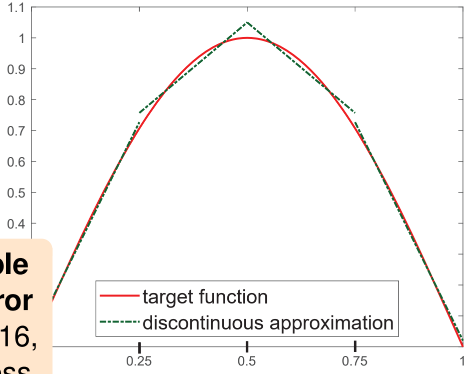
Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H^1_0(\Omega)$, **global**-best



Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local**-best

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**





comparable energy error
(Veeser 2016,
p-robustness
V. 2024)

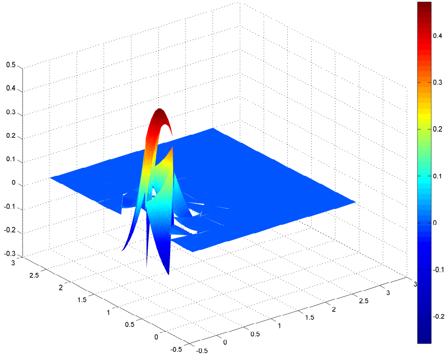
Best approximation by **continu**
piecewise polynomials in
$\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global**-best

approximation by **discontinuous**
piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$,
local-best

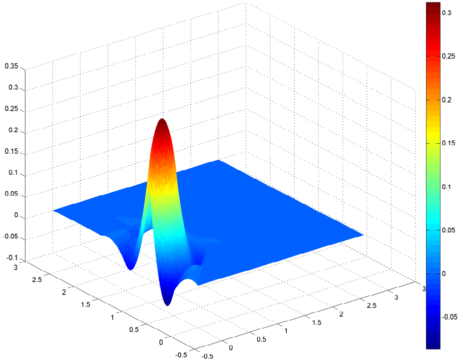
Outline

Outline

Potential reconstruction



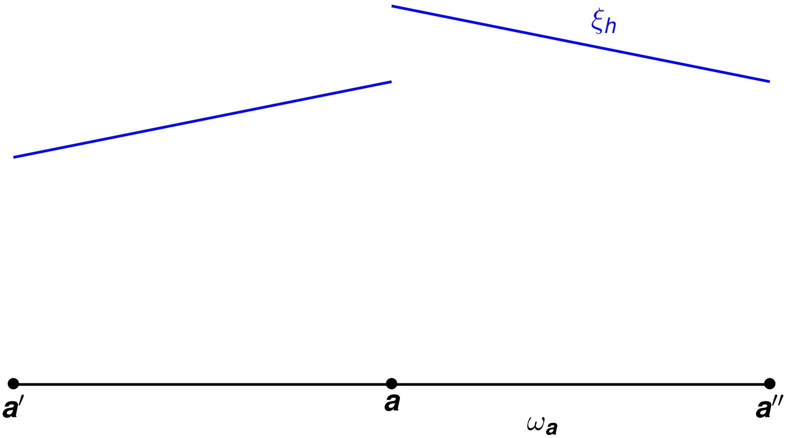
Potential ξ_h



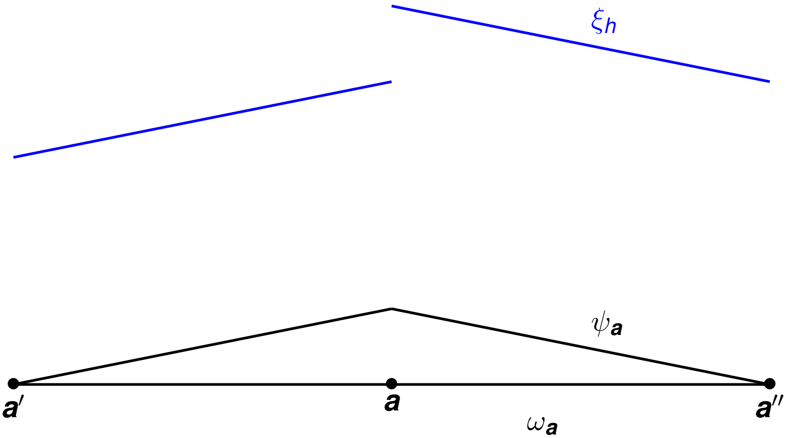
Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

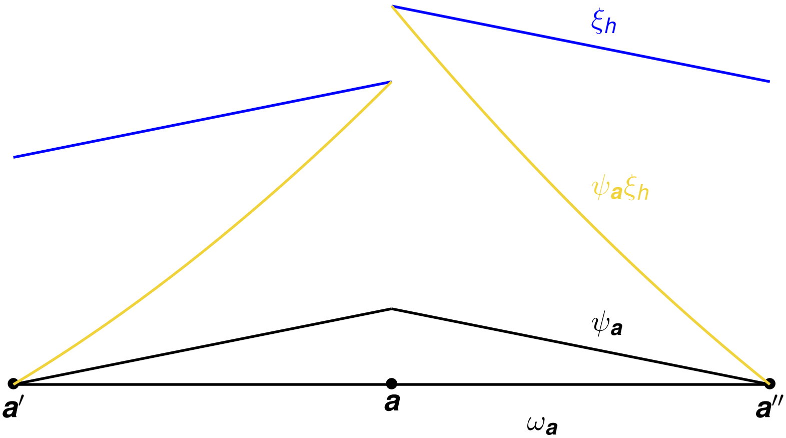
Potential reconstruction in 1D, $p = 1$, $p' = 2$



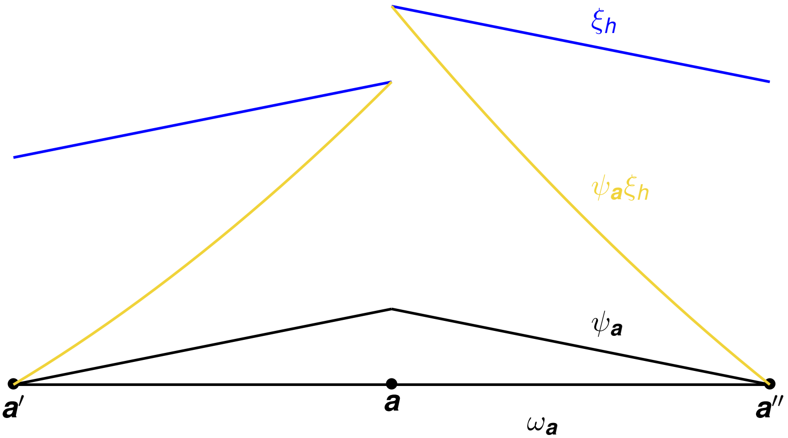
Potential reconstruction in 1D, $p = 1$, $p' = 2$



Potential reconstruction in 1D, $p = 1$, $p' = 2$



Potential reconstruction in 1D, $p = 1$, $p' = 2$



Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013)**)**

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathcal{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(\psi_{\boldsymbol{a}} \xi_h - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}}$$

Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla_h(\psi_{\boldsymbol{a}} \xi_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_{\boldsymbol{a}}$: $s_h \in \mathcal{P}_{p'}(\mathcal{T}_h) \cap H_0^1(\Omega)$
- $p' = p + 1$

Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local minimization problem**

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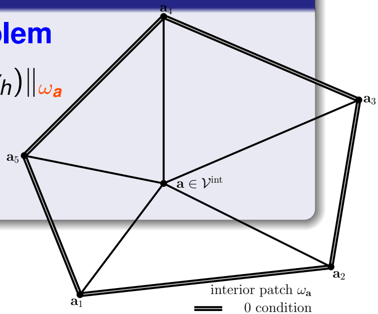

Equivalent form: conforming FEs

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Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
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- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
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Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

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and combine

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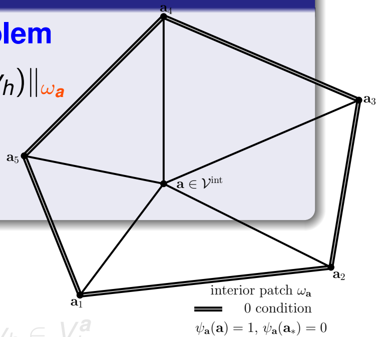

Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

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Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
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- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
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Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), $\approx$ Carstensen and Merdon (2013))

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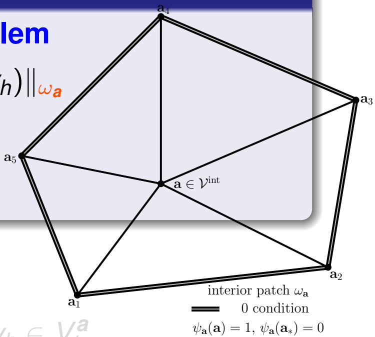
and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}}.$$



Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla_h I_{p'}(\psi_{\boldsymbol{a}} \xi_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
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- $p' = p + 1$

Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013)**)**

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local minimization problem**

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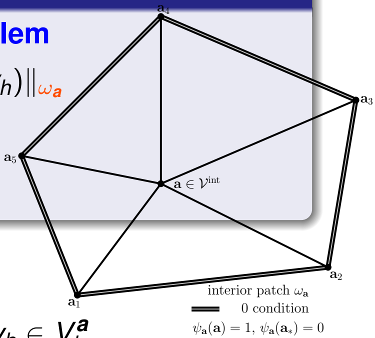

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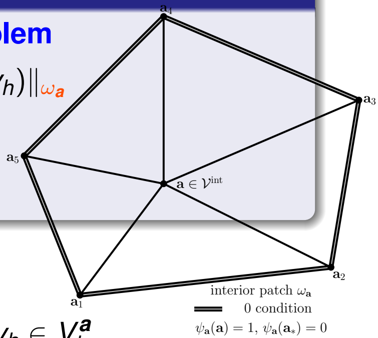

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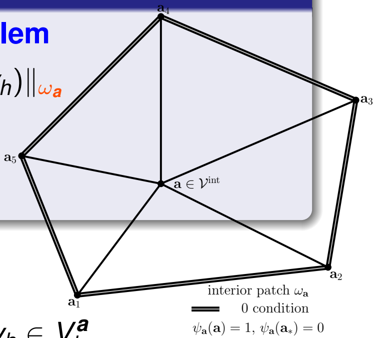

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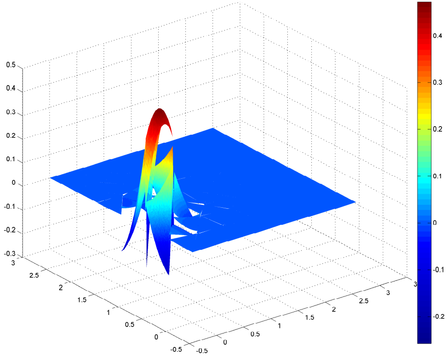
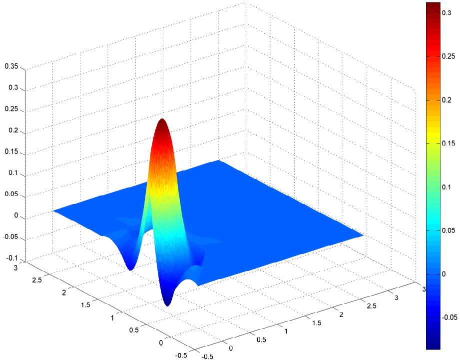
Potential reconstruction



Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H^1_0(\Omega)$$

Stability of the potential reconstruction

Theorem (Local stability Ern & V. (2015, 2020), using ▸ Tools**)**

There holds

$$\min_{v_h \in \mathcal{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H^1_0(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}} \lesssim \min_{v \in H^1_0(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v)\|_{\omega_{\mathbf{a}}}.$$

Stability of the potential reconstruction

Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

s_h so good that no $u \in H_0^1(\Omega)$ can do better

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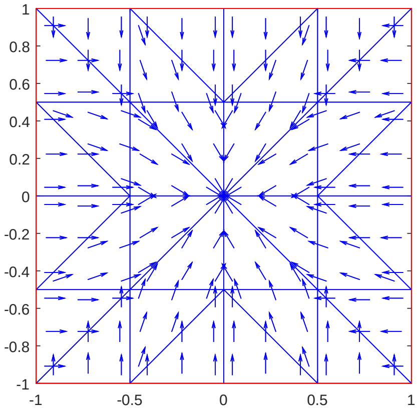
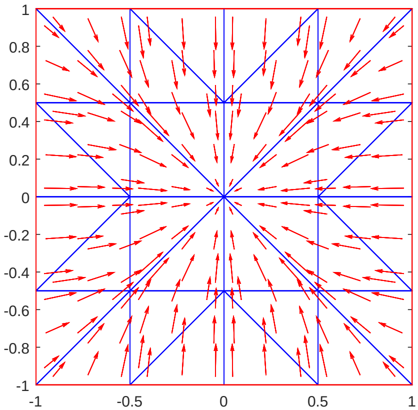
Outline

Equilibrated flux reconstruction



Flux ξ_h



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0 \ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

Flux reconstruction: $\xi_h \in \mathcal{RT}_p(\mathcal{T}_h)$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

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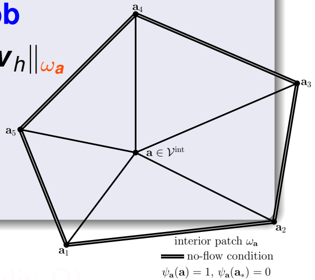

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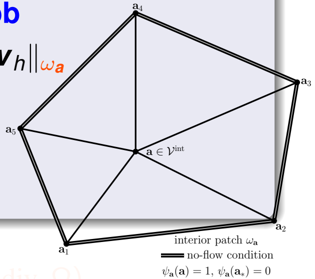

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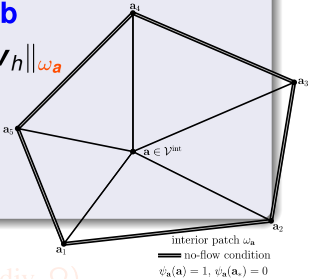

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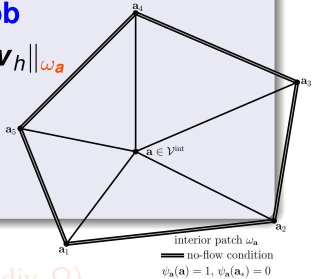

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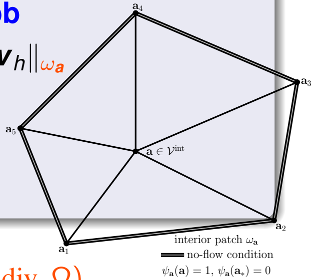

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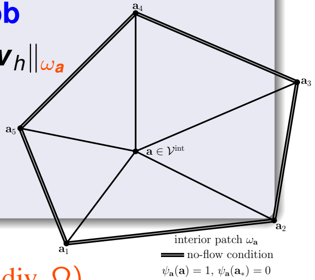

Key points

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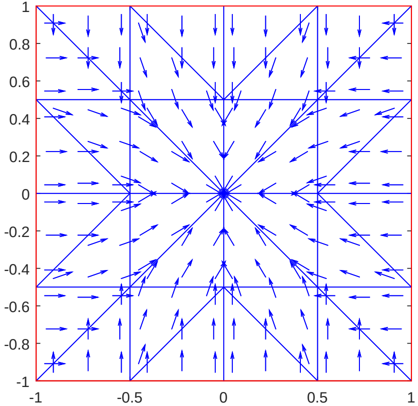
- divergence-constrained projection of the discontinuous $\psi_{\mathbf{a}} \xi_h$ to conf. space
- **equilibrium** $\nabla \cdot \boldsymbol{\sigma}_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}} = \sum_{\mathbf{a} \in \mathcal{V}_h} \Pi_{p'}(f \psi_{\mathbf{a}} + \xi_h \cdot \nabla \psi_{\mathbf{a}}) = \Pi_{p'} f$
- $p' = p + 1$ or $p' = p$

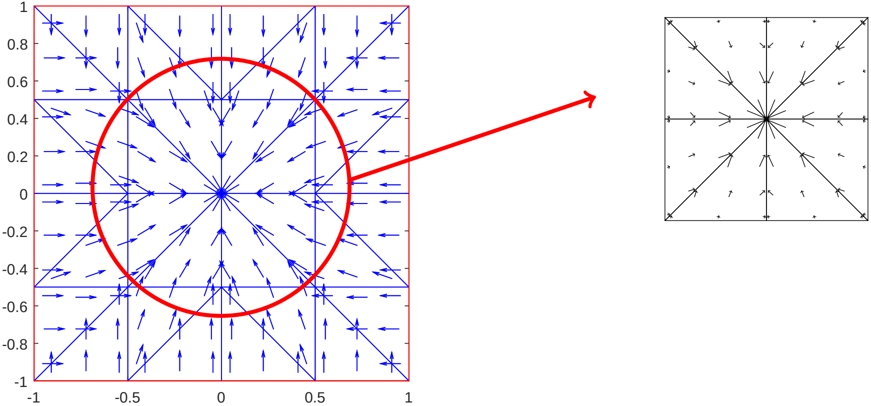
Equilibrated flux reconstruction



Flux ξ_h

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

Equilibrated flux reconstruction

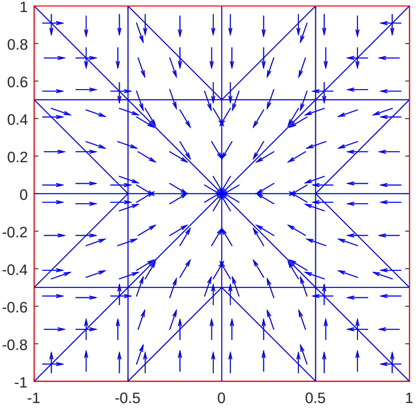


Flux ξ_h

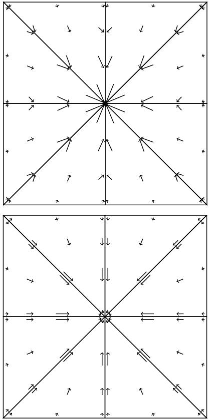
$\psi_{\boldsymbol{a}} \xi_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\xi_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}}$$

Equilibrated flux reconstruction



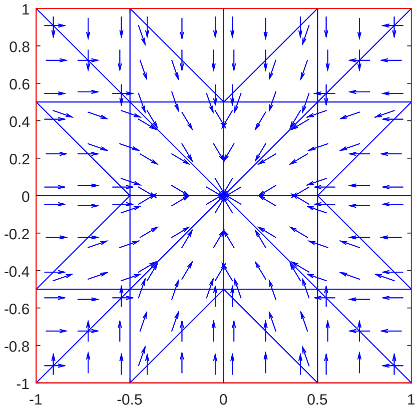
Flux ξ_h



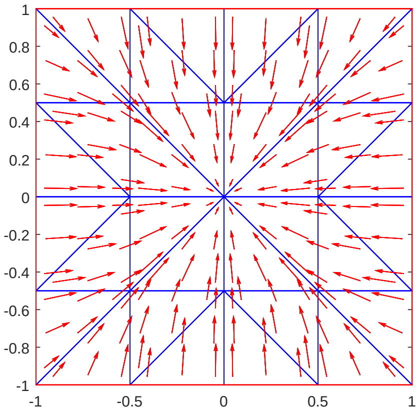
$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}} := \mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \xi_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\xi_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

Equilibrated flux reconstruction



Flux ξ_h



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \rightarrow \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

Stability of the flux reconstruction

Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), Ern & V. (2020; 3D), using ▸ Tools**)**

There holds

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h) - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Stability of the flux reconstruction

Corollary (Global stability; $p' = p + 1$)

$\boldsymbol{\sigma}_h$ *is closer to* $\boldsymbol{\xi}_h$ *than any* $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \boldsymbol{\sigma} = f$:

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

$\boldsymbol{\sigma}_h$ so good that no $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{\sigma} = f$ can do better

Stability of the flux reconstruction

Corollary (Global stability; $p' = p$ after a p-robust correction)

$\boldsymbol{\sigma}_h$ is closer to ξ_h than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla\cdot\boldsymbol{\sigma} = f$:

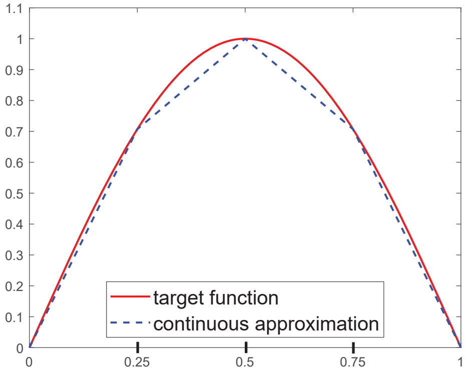
$$\|\xi_h - \boldsymbol{\sigma}_h\| \lesssim \|\xi_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

$\boldsymbol{\sigma}_h$ so good that no $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{\sigma} = f$ can do better

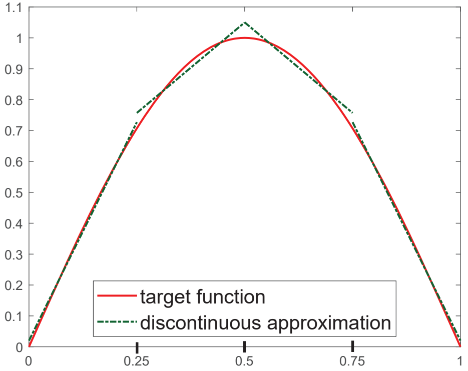
Outline

Outline

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D

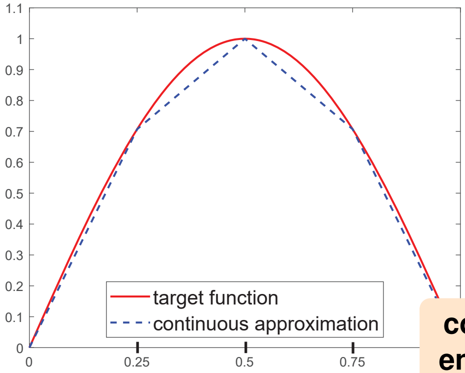


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global**-best

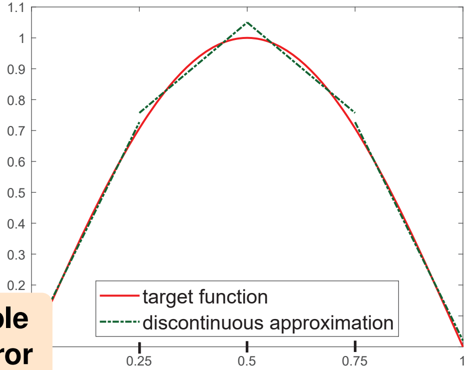


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local**-best

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



comparable energy error



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global**-best

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local**-best

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

$$\min_{\textit{smaller space}} \approx_p \min_{\textit{bigger space}}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

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$$\min_{\textit{CG space}} \approx_p \min_{\textit{DG space}}$$

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$

Theorem (Equivalence in H^1_0, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), V. (2024))

Let $u \in H^1_0(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \underbrace{\sum_{K \in \mathcal{T}_h} \min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|^2_K}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx_p$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$, and polynomial degree p
- proof taking $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}_h$, applying with $p' = p$, and using its

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Optimal a priori error estimate

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity)

Let $v \in H_0^1(\Omega)$ with

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$.

- $P_h^p : H_0^1(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$: a locally defined projector
- $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$: smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

Optimal a priori error estimate

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity)

Let $v \in H^1_0(\Omega)$ with

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for $s_K \geq 1$. Then

$$\|\nabla(v - P_h^p v)\|_K^2 \leq C(\kappa_{\mathcal{T}_h}, \kappa_p, d, s) \sum_{L \in \widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(\underline{p}_K, s_L - 1)}}{\underline{p}_K^{s_L - 1}} \|v\|_{H^{s_L}(L)} \right)^2 \qquad \forall K \in \mathcal{T}_h.$$

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Outline

Commuting de Rham diagram

Commuting de Rham diagram

$$H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

Commuting de Rham diagram

Commuting de Rham diagram

$$H^1_{0,\mathrm{N}}(\Omega) \quad \xrightarrow{\nabla} \quad \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \quad \xrightarrow{\nabla\times} \quad \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \quad \xrightarrow{\nabla\cdot} \quad L^2_*(\Omega)$$

$$\mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega)$$

Commuting de Rham diagram

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

Commuting de Rham diagram: operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

Commuting de Rham diagram: operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram: operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$
\begin{array}{ccc}
\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\Big\downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \Big\downarrow \Pi_h^p \\
\mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

Properties of $\boldsymbol{P}_h^{p,\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$** (**minimal regularity**, partial BCs)

Commuting de Rham diagram: operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc}
\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
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\end{array}$$

Properties of $\boldsymbol{P}_h^{p,\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$** (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)

Commuting de Rham diagram: operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_h^{p,\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$** (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)
3. is defined simply, starting from the **elementwise L^2-orthogonal projection**

Commuting de Rham diagram: operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

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1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$** (**minimal regularity**, partial BCs)
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Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
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- Licht (2019): essential boundary conditions on a part of $\partial\Omega$
- Arnold and Guzmán (2021): $\boldsymbol{L}^2$-stable
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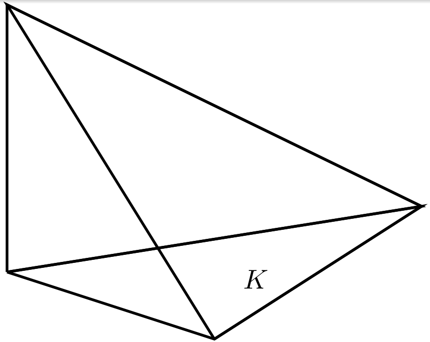
Classical elementwise interpolation



- $\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$
- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \Rightarrow \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{div}, K) \Rightarrow$ so interpolate $\boldsymbol{v}|_K$, elementwise interpolation

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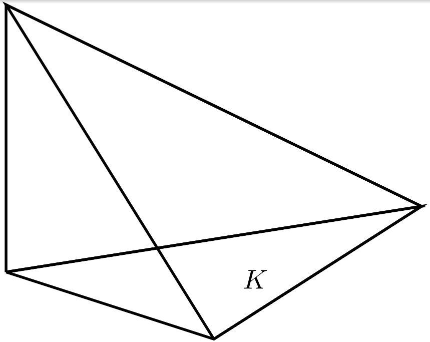

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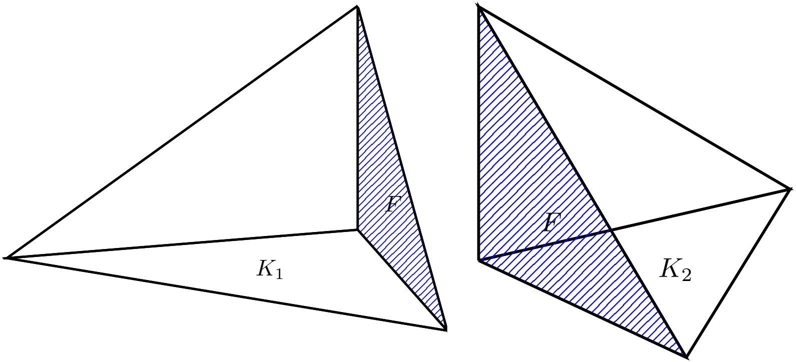
Classical elementwise interpolation: conformity enforcement



- $\boldsymbol{v}_h \in \mathcal{RT}_p(K_1, K_2) \cap \boldsymbol{H}(\mathrm{div}, K_1 \cup K_2)$ iff $\boldsymbol{v}_h|_{K_1} \in \mathcal{RT}_p(K_1)$, $\boldsymbol{v}_h|_{K_2} \in \mathcal{RT}_p(K_2)$, and $(\boldsymbol{v}_h|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}_h|_{K_2} \cdot \boldsymbol{n}_F)|_F$

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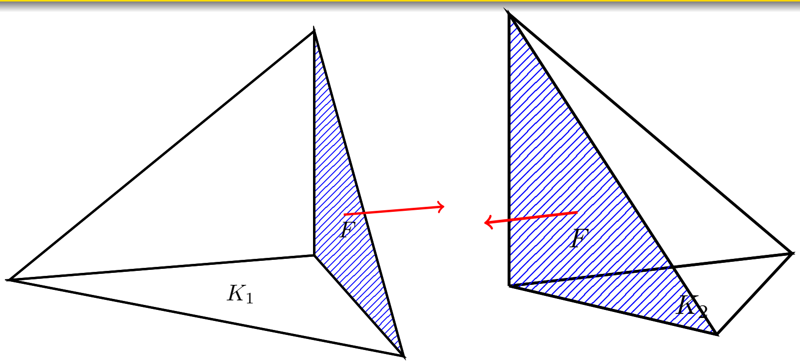

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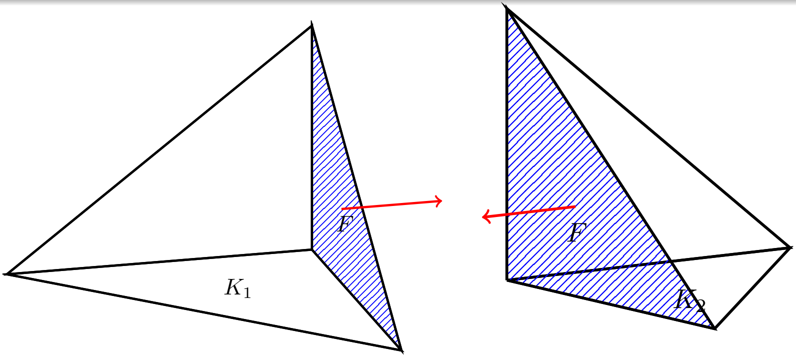

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Clash

Face normal trace integrals $\langle \boldsymbol{v} \cdot \boldsymbol{n}_F, 1 \rangle_F / |F|$ not available in $\boldsymbol{H}(\mathrm{div})$.

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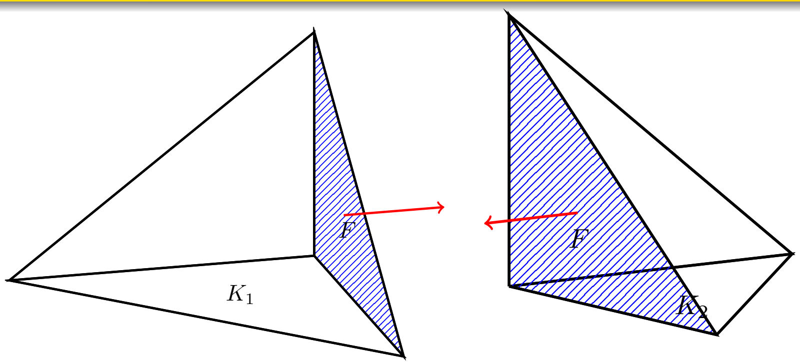

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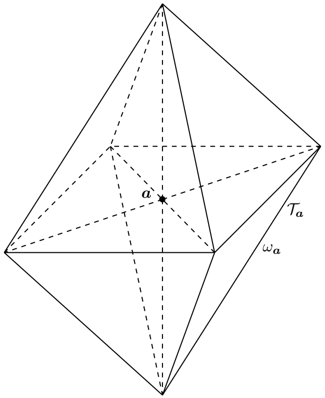
Conclusion

Not a single tetrahedron $K \in \mathcal{T}_h$ if the minimal regularity $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ requested.

Classical patchwise interpolation (Clément)



- some local-best polynomial approximation on $\omega_{\boldsymbol{a}}$
- values on $\omega_{\boldsymbol{a}}$ as coefficients for basis functions supported on $\omega_{\boldsymbol{a}}$

Conclusion

Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

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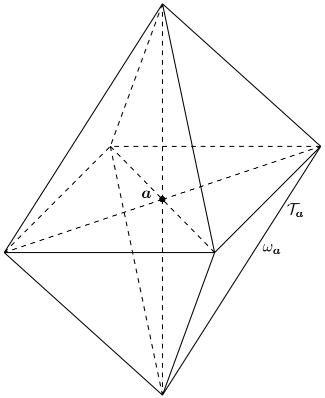

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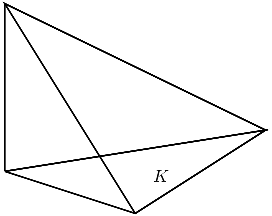

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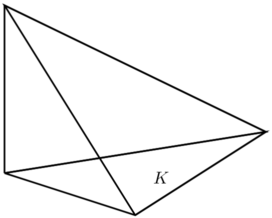

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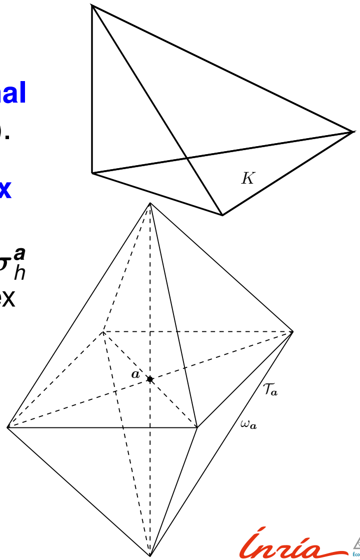

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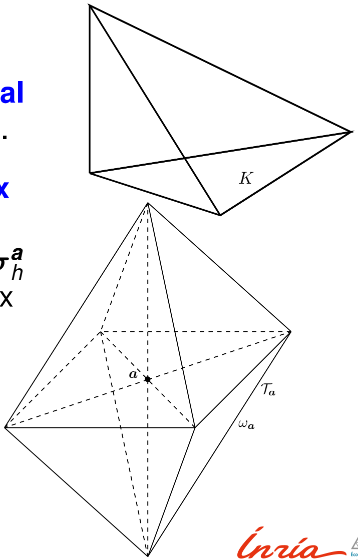

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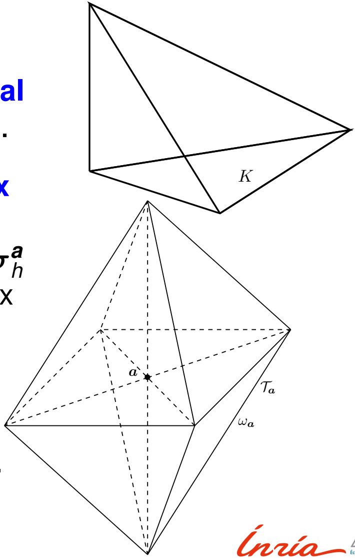
3. Apply a **p-stable decomposition** on extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$ to conforming projections of the reminder $\xi_h - \boldsymbol{\sigma}_h \Rightarrow$ correction ζ_h; $\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) := \boldsymbol{\sigma}_h + \zeta_h$.



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3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h, \boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

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(patchwise **flux equilibration** with an **additional orthogonality constraint**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_{h,\mathrm{alg}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h, \boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

(patchwise divergence-free remainder equilibration with an additional constraint)

$$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}'\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}'} \quad \text{with in particular} \quad \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$$

(patchwise p-**stable** remainder **decomposition**)

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

1. $\boldsymbol{\xi}_h|_K := \arg\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_{h,\mathrm{alg}} := \arg\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi_{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi_{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p \geq 1}} \left\| \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}_h \right\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_{h,\mathrm{alg}}$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

   (patchwise divergence-free remainder equilibration with an additional constraint)

   $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}' \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}'} \text{ with in particular } \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$$

   (patchwise p-**stable** remainder **decomposition**)

   $\boldsymbol{\zeta}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}$ (gluing patchwise correction contributions)

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}} := \boldsymbol{\sigma}_h + \boldsymbol{\zeta}_h$

1. $\boldsymbol{\xi}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_{h,\mathrm{alg}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi_{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi_{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d,\, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p\geq 1}} \left\|\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_{h,\mathrm{alg}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d,\, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

   (patchwise divergence-free remainder equilibration with an additional constraint)

   $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}'\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}'} \text{ with in particular } \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$$

   (patchwise ***p*-stable** remainder **decomposition**)

   $$\boldsymbol{\zeta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}$$ (gluing patchwise correction contributions)

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

Theorem ($\boldsymbol{P}_h^{p,\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_h^{p,\mathrm{div}}$ *is* ***commuting*** *since*

$$\nabla\cdot\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

Theorem ($\boldsymbol{P}_h^{p,\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

*$\boldsymbol{P}_h^{p,\mathrm{div}}$ is **commuting** and **projector** since*

$$\begin{aligned}
\nabla\cdot\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) &= \Pi_h^p(\nabla\cdot\boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),\\
\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) &= \boldsymbol{v} && \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega).
\end{aligned}$$

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

Theorem ($\boldsymbol{P}_h^{p,\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

*$\boldsymbol{P}_h^{p,\mathrm{div}}$ is **commuting** and **projector** since*

$$\nabla\cdot\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$

$$\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega).$$

*It has **p-robust local-best approximation property***

since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $K \in \mathcal{T}_h$,

$$\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\cdot(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}))\|_K\right)^2$$

$$\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \left\{ \min_{\boldsymbol{v}_h\in\mathcal{RT}_p(L)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2 \right\},$$

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

Theorem ($\boldsymbol{P}_h^{p,\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_h^{p,\mathrm{div}}$ *is* ***commuting*** *and* ***projector*** *since*

$$\nabla\cdot\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$
$$\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega).$$

It has ***p-robust local-best approximation property*** *and is* ***p-robustly L^2 stable*** *up to data oscillation, since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $K \in \mathcal{T}_h$,*

$$\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\cdot(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}))\|_K\right)^2$$
$$\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K}\left\{\min_{\boldsymbol{v}_h\in\mathcal{RT}_p(L)}\|\boldsymbol{v} - \boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2\right\},$$

$$\|\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K^2 \lesssim \sum_{L\in\widetilde{\mathcal{T}}_K}\left\{\|\boldsymbol{v}\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2\right\}.$$

Outline

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021))

bigger $\approx_p$ *smaller*

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021))

$$\min_{\textit{smaller space with constraints}} \approx_p \min_{\textit{bigger space without constraints}}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021))

$$\min_{\textit{MFE space with constraints}} \approx_p \min_{\textit{broken MFE space without constraints}}$$

Global-best approximation $\approx$ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021) Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot \boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v}-\Pi_p\nabla\cdot\boldsymbol{v}\|_K^2}_{\text{global-best on } \Omega}$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h\in\mathcal{RT}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v}-\Pi_p\nabla\cdot\boldsymbol{v}\|_K^2\right]}_{\text{local-best on each } K}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021) Demkowicz & V. (2024))

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$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

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$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p
- proof using flux reconstruction with $p' = p$ & $\boldsymbol{H}(\mathrm{div})$ stability

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_{\not p}$: only depends on d, shape-regularity of $\mathcal{T}_h$, ~~and p~~
- proof using flux reconstruction with $p' = p$ & $\boldsymbol{H}(\mathrm{div})$ stability

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_{\not p}$: only depends on d, shape-regularity of $\mathcal{T}_h$, ~~and p~~
- proof using ▸ flux reconstruction with $p' = p$ & ▸ $\boldsymbol{H}(\mathrm{div})$ stability

Outline

Optimal a priori error estimate

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla\cdot\boldsymbol{v})|_K \in [\mathcal{P}_p(K)]^3 \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$.

- varying polynomial degree can be added

Optimal a priori error estimate

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla{\cdot}\boldsymbol{v})|_K \in [\mathcal{P}_p(K)]^3 \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla{\cdot}\boldsymbol{v}_h = \nabla{\cdot}\boldsymbol{v}}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s)^2 \sum_{K \in \mathcal{T}_h} \left(\frac{h_K^{\min\{p+1, s_K\}}}{(p+1)^{s_K}} \|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)} \right)^2.$$

- varying polynomial degree can be added

Optimal a priori error estimate

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla \cdot \boldsymbol{v})|_K \in \boldsymbol{H}^{t_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ *and* $t_K \geq 0$.

- varying polynomial degree can be added

Optimal a priori error estimate

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla\cdot\boldsymbol{v})|_K \in H^{t_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ *and* $t_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h=\Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p(\nabla\cdot\boldsymbol{v})\|_K\right)^2\Bigg]$$

$$\leq C(\kappa_{\mathcal{T}_h},d,s,t)^2\sum_{K\in\mathcal{T}_h}\left[\left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2+\left(\frac{h_K}{p+1}\frac{h_K^{\min\{p+1,t_K\}}}{(p+1)^{t_K}}\|\nabla\cdot\boldsymbol{v}\|_{H^{t_K}(K)}\right)^2\right].$$

- varying polynomial degree can be added

Optimal a priori error estimate

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla\cdot\boldsymbol{v})|_K \in H^{t_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ *and* $t_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p(\nabla\cdot\boldsymbol{v})\|_K\right)^2]$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s, t)^2 \sum_{K\in\mathcal{T}_h}\left[\left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2 + \left(\frac{h_K}{p+1}\frac{h_K^{\min\{p+1,t_K\}}}{(p+1)^{t_K}}\|\nabla\cdot\boldsymbol{v}\|_{H^{t_K}(K)}\right)^2\right].$$

- varying polynomial degree can be added

Outline

Outline

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Theorem (A guaranteed a posteriori error estimate Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), V. (2007), . . .)

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
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$$(\nabla_h u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}};$$

- *$\xi_h := u_h$: $s_h \in \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_0^1(\Omega)$* potential reconstruction;
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Theorem (Polynomial-degree-robust efficiency; $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$ for simplicity Braess, Pillwein, and Schöberl (2009), Ern & V. (2015, 2020))

Let $u \in H^1_0(\Omega)$ be the weak solution. Then

$$\|\nabla_h(u_h - s_h)\| \lesssim \|\nabla_h(u - u_h)\| + \left\{ \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![u_h]\!]\|_F^2 \right\}^{1/2},$$

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Remarks

- immediate consequence of H^1 stability and $\boldsymbol{H}(\mathrm{div})$ stability with $p' = p + 1$
- p-robustness
- local efficiency on patches
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Outline

How **large** is the error? (numerical simulation, known solution)

$h \approx 1/\lvert\mathcal{T}_\ell\rvert^{\frac{1}{2}}$	p	relative error estimate $\frac{\eta(u_h)}{\lVert\nabla u_h\rVert}$	relative error $\frac{\lVert\nabla(u-u_h)\rVert}{\lVert\nabla u_h\rVert}$	effectivity index $\frac{\eta(u_h)}{\lVert\nabla(u-u_h)\rVert}$
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A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

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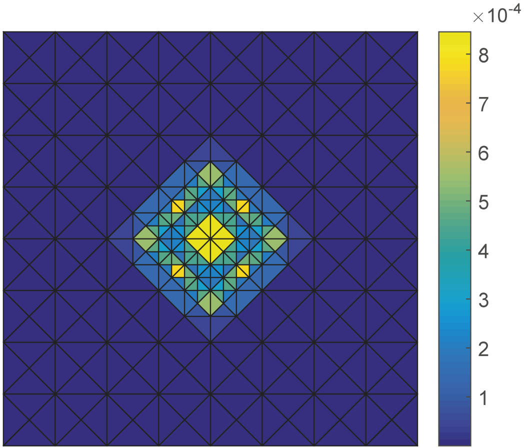
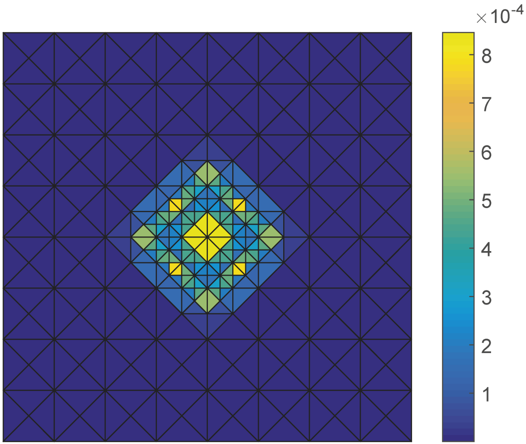
Where (in space) is the error **localized**?



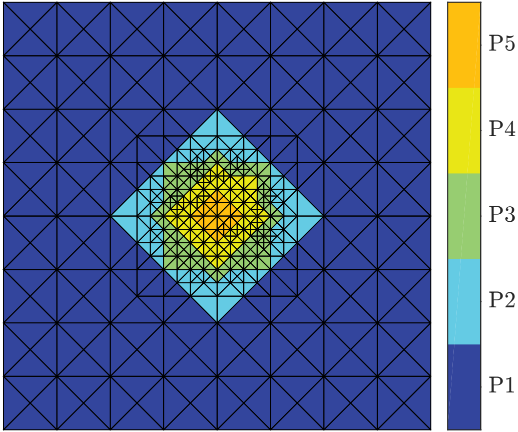
Estimated error distribution $\eta_K(u_h)$



Exact error distribution $\|\nabla(u - u_h)\|_K$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently? *hp* adaptivity, (**smooth** solution)



Mesh $\mathcal{T}_\ell$ and pol. degrees p_K

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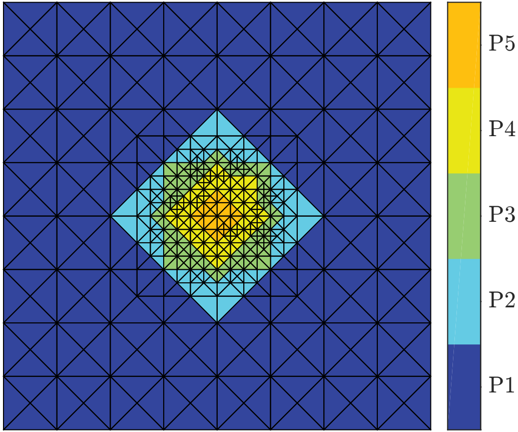

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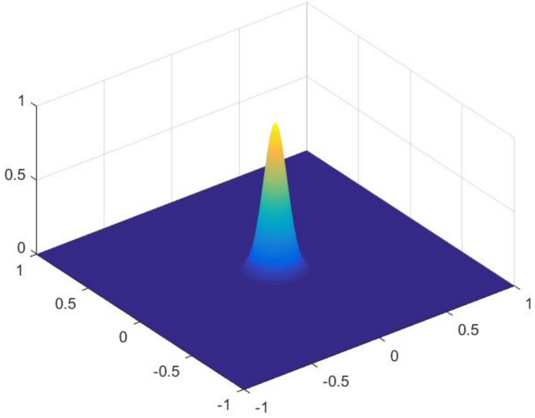

Exact solution

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

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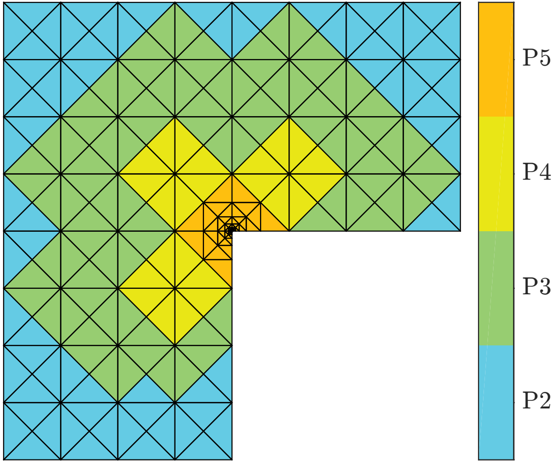

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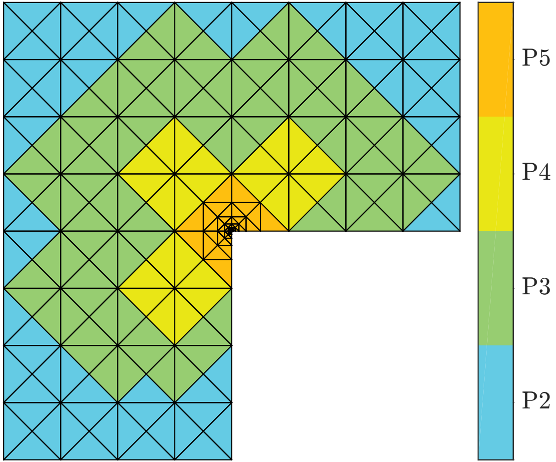

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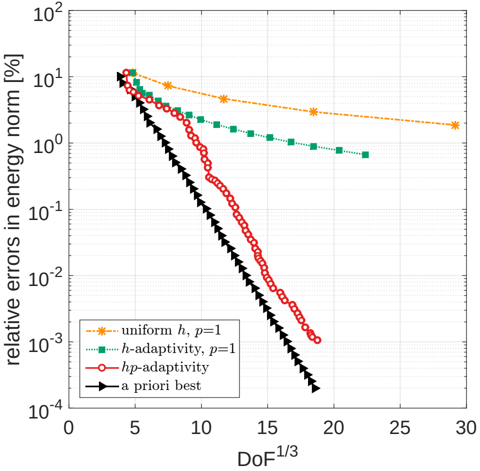

Relative error as a function of DoF

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Outline

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Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

Let $p \geq 1$, $K \in \mathcal{T}_h$, *and* $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. *Let* $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{D}})$ *be continuous on* $\mathcal{F}_K^{\mathrm{D}}$. *Then*

$$\min_{\substack{v_h \in \mathcal{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathcal{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Potentials: vertex patch

Theorem (Broken H^1 polynomial extension on a vertex patch Ern & V. (2015, 2020)**)**

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, let $r \in \mathcal{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}})$. Suppose the compatibility

$$
\begin{aligned}
r_F|_{F \cap \partial\omega_{\boldsymbol{a}}} &= 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}},\\
\sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e &= 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}.
\end{aligned}
$$

Then

$$
\min_{\substack{v_h \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})\\ v_h = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}}\\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\boldsymbol{a}})\\ v = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}}\\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.
$$

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2020)**)**

Let $p \geq 0$, $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K \quad .$$

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$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K \quad .$$

Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

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$$\|\boldsymbol{\varphi}_{h,K}\|_K \overset{MFEs}{=} \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K = \|\boldsymbol{\varphi}_K\|_K.$$

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$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

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Fluxes: vertex patch

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a vertex patch Braess, Pillwein, & Schöberl (2009; 2D), Ern & V. (2020; 3D))

For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, let $r \in \mathcal{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Outline

$\boldsymbol{H}(\mathrm{div})$ stable decomposition

Theorem ($\boldsymbol{H}(\mathrm{div})$ stable decomposition in 2D; in extension of Schöberl, Melenk, Pechstein, & Zaglmayr (2008))

Let $d = 2$ and let $\overline{\Omega}$ be contractible. Let

$$\boldsymbol{\delta}_p \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \quad \text{with} \quad \nabla{\cdot}\boldsymbol{\delta}_p = 0, \qquad \textit{div-free}$$

$$(\boldsymbol{\delta}_p, \boldsymbol{r}_h)_K = 0 \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d,\ \forall K \in \mathcal{T}_h. \qquad \textit{vanishing means}$$

Then there exists a decomposition of $\boldsymbol{\delta}_p$ as

$$\boldsymbol{\delta}_p = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_p^{\boldsymbol{a}}, \qquad \textit{decomposition}$$

where

$\boldsymbol{\delta}_p^{\boldsymbol{a}}$ are supported on the vertex patch subdomains $\omega_{\boldsymbol{a}}$, linearly depend on $\boldsymbol{\delta}_p$ on the extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$,

and satisfy

$$\boldsymbol{\delta}_p^{\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \quad \text{with} \quad \nabla{\cdot}\boldsymbol{\delta}_p^{\boldsymbol{a}} = 0, \qquad \textit{local}$$

$$\|\boldsymbol{\delta}_p^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\delta}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \qquad \textit{p-stable}$$

Outline

Conclusions and outlook

Conclusions

- p-stable local commuting projectors
- p-robust global-best – local-best equivalence in H^1
- p-robust global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$, removing constraints
- optimal *hp* localized a priori error estimates under minimal elementwise regularity
- p-robust a posteriori error estimates (unified framework for all classical numerical schemes)
- extensions to nonmatching meshes (robust wrt number of hanging nodes), mixed parallelepipedal–simplicial meshes, varying polynomial degree, general BCs, H^{-1} source terms, splines and IGA, and others carried out

Ongoing work

- extensions to other settings

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Thank you for your attention!

Outline

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\rightarrow$ continuous pw polynomial ▸ potential reconstruction

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- discontinuous pw polynomial $\rightarrow$ continuous pw polynomial ▸ potential reconstruction
- a posteriori analysis of mixed and nonconforming FEs:

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estimate ≈ error: guaranteed & p-robust bounds Braess, Pillwein, Schöberl (2009), Ern, V. (2015, 2020)

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Outline

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The Poisson model problem and its Galerkin approximation

The Poisson problem

Find $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, *such that*

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Weak formulation

Find $u \in H_0^1(\Omega)$ *such that*

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \text{for all } v \in H_0^1(\Omega).$$

Galerkin approximation

Find $u_h \in V_h \subset H_0^1(\Omega)$ *such that*

$$V_h = Q^p(\mathcal{T}_h) \cap C^{p-m}(\Omega)$$

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Outline

8 Potential and flux reconstructions

9 Application to IGA
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Partition of unity, $V_h = \mathcal{Q}^p(\mathcal{T}_h) \cap C^{p-1}(\Omega)$

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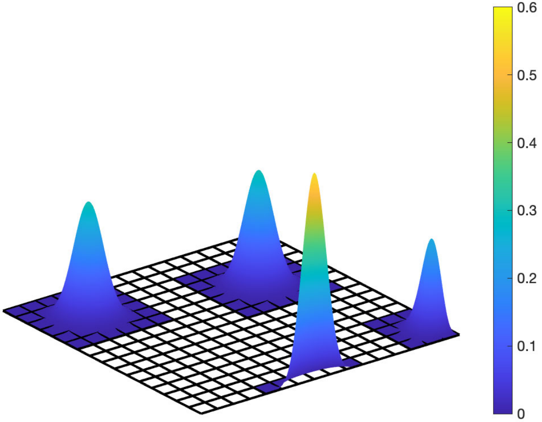

Spline basis functions $\psi_{\mathbf{a}} \in \mathcal{Q}^p(\mathcal{T}_h) \cap C^{p-1}(\Omega) \subset V_h$

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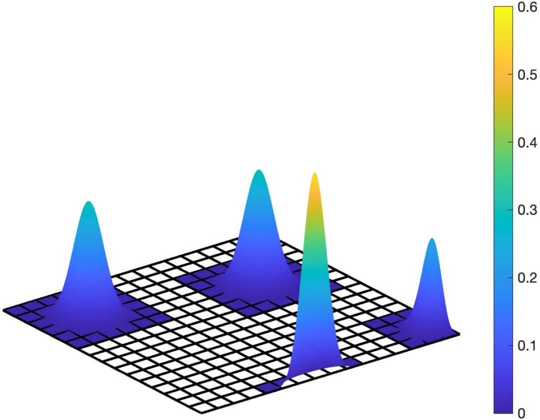
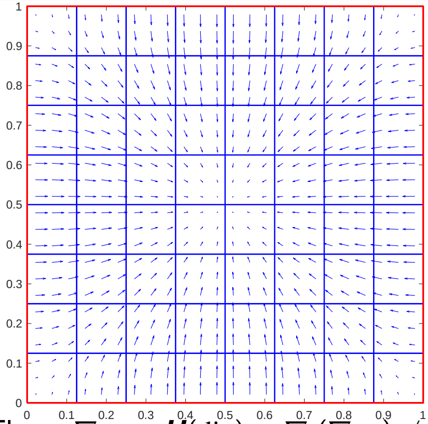


$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1$$

Spline basis functions $\psi_{\mathbf{a}} \in \mathcal{Q}^p(\mathcal{T}_h) \cap C^{p-1}(\Omega) \subset V_h$

Equilibrated flux reconstruction in IGA (a first idea)



Flux $-\nabla u_h \in \boldsymbol{H}(\mathrm{div})$, $-\nabla\cdot(\nabla u_h) \neq f$

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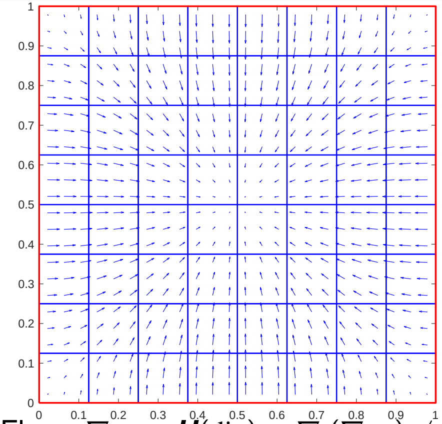

Flux $-\nabla u_h \in \boldsymbol{H}(\mathrm{div}),\ -\nabla\cdot(\nabla u_h) \neq f$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{Q}^{p-1}(\mathcal{T}_h)}$$

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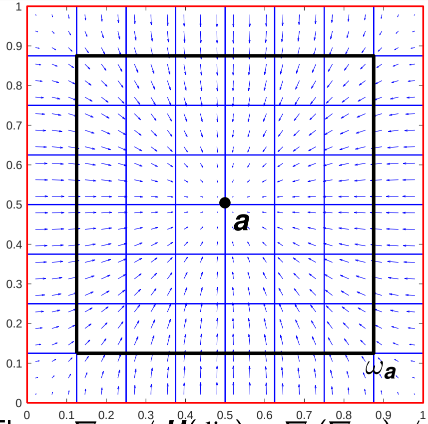
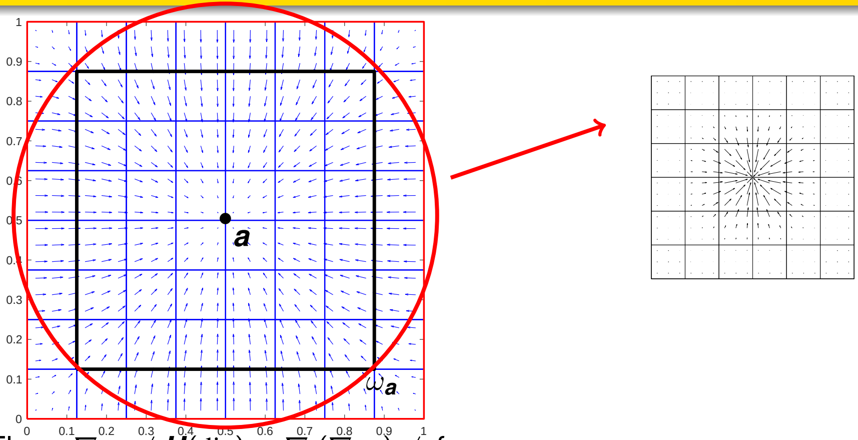


Flux $-\nabla u_h \notin \boldsymbol{H}(\mathrm{div})$, $-\nabla\cdot(\nabla u_h) \neq f$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{Q}^{p-1}(\mathcal{T}_h)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}}$$

Equilibrated flux reconstruction in IGA (a first idea)



$-\psi_{\boldsymbol{a}} \nabla u_h$

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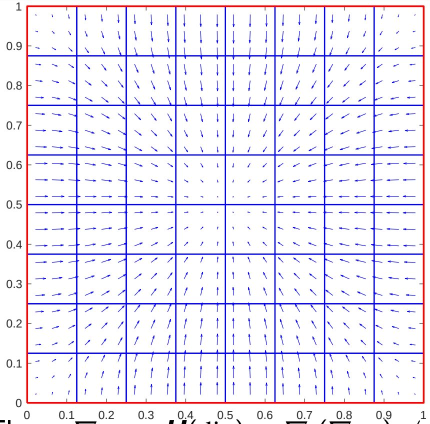

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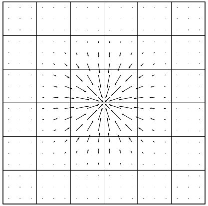

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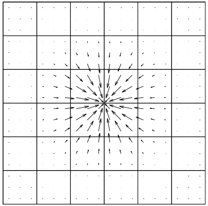
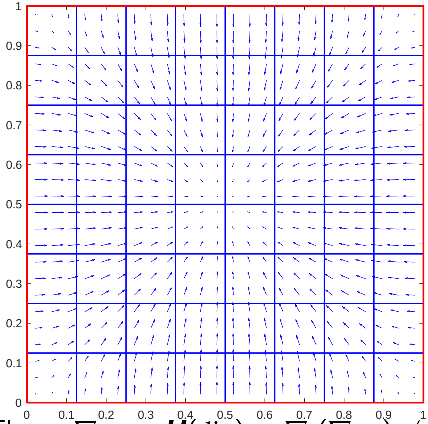


$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{Q}^{p-1}(\mathcal{T}_h)} \qquad \boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{2p}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = f\psi_{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi_{\boldsymbol{a}}}} \|\psi_{\boldsymbol{a}} \nabla u_h + \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

Equilibrated flux reconstruction in IGA (a first idea)



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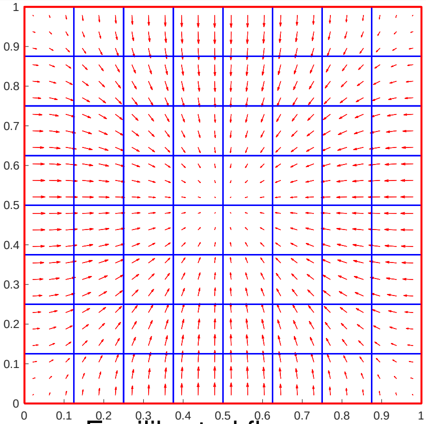
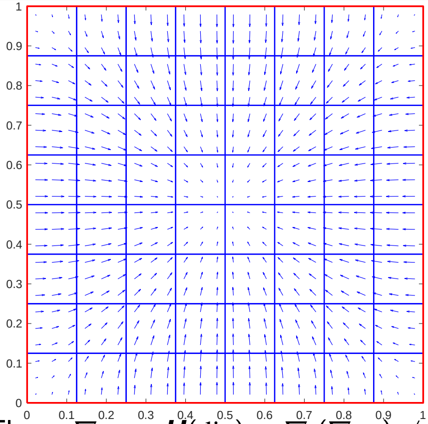


Equilibrated flux $\boldsymbol{\sigma}_h$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{Q}^{p-1}(\mathcal{T}_h)} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \mathcal{RT}_{2p}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}), \nabla\cdot\boldsymbol{\sigma}_h = f$$

Equilibrated flux reconstruction in IGA (a first idea)



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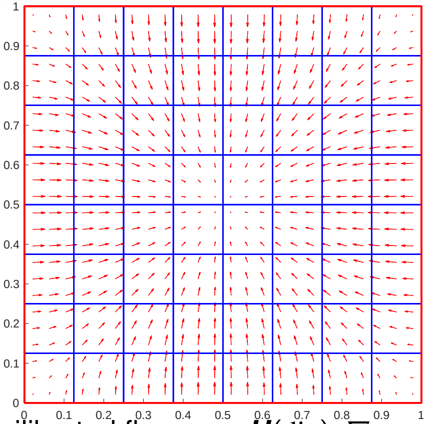

Equilibrated flux $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_h = f$

Equilibrated flux reconstruction in IGA (a first idea)

Observations

- ✓ works in principle

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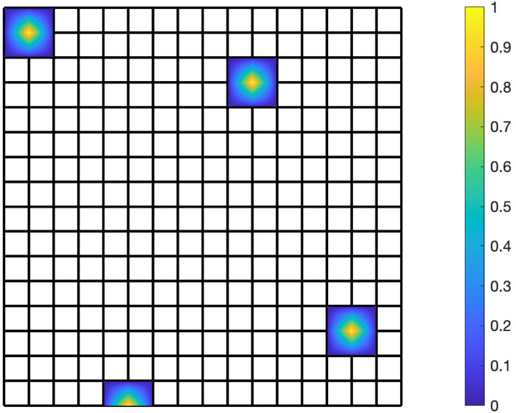
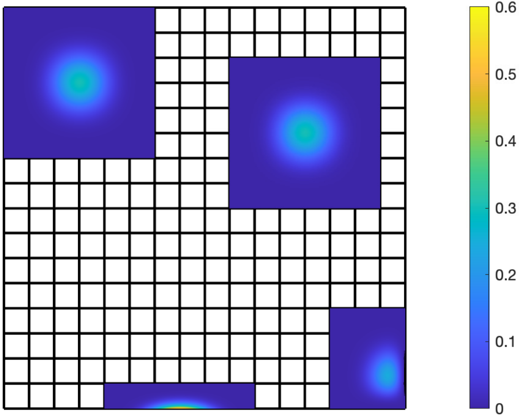
- ✗ p-robustness possibly upon extension of available tools to the large patches

Equilibration patches and partition of unity functions $\psi_{\mathbf{a}}$



$\psi_{\mathbf{a}} \in \mathcal{Q}^1(\mathcal{T}_h) \cap C^0(\Omega)$, p arbitrary



$\psi_{\mathbf{a}} \in \mathcal{Q}^p(\mathcal{T}_h) \cap C^{p-1}(\Omega)$, $p = 5$

Outline

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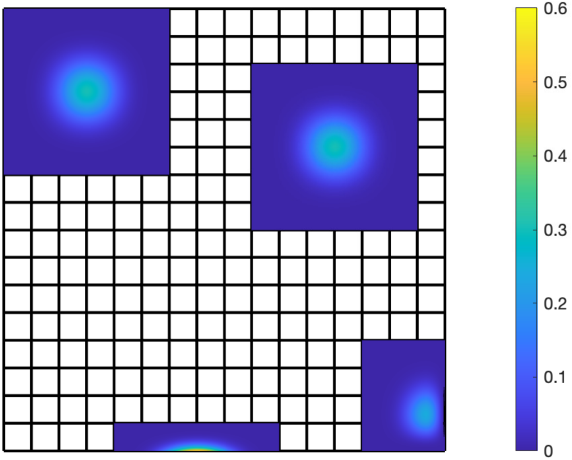
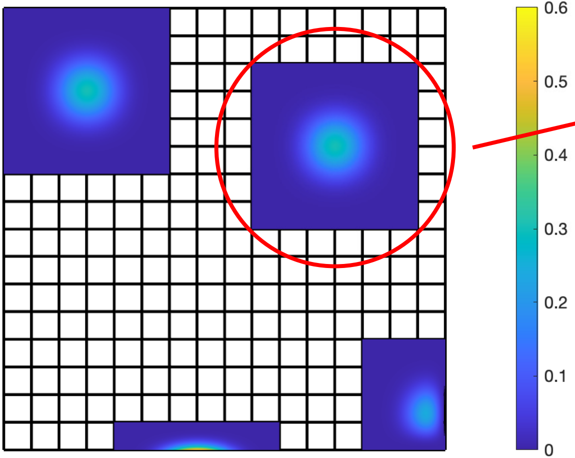
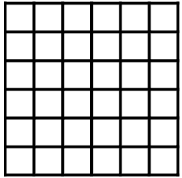
Breaking the large patch problems



1) consider the large patches (supports of $\psi_{\boldsymbol{a}}$)

Breaking the large patch problems





2) extract the submeshes $\mathcal{T}_{\boldsymbol{a}}$

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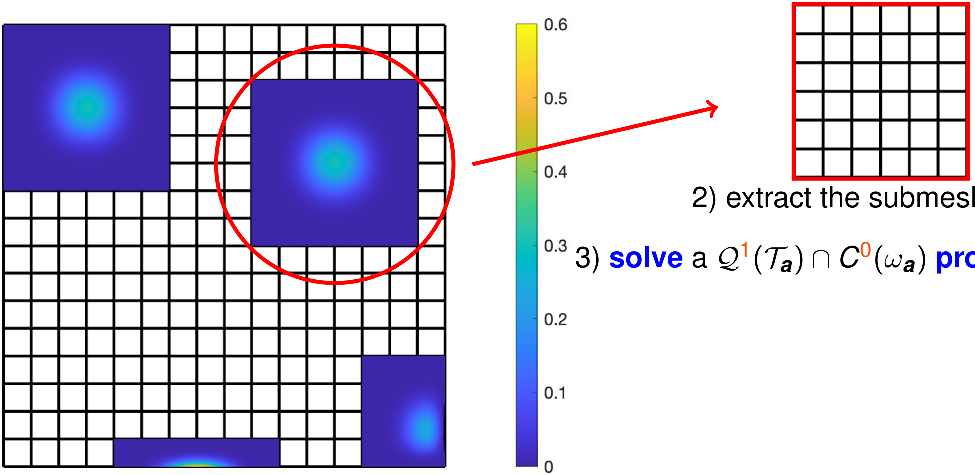
Breaking the large patch problems

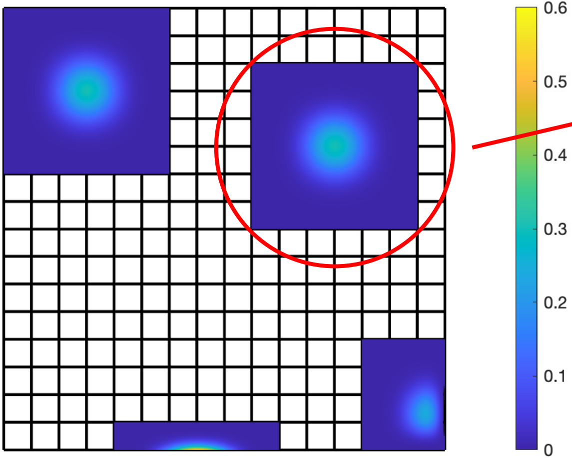


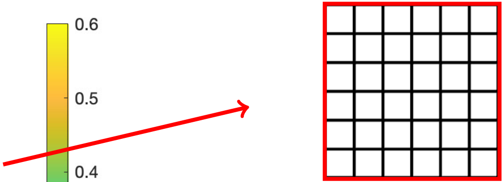
1) consider the large patches (supports of $\psi_{\boldsymbol{a}}$)

2) extract the submeshes $\mathcal{T}_{\boldsymbol{a}}$

3) **solve** a $\mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})$ **problem** on $\omega_{\boldsymbol{a}}$

Breaking the large patch problems





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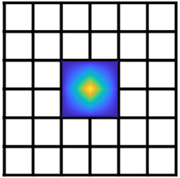
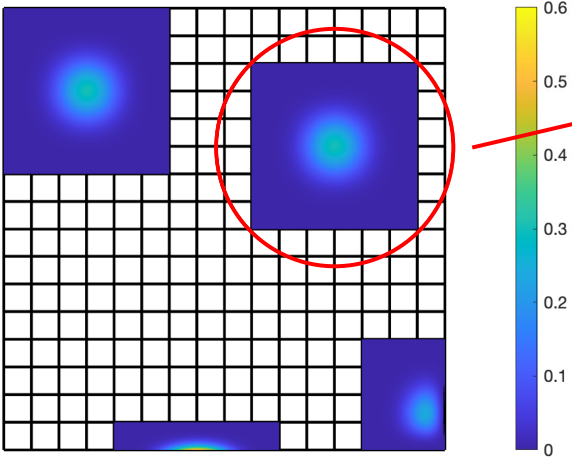


1) consider the large patches (supports of $\psi_{\mathbf{a}}$)

4) consider the hat b.f. $\psi_{\mathbf{a}'} \in \mathcal{Q}^1(\mathcal{T}_{\mathbf{a}}) \cap C^0(\omega_{\mathbf{a}})$

Breaking the large patch problems



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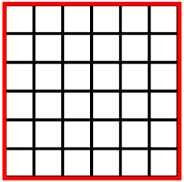

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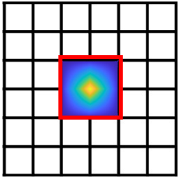
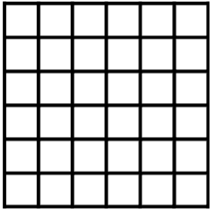


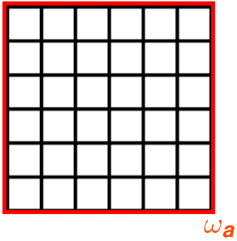
4) consider the hat b.f. $\psi_{\mathbf{a}'} \in \mathcal{Q}^1(\mathcal{T}_{\mathbf{a}}) \cap C^0(\omega_{\mathbf{a}})$

5) **perform equilibration** on $\omega_{\mathbf{a}'}$

Breaking the large patch problems

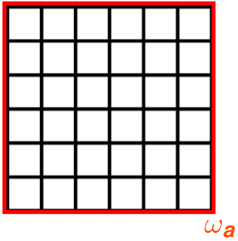


Breaking the large patch problems



3) **solve** the $\boxed{V_h^{\boldsymbol{a}} := \mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})}$ **problem**:

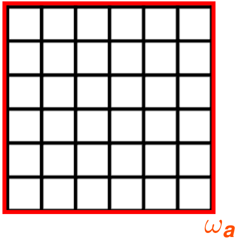
Breaking the large patch problems

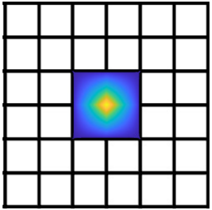


3) **solve** the $\boxed{V_h^{\boldsymbol{a}} := \mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})}$ **problem**: find $r_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that, for all $v_h \in V_h^{\boldsymbol{a}}$,

$$\boxed{(\nabla r_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (f, v_h \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(v_h \psi_{\boldsymbol{a}}))_{\omega_{\boldsymbol{a}}}}$$

Breaking the large patch problems





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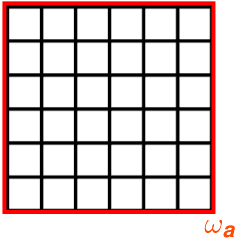
Breaking the large patch problems



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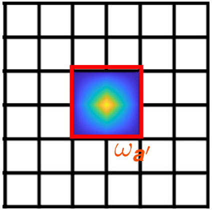
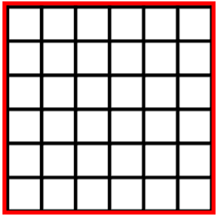


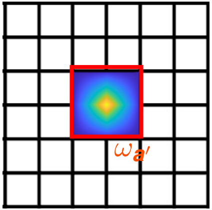
5) **perform equilibration** on $\omega_{\mathbf{a}'}$:

$$\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{a}'} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_{2p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\mathrm{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = \Upsilon_{Q_h^{\mathbf{a},\mathbf{a}'}} (f \psi_{\mathbf{a}} \psi_{\mathbf{a}'} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}} \psi_{\mathbf{a}'}) - \nabla r_h^{\mathbf{a}} \cdot \nabla \psi_{\mathbf{a}'})}} \|\psi_{\mathbf{a}'}(\psi_{\mathbf{a}} \nabla u_h + \nabla r_h^{\mathbf{a}}) + \mathbf{v}_h\|_{\omega_{\mathbf{a}'}}$$

Breaking the large patch problems



$\omega_{\boldsymbol{a}}$



3) **solve** the $\boxed{V_h^{\boldsymbol{a}} := \mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})}$ **problem**: find $r_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that, for all $v_h \in V_h^{\boldsymbol{a}}$,

$$\boxed{(\nabla r_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (f, v_h \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(v_h \psi_{\boldsymbol{a}}))_{\omega_{\boldsymbol{a}}}}$$

5) **perform equilibration** on $\omega_{\boldsymbol{a}'}$:

$$\boxed{\boldsymbol{\sigma}_h^{\boldsymbol{a},\boldsymbol{a}'} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{2p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Upsilon_{Q_h^{\boldsymbol{a},\boldsymbol{a}'}}(f \psi_{\boldsymbol{a}} \psi_{\boldsymbol{a}'} - \nabla u_h \cdot \nabla(\psi_{\boldsymbol{a}} \psi_{\boldsymbol{a}'}) - \nabla r_h^{\boldsymbol{a}} \cdot \nabla \psi_{\boldsymbol{a}'})}} \|\psi_{\boldsymbol{a}'}(\psi_{\boldsymbol{a}} \nabla u_h + \nabla r_h^{\boldsymbol{a}}) + \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}'}}}$$

6) **combine**:

$$\boxed{\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \sum_{\boldsymbol{a}' \in \mathcal{V}_h^{\boldsymbol{a}}} \boldsymbol{\sigma}_h^{\boldsymbol{a},\boldsymbol{a}'}, \qquad \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}}$$

Breaking the large patch problems

Same building principles

Additive Schwarz smoother/preconditiner Schöberl, Melenk, Pechstein, & Zaglmayr (2008): only $\mathcal{P}_1$ global problem, then high-order patch remainders

H^{-1} problems and parabolic time stepping Ern, Smears, & Vohralík (2017): arbitrary coarsening

Details

GANTNER G., VOHRALÍK M. Inexpensive polynomial-degree-robust equilibrated flux a posteriori estimates for isogeometric analysis. *Math. Models Methods Appl. Sci.* **34** (2024), 477–522.

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