

Estimation d'erreur a posteriori : principe et applications

Martin Vohralík

Inria Paris & Ecole des Ponts ParisTech

CEA list, April 29, 2024



Outline

- 1 Introduction: a posteriori error control and adaptivity
- 2 Laplace equation: discretization error control and mesh adaptivity
 - A posteriori error control (discretization)
 - Potential reconstruction
 - Flux reconstruction
 - Balancing error components: mesh adaptivity
- 3 Nonlinear Laplace equation: overall error control and solver adaptivity
 - A posteriori error control (overall and components)
 - Balancing error components: solver adaptivity
- 4 Reaction–diffusion equation: robustness wrt parameters
- 5 Heat equation: robustness wrt final time and space–time localization
- 6 Helmholtz equation: asymptotic robustness
- 7 Conclusions

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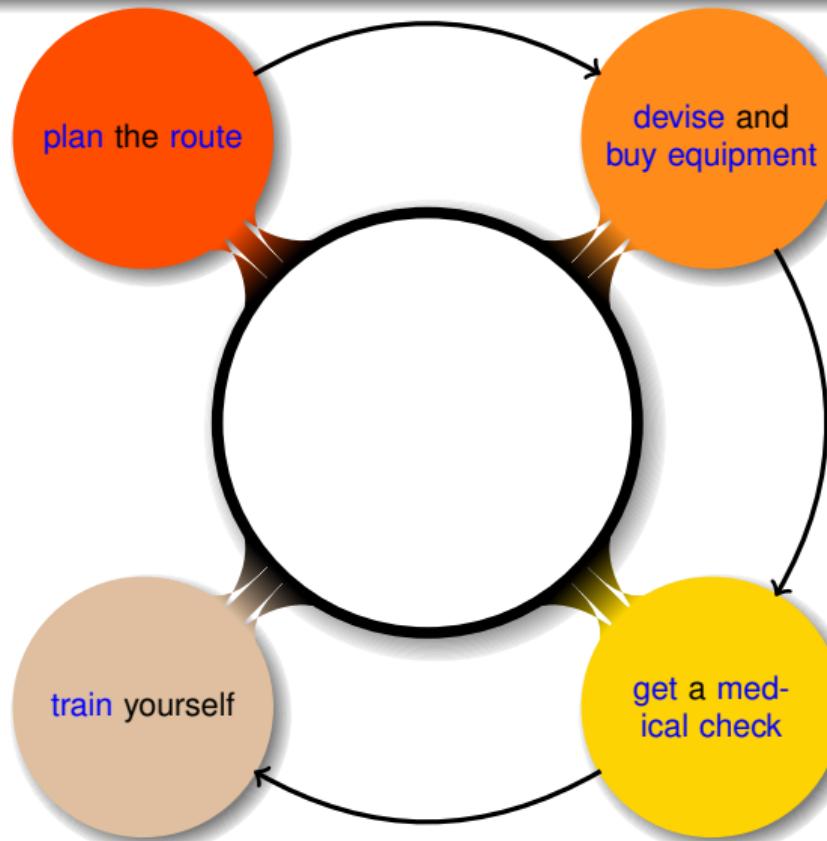
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Control the error and act adaptively: real life

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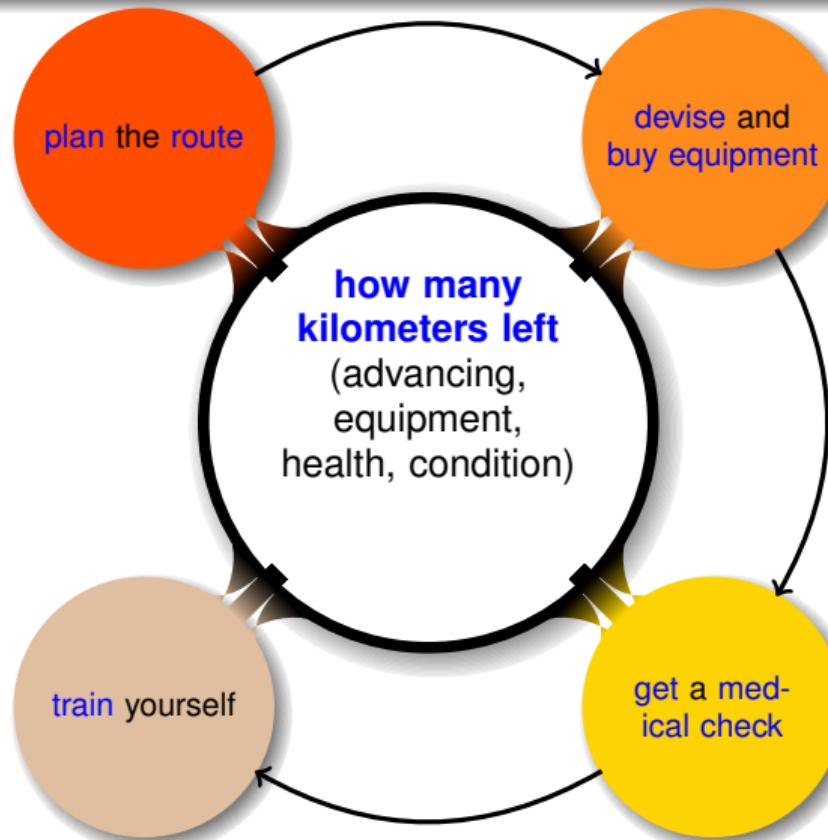
wandering Paris–Santiago
de Compostela

Control the error and act adaptively: real life



wandering Paris–Santiago de Compostela

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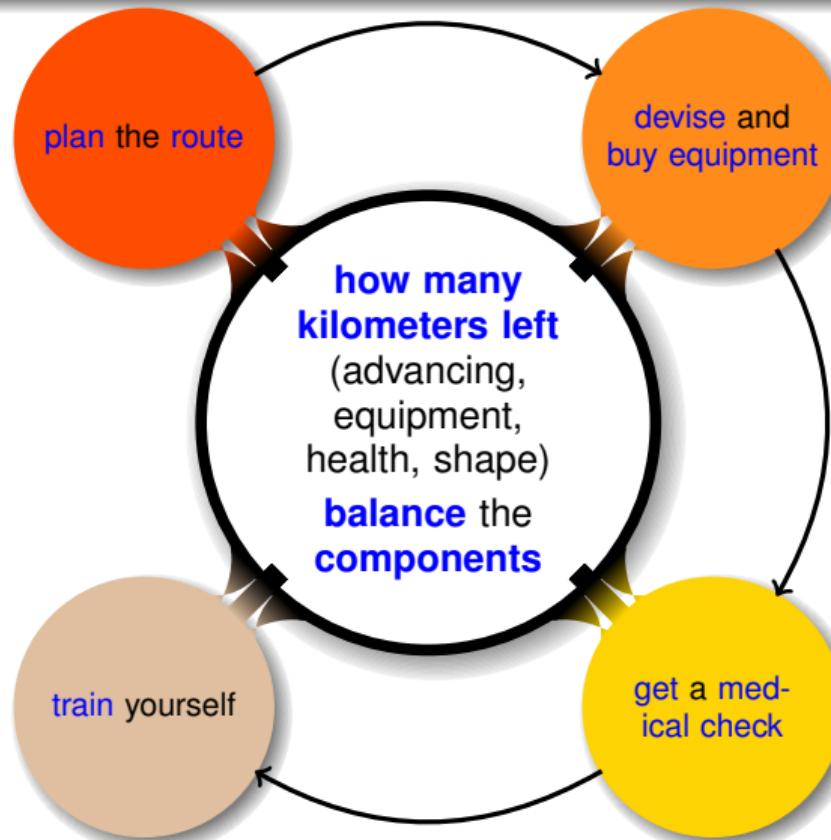


control the error

possible since

- target known
-

Control the error and act adaptively: real life



wandering Paris–Santiago de Compostela

↔ control the error

↔ act adaptively

possible since

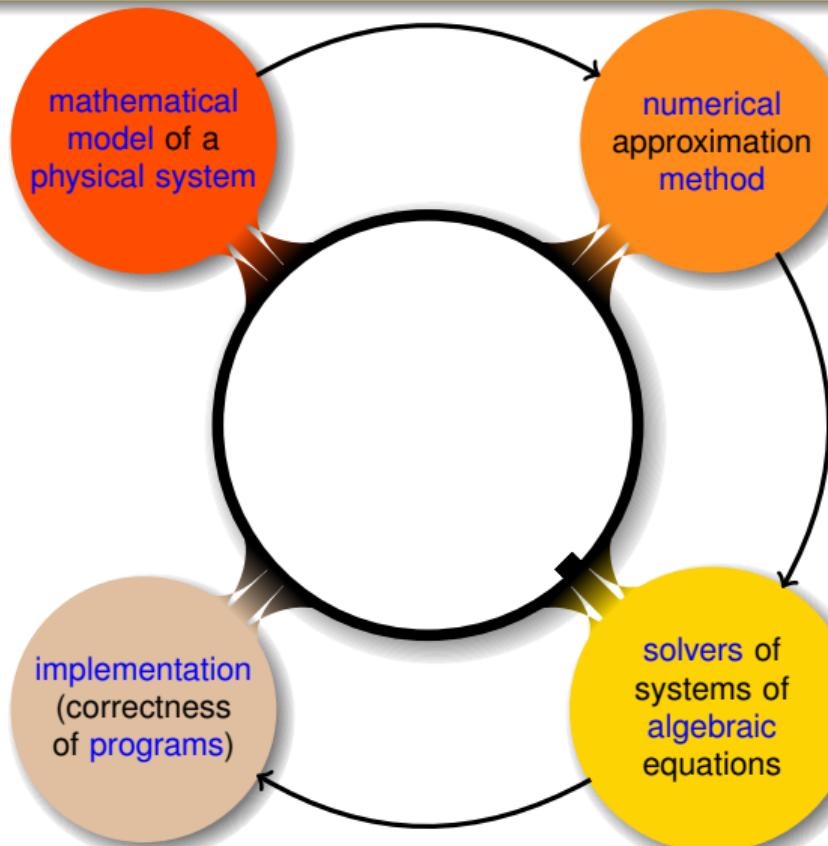
- target known
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Control the error and act adaptively: numerical simulations

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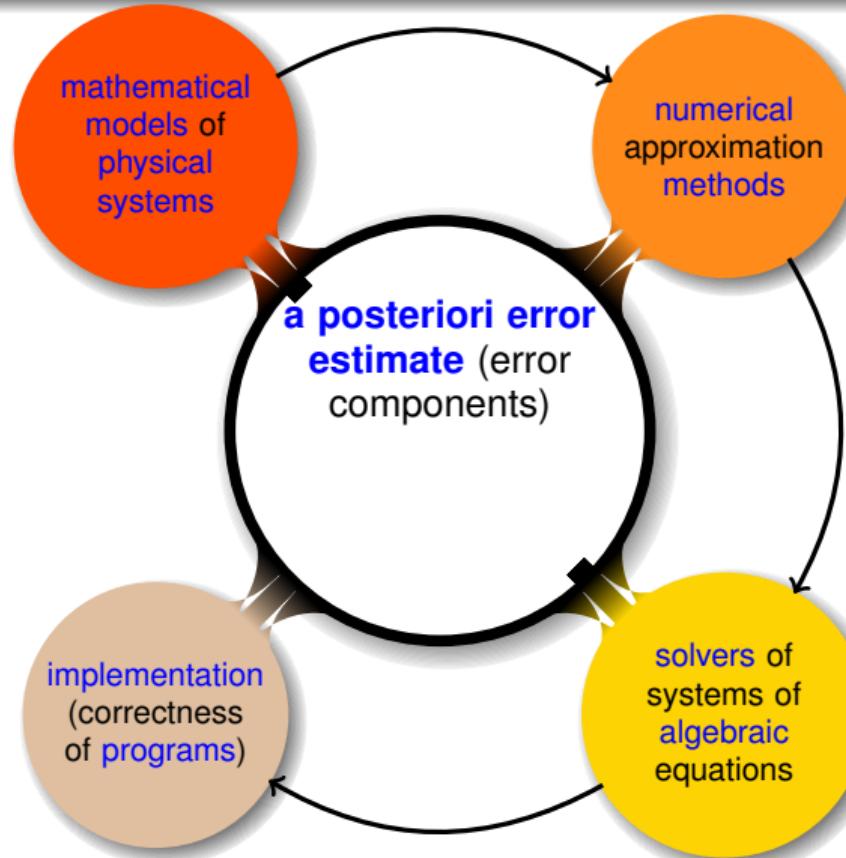
numerical simulation

Control the error and act adaptively: numerical simulations



numerical simulation

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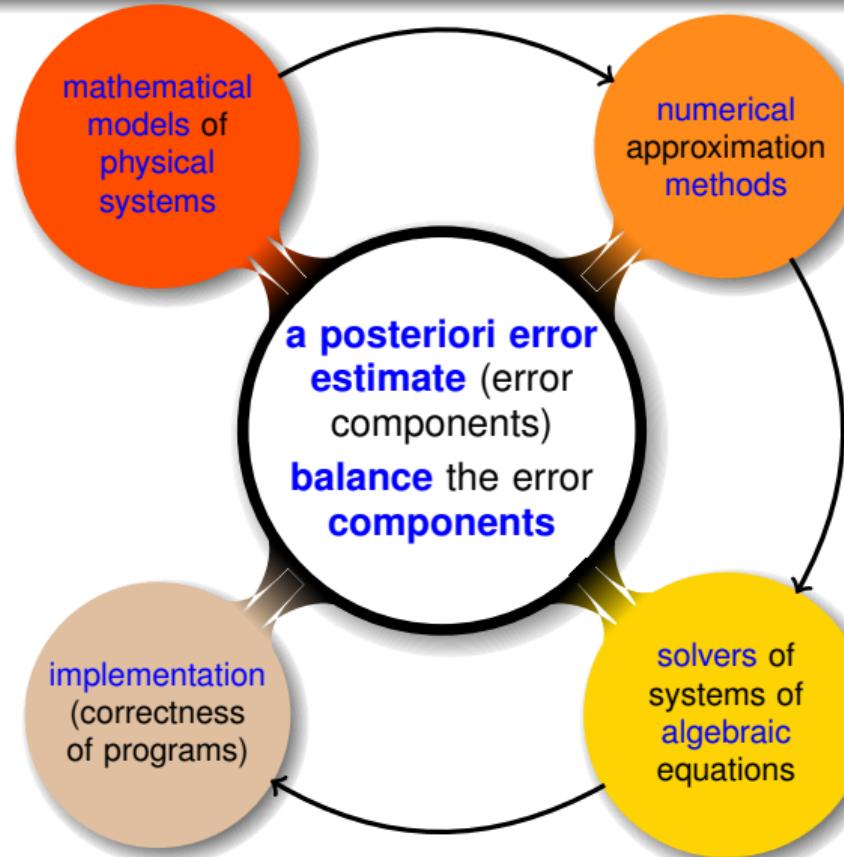


control the error (reliability)

hard since

- target unknown
-

Control the error and act adaptively: numerical simulations



numerical simulation

↔
control the error (reliability)

↔
act adaptively (efficiency)

hard since

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- components known

Numerical approximations of PDEs:

Setting

- u : unknown exact PDE solution
- u_h : known numerical approximation on mesh \mathcal{T}_h

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Numerical approximations of PDEs: 3 crucial questions

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Crucial questions

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- ② **Where** (model/space/time/linearization/algebra) is it **localized**?

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Numerical approximations of PDEs: 3 crucial questions & suggested answers

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Suggested answers

- ① Computable **a posteriori** error estimates.

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- ② Identification of **error components**.
- ③ **Balancing** error components, **adaptivity** (working where needed).

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A posteriori error estimates: error control

Laplace equation in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $f \in L^2(\Omega)$

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Guaranteed error upper bound (reliability) ($u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, FEs)

$$\underbrace{\|\nabla(u - u_h)\|}_{\text{unknown error}} \quad \underbrace{\eta(u_h)}_{\text{computable estimator}}$$

error lower bound (efficiency, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

$$\eta(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|$$

- C_{eff} a generic constant only dependent on shape regularity of \mathcal{T}_h and thus independent of Ω , u , u_h , h , p

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Local error lower bound (efficiency, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

$$\eta_K(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\omega_K} \quad \forall K \in \mathcal{T}_h$$

- C_{eff} a generic constant only dependent on shape regularity of \mathcal{T}_h and thus independent of Ω , u , u_h , h , p
- computable bound on C_{eff} available, $C_{\text{eff}} \approx 5$

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⇒ Poincaré-Friedrichs inequality, Lax-Milgram theorem, Babuška & Rheinboldt (1973).

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How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$L^2 = \frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$						
$\approx h_0/4$						
$\approx h_0/8$						
$\approx h_0/16$						
$\approx h_0/32$						
$\approx h_0/64$						
$\approx h_0/128$						

A. Ern, M. Vohralík, Réduire l'erreur en évaluant localement les erreurs
de discrépance et de quadrature pour la résolution des équations aux dérivées partielles

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h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}\ }$	$E = \frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$	6.07 $\times 10^{-1}$					
$\approx h_0/4$	3.10 $\times 10^{-1}$					
$\approx h_0/8$	1.45 $\times 10^{-1}$					
$\approx h_0/16$	4.23 $\times 10^{-2}$					
$\approx h_0/32$	1.06 $\times 10^{-2}$					
$\approx h_0/64$	2.65 $\times 10^{-3}$					
$\approx h_0/128$	6.63 $\times 10^{-4}$					
$\approx h_0/256$	1.66 $\times 10^{-4}$					
$\approx h_0/512$	4.15 $\times 10^{-5}$					
$\approx h_0/1024$	1.04 $\times 10^{-5}$					

A. Ern, M. Vohralík, *Réduire l'erreur en éléments finis par la méthode d'erreur a posteriori*, *ESAIM: Mathematical Modelling and Numerical Analysis*, 2010, 44(2), 321–344.

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\textcolor{brown}{P^h} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$	6.07 $\times 10^{-1}$	1.93	16%	0.53	10%	3.64
$\approx h_0/4$	3.10 $\times 10^{-1}$	3.02	10%	0.26	5%	11.54
$\approx h_0/8$	1.45 $\times 10^{-1}$	4.96	6%	0.13	3%	34.92
$\approx h_0/16$	4.23 $\times 10^{-2}$	7.95	4%	0.065	2%	109.71
$\approx h_0/32$	1.06 $\times 10^{-2}$	11.98	3%	0.0325	1%	359.37
$\approx h_0/64$	2.65 $\times 10^{-3}$	16.95	2%	0.01625	0.5%	1078.12
$\approx h_0/128$	6.63 $\times 10^{-4}$	22.93	1%	0.008125	0.25%	3234.37
$\approx h_0/256$	1.66 $\times 10^{-4}$	30.91	0.5%	0.0040625	0.125%	9709.11
$\approx h_0/512$	4.15 $\times 10^{-5}$	41.88	0.25%	0.00203125	0.0625%	32343.70
$\approx h_0/1024$	1.04 $\times 10^{-5}$	55.85	0.125%	0.001015625	0.03125%	107091.23
$\approx h_0/2048$	2.60 $\times 10^{-6}$	73.73	0.0625%	0.0005078125	0.015625%	323437.00
$\approx h_0/4096$	6.50 $\times 10^{-7}$	95.60	0.03125%	0.00025390625	0.0078125%	1070912.30
$\approx h_0/8192$	1.62 $\times 10^{-7}$	121.48	0.015625%	0.000126953125	0.00390625%	3234370.00
$\approx h_0/16384$	4.05 $\times 10^{-8}$	152.35	0.0078125%	0.0000634765625	0.001953125%	10709123.00
$\approx h_0/32768$	1.01 $\times 10^{-8}$	188.22	0.00390625%	0.0000317381640625	0.0009765625%	32343700.00
$\approx h_0/65536$	2.53 $\times 10^{-9}$	232.10	0.001953125%	0.00001586908203125	0.00048828125%	107091230.00
$\approx h_0/131072$	6.32 $\times 10^{-10}$	281.98	0.0009765625%	0.000007934541015625	0.000244140625%	323437000.00
$\approx h_0/262144$	1.58 $\times 10^{-10}$	341.85	0.00048828125%	0.0000039672705078125	0.0001220703125%	1070912300.00
$\approx h_0/524288$	3.95 $\times 10^{-11}$	415.73	0.000244140625%	0.00000198363525390625	0.00006103515625%	3234370000.00
$\approx h_0/1048576$	9.88 $\times 10^{-12}$	504.60	0.0001220703125%	0.000000991817626953125	0.000030517578125%	10709123000.00
$\approx h_0/2097152$	2.47 $\times 10^{-12}$	610.48	0.00006103515625%	0.0000004959088134765625	0.0000152587890625%	32343700000.00
$\approx h_0/4194304$	6.18 $\times 10^{-13}$	736.35	0.000030517578125%	0.000000247954406734375	0.0000152587890625%	107091230000.00
$\approx h_0/8388608$	1.54 $\times 10^{-13}$	880.22	0.0000152587890625%	0.0000001239772033671875	0.00000762939453125%	323437000000.00
$\approx h_0/16777216$	3.86 $\times 10^{-14}$	1044.09	0.00000762939453125%	0.000000061988601683890625	0.000003814677734375%	1070912300000.00
$\approx h_0/33554432$	9.65 $\times 10^{-15}$	1230.96	0.000003814677734375%	0.000000030994300841948438	0.0000019073388671875%	3234370000000.00
$\approx h_0/67108864$	2.41 $\times 10^{-15}$	1440.83	0.0000019073388671875%	0.000000015497150420974219	0.000000953669434375%	10709123000000.00
$\approx h_0/134217728$	6.03 $\times 10^{-16}$	1672.70	0.000000953669434375%	0.000000007748577210487109	0.0000004768347171875%	32343700000000.00
$\approx h_0/268435456$	1.51 $\times 10^{-16}$	2028.57	0.0000004768347171875%	0.000000003874288605243554	0.0000002384173584375%	107091230000000.00
$\approx h_0/516870912$	3.78 $\times 10^{-17}$	2418.44	0.0000002384173584375%	0.000000001937144302621777	0.00000011920867921875%	323437000000000.00
$\approx h_0/1033741824$	9.45 $\times 10^{-18}$	2844.31	0.00000011920867921875%	0.000000000968572151310888	0.000000059604339609375%	1070912300000000.00
$\approx h_0/2067483648$	2.36 $\times 10^{-18}$	3306.18	0.000000059604339609375%	0.000000000484286075655444	0.000000029802169805625%	3234370000000000.00
$\approx h_0/4134967296$	6.00 $\times 10^{-19}$	3810.05	0.000000029802169805625%	0.000000000242143037827722	0.0000000149010849028125%	10709123000000000.00
$\approx h_0/8269934592$	1.50 $\times 10^{-19}$	4400.92	0.0000000149010849028125%	0.000000000121071518913861	0.00000000745054245140625%	32343700000000000.00
$\approx h_0/16539869184$	3.75 $\times 10^{-20}$	5040.79	0.00000000745054245140625%	0.000000000060535759346931	0.00000000472527122578125%	107091230000000000.00
$\approx h_0/33079738368$	9.38 $\times 10^{-21}$	5744.66	0.00000000472527122578125%	0.000000000030267879673465	0.000000002362635612890625%	323437000000000000.00
$\approx h_0/66159476736$	2.34 $\times 10^{-21}$	6504.53	0.000000002362635612890625%	0.000000000015133939346733	0.0000000011813178064453125%	1070912300000000000.00
$\approx h_0/132318953472$	5.85 $\times 10^{-22}$	7318.40	0.0000000011813178064453125%	0.000000000007566969673366	0.0000000005906589032234375%	3234370000000000000.00
$\approx h_0/264637906944$	1.46 $\times 10^{-22}$	8182.27	0.0000000005906589032234375%	0.000000000003783294836683	0.000000000305329476611875%	10709123000000000000.00
$\approx h_0/529275813888$	3.65 $\times 10^{-23}$	9106.14	0.000000000305329476611875%	0.000000000001891647383342	0.000000000152662381305625%	32343700000000000000.00
$\approx h_0/1058551627776$	9.13 $\times 10^{-24}$	10070.01	0.000000000152662381305625%	0.000000000000945823941671	0.0000000000763311906528125%	107091230000000000000.00
$\approx h_0/2117103255552$	2.28 $\times 10^{-24}$	11083.88	0.0000000000763311906528125%	0.000000000000472915970835	0.0000000000381655953264375%	323437000000000000000.00
$\approx h_0/4234206511104$	5.70 $\times 10^{-25}$	12147.75	0.0000000000381655953264375%	0.000000000000236582975418	0.0000000000190827466411875%	1070912300000000000000.00
$\approx h_0/8468413022208$	1.43 $\times 10^{-25}$	13251.62	0.0000000000190827466411875%	0.000000000000118413732709	0.000000000011813178064453125%	3234370000000000000000.00
$\approx h_0/16936826044416$	3.58 $\times 10^{-26}$	14415.49	0.000000000011813178064453125%	0.000000000000060206866355	0.0000000000060535759346733125%	10709123000000000000000.00
$\approx h_0/33873652088832$	8.95 $\times 10^{-27}$	15629.36	0.0000000000060535759346733125%	0.000000000000030103433178	0.0000000000030267879673465625%	32343700000000000000000.00
$\approx h_0/67747304177664$	2.24 $\times 10^{-27}$	16883.23	0.0000000000030267879673465625%	0.0000000000000151339393467333125	0.0000000000015133939346733125%	107091230000000000000000.00
$\approx h_0/135494608355328$	5.60 $\times 10^{-28}$	18187.10	0.0000000000015133939346733125%	0.000000000000007566969673366625	0.0000000000007566969673366625%	323437000000000000000000.00
$\approx h_0/270989216710656$	1.40 $\times 10^{-28}$	19541.97	0.0000000000007566969673366625%	0.0000000000000037832948366833125	0.00000000000037832948366833125%	1070912300000000000000000.00
$\approx h_0/541978433421312$	3.49 $\times 10^{-29}$	21045.84	0.00000000000037832948366833125%	0.00000000000018916473833426625	0.00000000000018916473833426625%	3234370000000000000000000.00
$\approx h_0/1083956866842624$	8.73 $\times 10^{-30}$	22659.71	0.00000000000018916473833426625%	0.000000000000094582394167333125	0.000000000000094582394167333125%	10709123000000000000000000.00
$\approx h_0/2167913733685248$	2.18 $\times 10^{-30}$	24373.58	0.000000000000094582394167333125%	0.000000000000047291597083316625	0.000000000000047291597083316625%	32343700000000000000000000.00
$\approx h_0/4335827467370496$	5.45 $\times 10^{-31}$	26187.45	0.000000000000047291597083316625%	0.0000000000000236582975416683125	0.0000000000000236582975416683125%	107091230000000000000000000.00
$\approx h_0/8671654934740992$	1.36 $\times 10^{-31}$	28091.32	0.0000000000000236582975416683125%	0.0000000000000118413732708341625	0.0000000000000118413732708341625%	323437000000000000000000000.00
$\approx h_0/17343309869481984$	3.40 $\times 10^{-32}$	30095.19	0.0000000000000118413732708341625%	0.00000000000000592068663541708125	0.00000000000000592068663541708125%	1070912300000000000000000000.00
$\approx h_0/34686619738963968$	8.50 $\times 10^{-33}$	32199.06	0.00000000000000592068663541708125%	0.000000000000002960343317708540625	0.000000000000002960343317708540625%	3234370000000000000000000000.00
$\approx h_0/69373239477927936$	2.12 $\times 10^{-33}$	34403.93	0.000000000000002960343317708540625%	0.0000000000000014801716588542703125	0.0000000000000014801716588542703125%	10709123000000000000000000000.00
$\approx h_0/138746478955855872$	5.31 $\times 10^{-34}$	36707.80	0.0000000000000014801716588542703125%	0.00000000000000074008582942723515625	0.00000000000000074008582942723515625%	32343700000000000000000000000.00
$\approx h_0/277492957911711744$	1.33 $\times 10^{-34}$	39111.67	0.00000000000000074008582942723515625%	0.000000000000000370042914713617578125	0.000000000000000370042914713617578125%	107091230000000000000000000000.00
$\approx h_0/554985915823423488$	3.33 $\times 10^{-35}$	41615.54	0.000000000000000370042914713617578125%	0.0000000000000001850214573568087890625	0.0000000000000001850214573568087890625%	323437000000000000000000000000.00
$\approx h_0/1109971831646846976$	8.33 $\times 10^{-36}$	44219.41	0.0000000000000001850214573568087890625%	0.00000000000000009251072867840439453125	0.00000000000000009251072867840439453125%	1070912300000000000000000000000.00
$\approx h_0/2219943663293693952$	2.08 $\times 10^{-36}$	46923.28	0.00000000000000009251072867840439453125%	0.000000000000000046255364339202197265625	0.000000000000000046255364339202197265625%	3234370000000000000000000000000.00
$\approx h_0/4439887326587387904$	5.20 $\times 10^{-37}$	49727.15	0.000000000000000046255364339202197265625%	0.0000000000000000231276821696010986328125	0.0000000000000000231276821696010986328125%	107091230000000000000000000000000.00
$\approx h_0/8879774653174775808$	1.30 $\times 10^{-37}$	52631.02	0.0000000000000000231276821696010986328125%	0.000000000000000011563841084800549314453125	0.000000000000000011563841084800549314453125%	32343700000000000000000000000000.00
$\approx h_0/17759549306349551616$	3.25 $\times 10^{-38}$	55634.89	0.000000000000000011563841084800549314453125%	0.0000000000000000057819205424002746572265625	0.0000000000000000057819205424002746572265625%	1070912300000000000000000000000000.00
$\approx h_0/35519098612699103232$	8.13 $\times 10^{-39}$	58738.76	0.0000000000000000057819205424002746572265625%	0.0000000000000000028909602712001373286328125	0.0000000000000000028909602712001373286328125%	3234370000000000000000000000000000.00
$\approx h_0/71038197225398206464$	2.03 $\times 10^{-39}$	61942.63	0.0000000000000000028909602712001373286328125%	0.0000000000000000014454801356000686443165625	0.0000000000000000014454801356000686443165625%	10709123000000000000000000000000000.00
$\approx h_0/142076394450796412928$	5.08 $\times 10^{-40}$	65246.50	0.0000000000000000014454801356000686443165625%	0.0000000000000000007227400678000343221582812		

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$r^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	
$\approx h_0/4$		3.10×10^{-1}	7.6%	2.92×10^{-1}	7.3%	
$\approx h_0/8$		1.45×10^{-1}	4.8%	1.39×10^{-1}	4.6%	
$\approx h_0/16$		4.23×10^{-2}	1.0%	4.07×10^{-2}	1.0%	
$\approx h_0/32$		1.06×10^{-2}	0.25%	1.05×10^{-2}	0.25%	
$\approx h_0/64$		2.66×10^{-3}	0.06%	2.65×10^{-3}	0.06%	
$\approx h_0/128$		6.66×10^{-4}	0.015%	6.65×10^{-4}	0.015%	

A. Ern, M. Vohralík, *Mathematics of Computation*, 2019
DOI 10.1090/mcom/3370 ©2019 American Mathematical Society
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How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$r^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.08
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.02
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.01
$\approx h_0/16$		4.23×10^{-2}	0.5 $\times 10^{-1}\%$	4.07×10^{-2}	0.2 $\times 10^{-1}\%$	1.00
$\approx h_0/32$		1.06×10^{-2}	0.1 $\times 10^{-2}\%$	1.06×10^{-2}	0.1 $\times 10^{-2}\%$	1.00
$\approx h_0/64$		2.66×10^{-3}	0.03 $\times 10^{-3}\%$	2.66×10^{-3}	0.03 $\times 10^{-3}\%$	1.00
$\approx h_0/128$		6.66×10^{-4}	0.008 $\times 10^{-4}\%$	6.66×10^{-4}	0.008 $\times 10^{-4}\%$	1.00

A. Ern, M. Vohralík, "Robust a posteriori error estimation for the finite element method on anisotropic meshes", 2010.

DOI: 10.1007/s00211-010-0340-0, Journal of Numerical Mathematics, 2010.

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\text{P}^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.09
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.04
$\approx h_0/16$		4.23×10^{-2}	0.5 $\times 10^{-2}\%$	4.07×10^{-2}	0.2 $\times 10^{-2}\%$	1.03
$\approx h_0/32$		1.06×10^{-2}	0.1 $\times 10^{-3}\%$	1.06×10^{-2}	0.1 $\times 10^{-3}\%$	1.02
$\approx h_0/64$		2.66×10^{-3}	0.03 $\times 10^{-4}\%$	2.66×10^{-3}	0.03 $\times 10^{-4}\%$	1.01
$\approx h_0/128$		6.66×10^{-4}	0.008 $\times 10^{-5}\%$	6.66×10^{-4}	0.008 $\times 10^{-5}\%$	1.00

A. Ern, M. Vohralík, "Robust a posteriori error estimation for discontinuous Galerkin methods", *Journal of Numerical Mathematics*, 2010.A. Ern, M. Vohralík, "Robust a posteriori error estimation for discontinuous Galerkin methods", *Journal of Numerical Mathematics*, 2010.

How **large** is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$I^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.09
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.04
$\approx h_0/16$	2	4.23×10^{-2}	$9.5 \times 10^{-4}\%$	4.07×10^{-2}	$9.2 \times 10^{-4}\%$	1.04
$\approx h_0/32$		2.62×10^{-2}	5.8 $\times 10^{-4}\%$	2.56×10^{-2}	$5.9 \times 10^{-4}\%$	1.04
$\approx h_0/64$	4	2.60×10^{-7}	$5.9 \times 10^{-14}\%$	2.58×10^{-7}	$5.8 \times 10^{-14}\%$	1.04

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\textcolor{red}{f}_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.09
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.04
$\approx h_0/2$	2	4.23×10^{-2}	$9.5 \times 10^{-1}\%$	4.07×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	2.62×10^{-4}	$5.9 \times 10^{-3}\%$	2.60×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
$\approx h_0/8$		1.60×10^{-5}	$3.5 \times 10^{-5}\%$	2.54×10^{-5}	$3.9 \times 10^{-5}\%$	1.00

A. Ern, M. Vohralík, Reliable a posteriori error estimation for discontinuous Galerkin methods, Journal of Numerical Mathematics, 2010.

DOI: 10.1515/jnumath.2010.001, URL: <http://www.degruyter.com/view/j/jnumat.2010.001>

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\textcolor{red}{f^{\text{eff}}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.09
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.04
$\approx h_0/2$	2	4.23×10^{-2}	$9.5 \times 10^{-1}\%$	4.07×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	2.62×10^{-4}	$5.9 \times 10^{-3}\%$	2.60×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
$\approx h_0/8$	4	2.60×10^{-7}	$5.9 \times 10^{-6}\%$	2.58×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\text{f}^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.09
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.04
$\approx h_0/2$	2	4.23×10^{-2}	$9.5 \times 10^{-1}\%$	4.07×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	2.62×10^{-4}	$5.9 \times 10^{-3}\%$	2.60×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
$\approx h_0/8$	4	2.60×10^{-7}	$5.9 \times 10^{-6}\%$	2.58×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\textcolor{red}{I}^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.09
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.04
$\approx h_0/2$	2	4.23×10^{-2}	$9.5 \times 10^{-1}\%$	4.07×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	2.62×10^{-4}	$5.9 \times 10^{-3}\%$	2.60×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
$\approx h_0/8$	4	2.60×10^{-7}	$5.9 \times 10^{-6}\%$	2.58×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

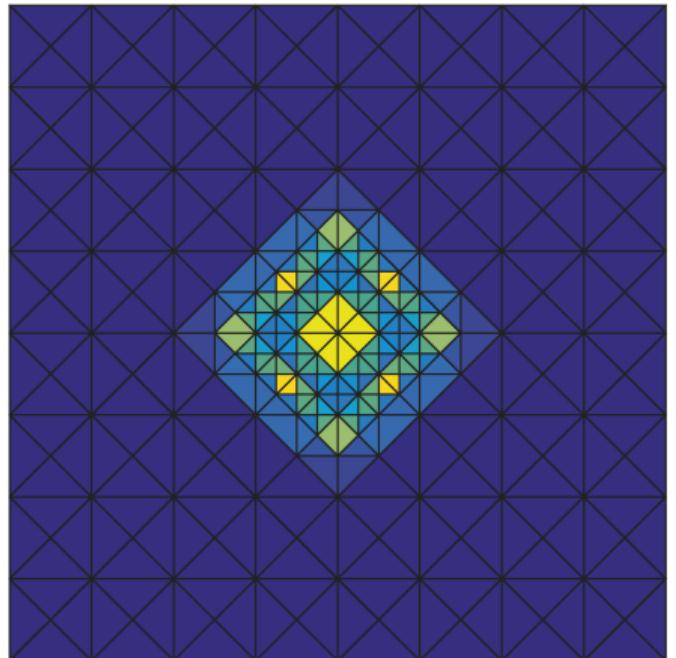
A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known smooth solution)

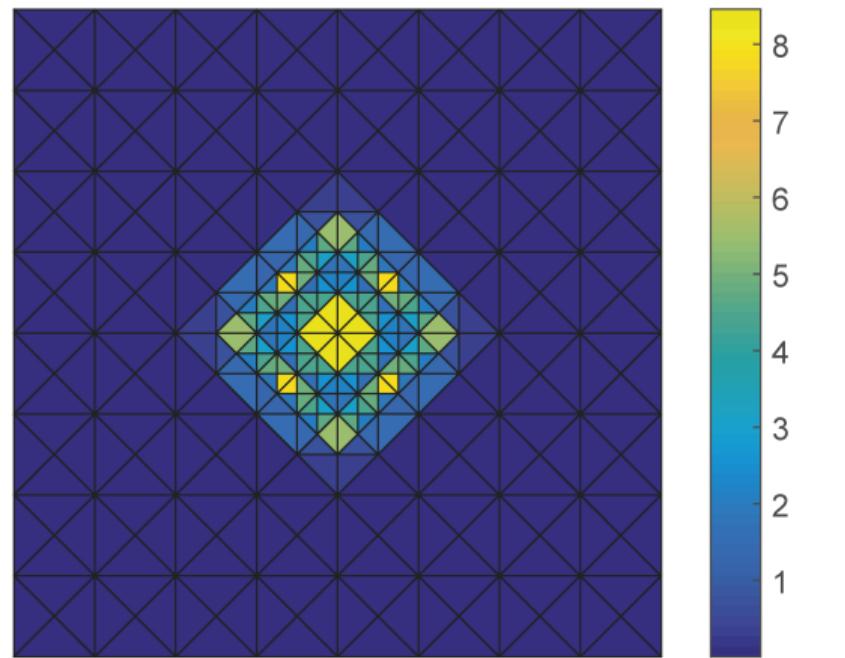
h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\text{f}^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	1.09
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	1.04
$\approx h_0/2$	2	4.23×10^{-2}	$9.5 \times 10^{-1}\%$	4.07×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	2.62×10^{-4}	$5.9 \times 10^{-3}\%$	2.60×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
$\approx h_0/8$	4	2.60×10^{-7}	$5.9 \times 10^{-6}\%$	2.58×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

Where (in space) is the error **localized**? (known smooth solution)



Estimated error distribution $\eta_K(u_h)$



Exact error distribution $\|\nabla(u - u_h)\|_K$

Error characterization

Theorem (Error characterization)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\|\nabla(u - u_h)\|^2 = \underbrace{\min_{\substack{\sigma \in H(\text{div}, \Omega) \\ \nabla \cdot \sigma = f}} \|\nabla u_h + \sigma\|^2}_{= \max_{\substack{\varphi \in H_0^1(\Omega) \\ \|\nabla \varphi\| = 1}} [(f, \varphi) - (\nabla u_h, \nabla \varphi)]^2} + \underbrace{\min_{s \in H_0^1(\Omega)} \|\nabla(u_h - s)\|^2}_{\text{distance to } H_0^1(\Omega)}.$$

dual norm of the residual

Comments

- It is enough to choose suitable (discrete, piecewise polynomial) $\sigma_h \in H(\text{div}, \Omega)$ with $\nabla \cdot \sigma_h = f$ and $s_h \in H_0^1(\Omega)$ to get a guaranteed upper bound.

Error characterization

Theorem (Error characterization)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\begin{aligned} \|\nabla(u - u_h)\|^2 &= \underbrace{\min_{\substack{\sigma \in \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \sigma = f}} \|\nabla u_h + \sigma\|^2}_{= \max_{\substack{\varphi \in H_0^1(\Omega) \\ \|\nabla \varphi\| = 1}} [(f, \varphi) - (\nabla u_h, \nabla \varphi)]^2} + \underbrace{\min_{s \in H_0^1(\Omega)} \|\nabla(u_h - s)\|^2}_{\text{distance to } H_0^1(\Omega)}. \end{aligned}$$

dual norm of the residual

Comments

- It is enough to choose suitable (discrete, piecewise polynomial) $\sigma_h \in \mathbf{H}(\text{div}, \Omega)$ with $\nabla \cdot \sigma_h = f$ and $s_h \in H_0^1(\Omega)$ to get a guaranteed upper bound.

Error characterization

Theorem (Error characterization)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

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dual norm of the residual

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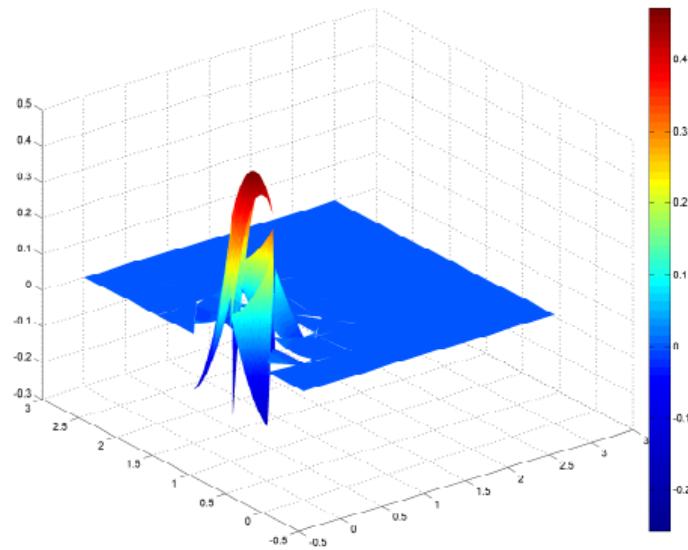
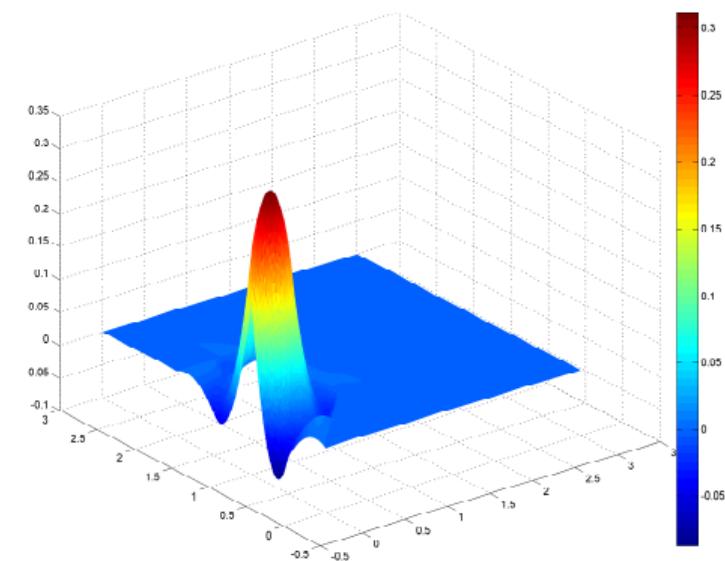
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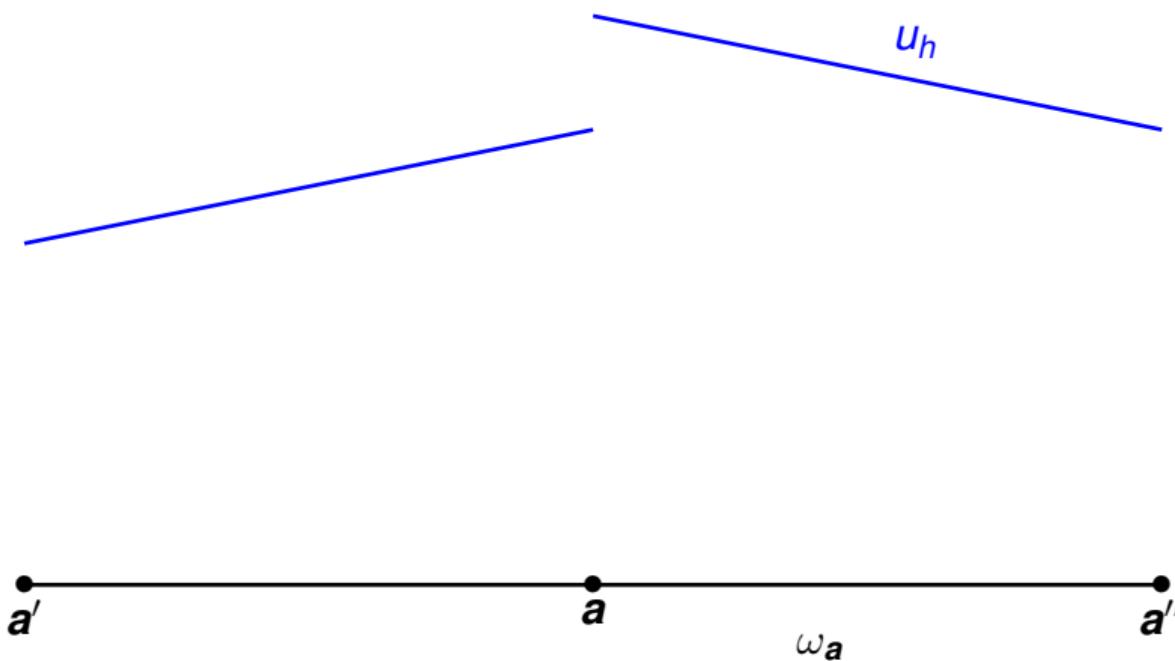
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Potential reconstruction

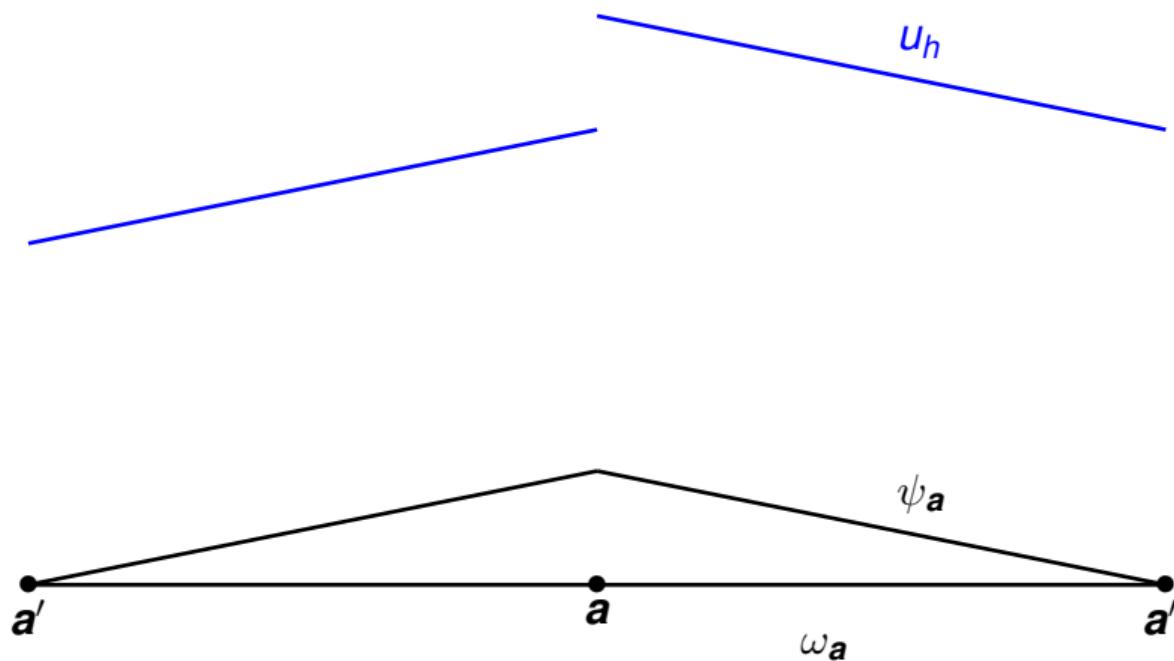
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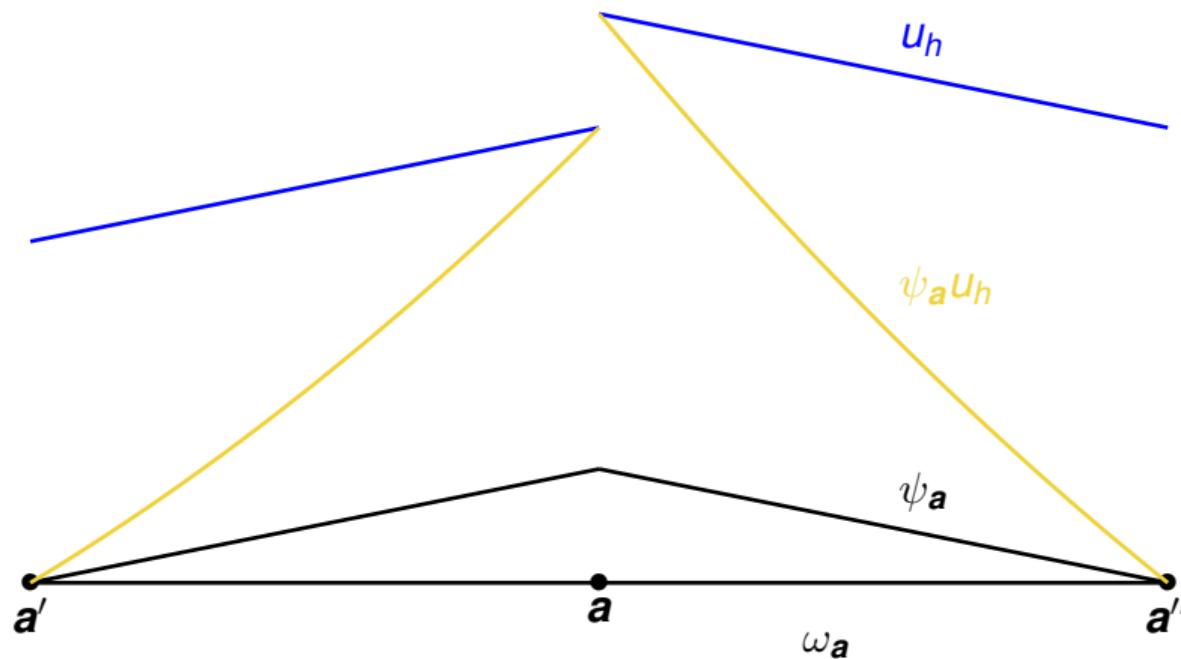
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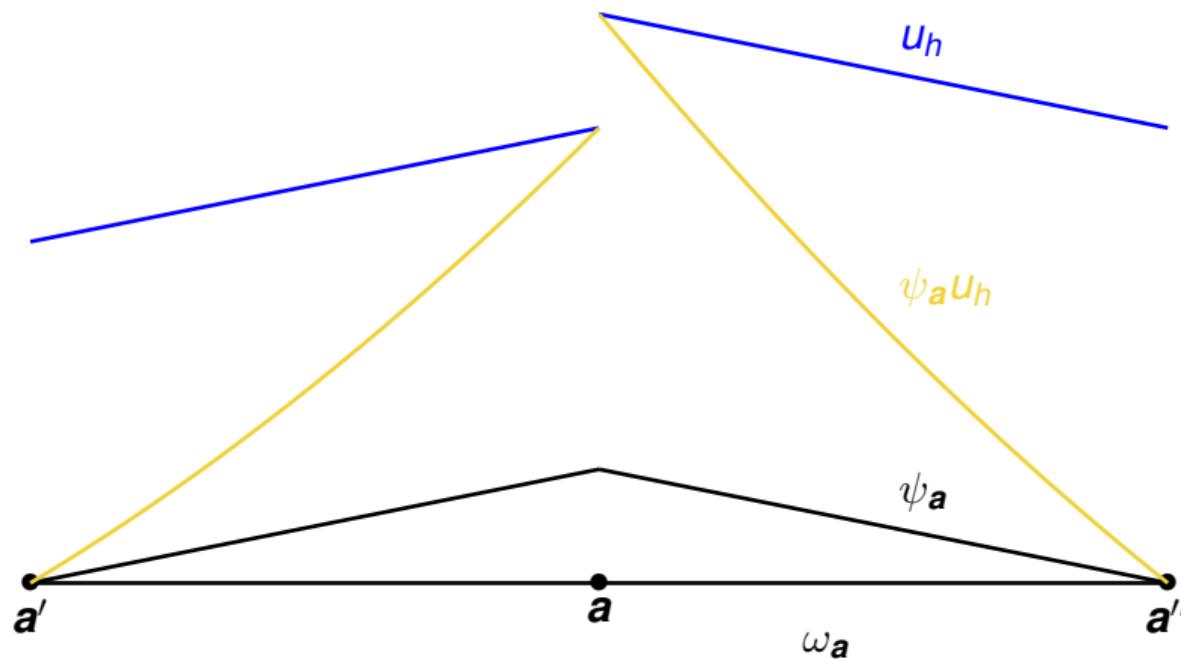
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Equivalent form: conforming FEs

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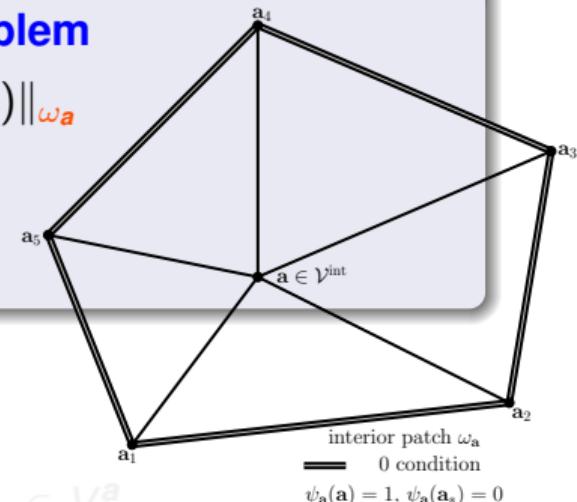
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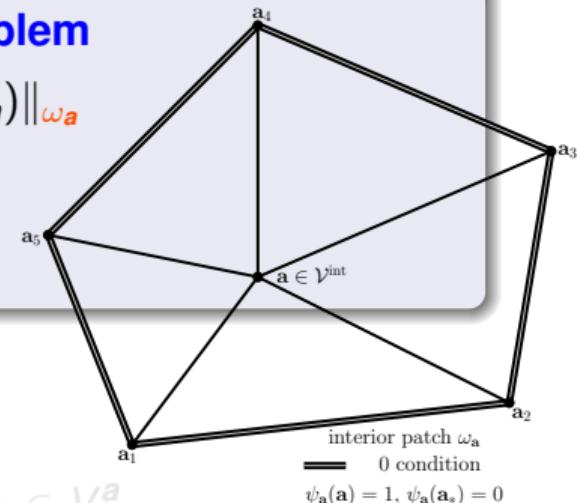
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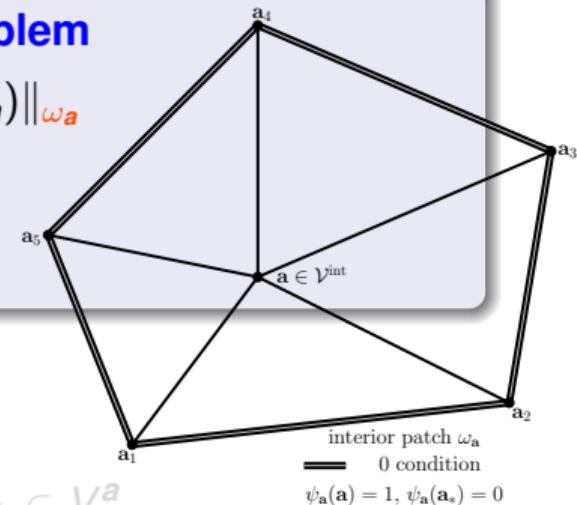
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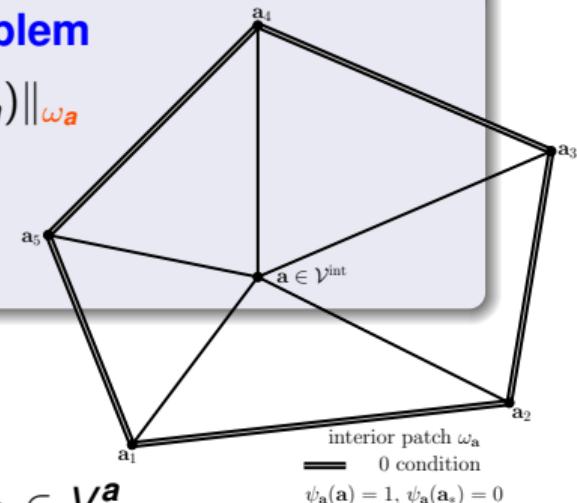
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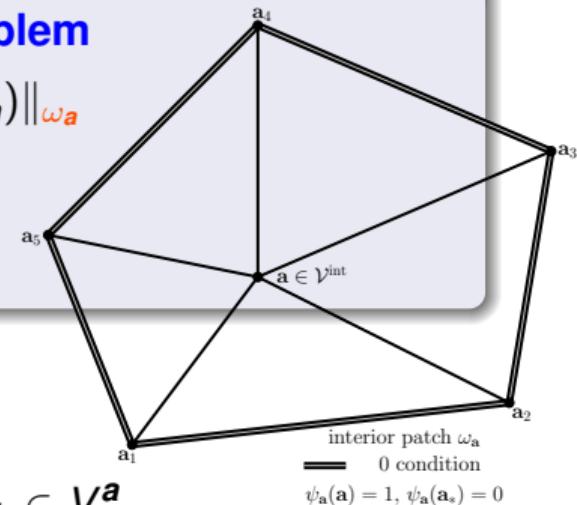
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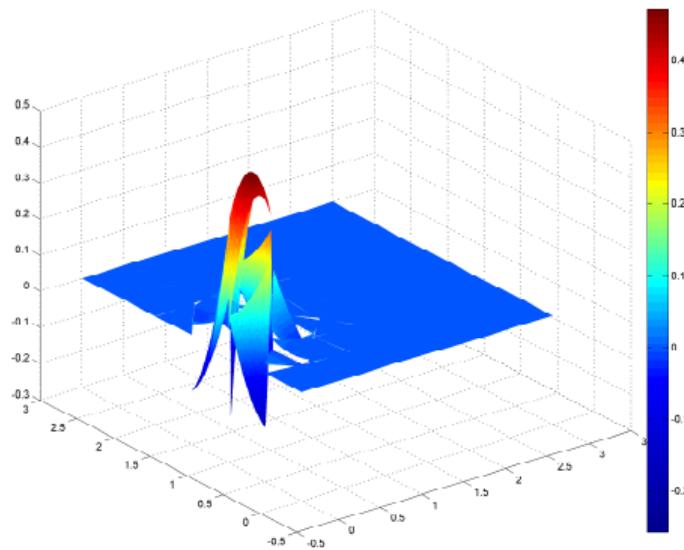
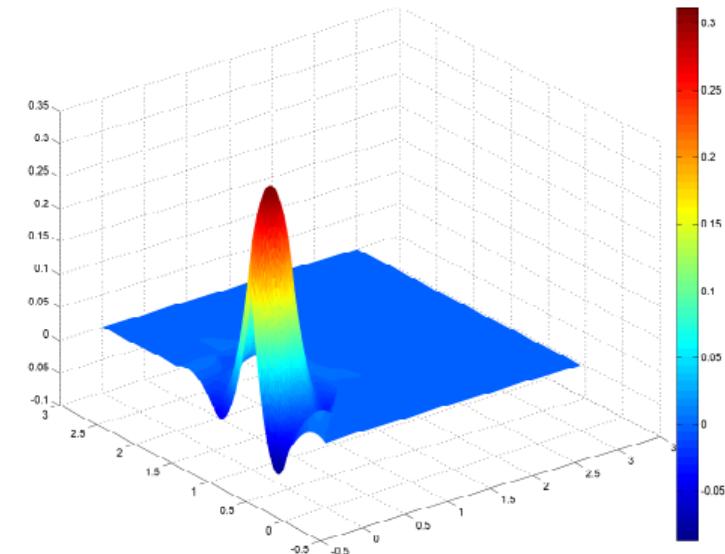
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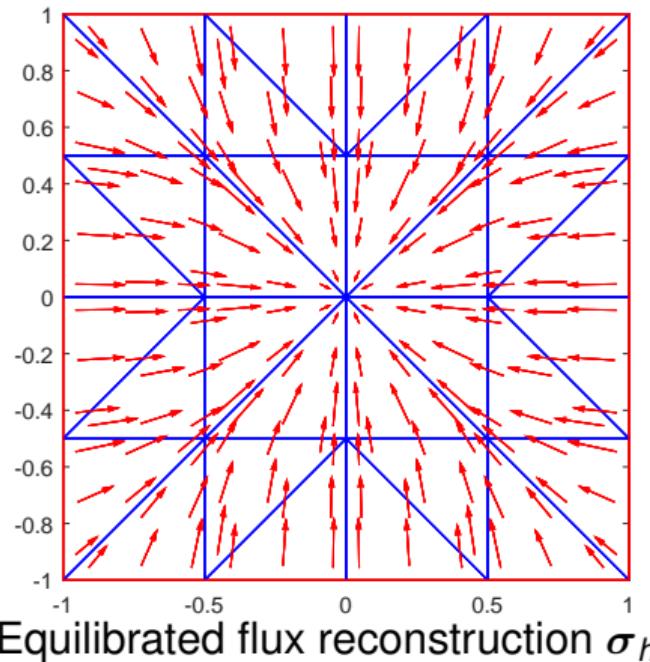
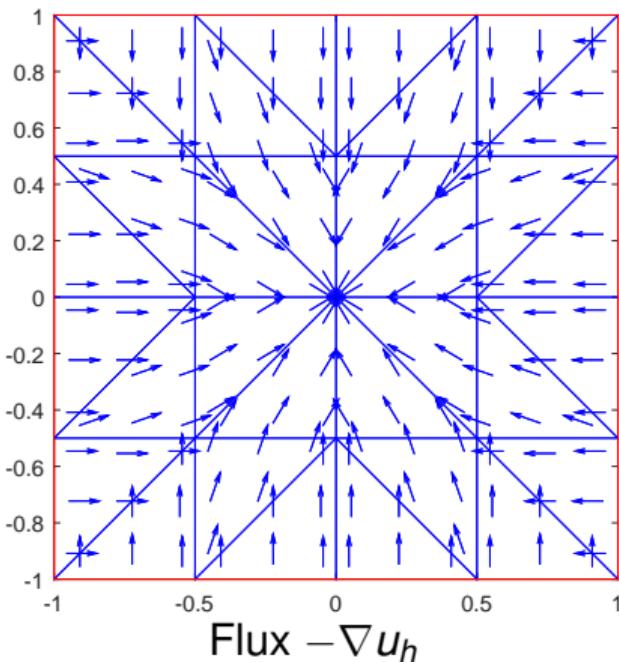
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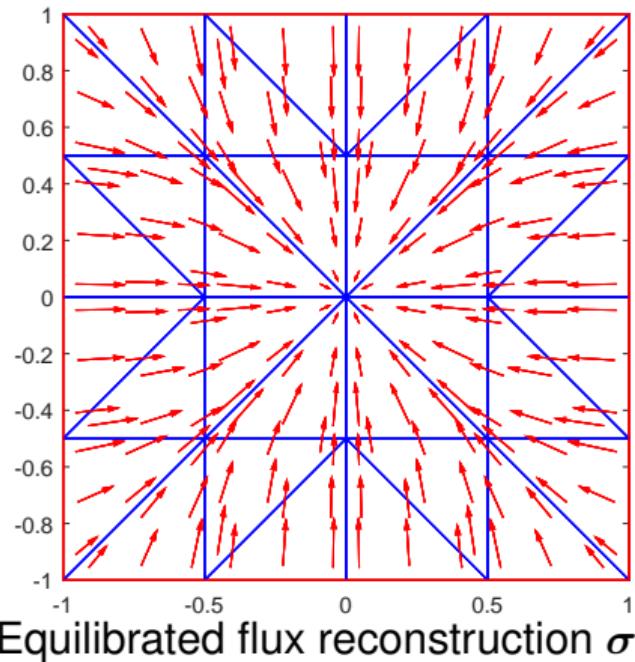
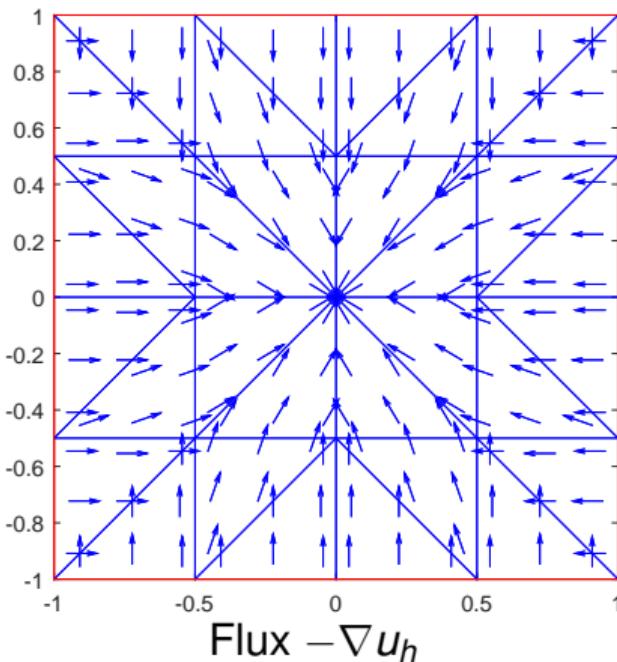
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Equilibrated flux reconstruction



$$\underbrace{-\nabla u_h \in \mathcal{RT}_{p-1}(\mathcal{T}_h), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} - (\nabla u_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}_h^{\text{int}}} \rightarrow \sigma_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap H(\text{div}, \Omega), \nabla \cdot \sigma_h = \Pi_p f$$

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Definition (Construction of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}_h$, solve the **local constrained minimization pb**

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Properties:

Key points

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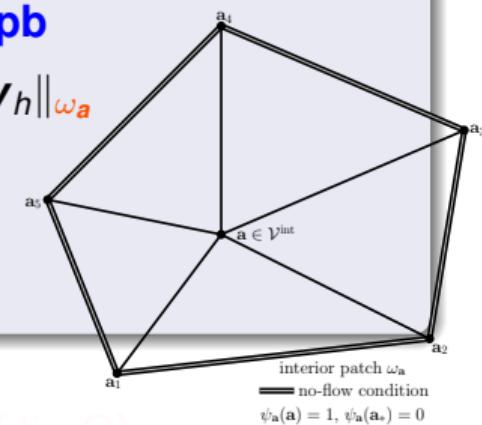
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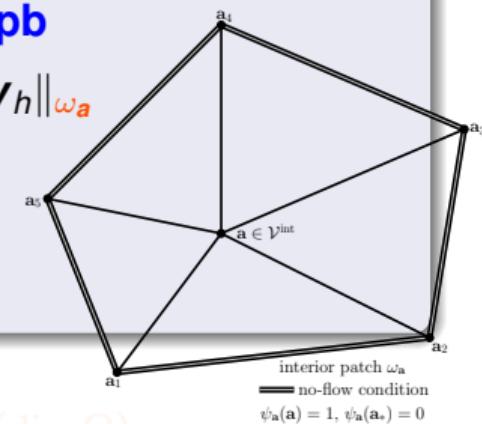
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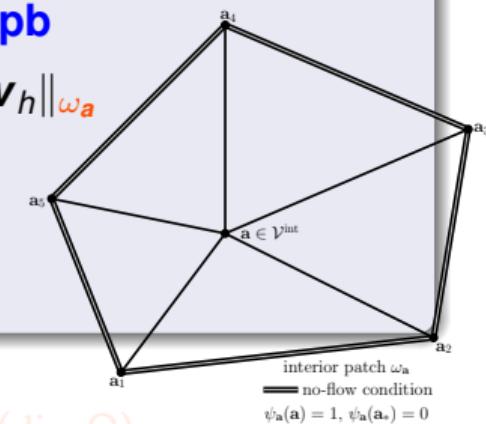
Definition (Construction of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{V}_h^a := \mathcal{RT}_p(\mathcal{T}^a) \cap H_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(f \psi_a - \nabla u_h \cdot \nabla \psi_a)}} \|\psi_a \nabla u_h + \mathbf{v}_h\|_{\omega_a}$$

and combine

$$\sigma_h := \sum_{a \in \mathcal{V}_h} \sigma_h^a.$$



Key points

- homogeneous Neumann BC on $\partial \omega_a$: $\sigma_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap H(\text{div}, \Omega)$
- equilibrium** $\nabla \cdot \sigma_h = \sum_{a \in \mathcal{V}_h} \nabla \cdot \sigma_h^a = \sum_{a \in \mathcal{V}_h} \Pi_p(f \psi_a - \nabla u_h \cdot \nabla \psi_a) = \Pi_p f$

Equilibrated flux reconstruction: $-\nabla u_h \in \mathcal{RT}_{p-1}(\mathcal{T}_h)$, $p \geq 1$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

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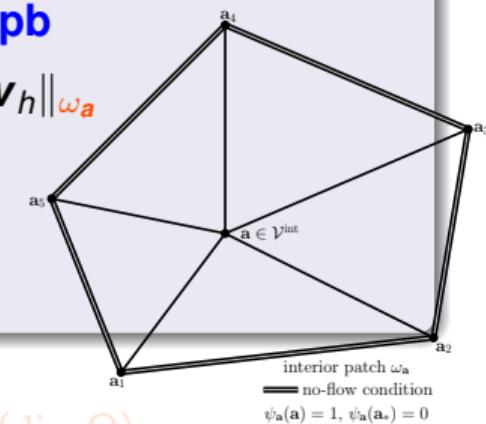
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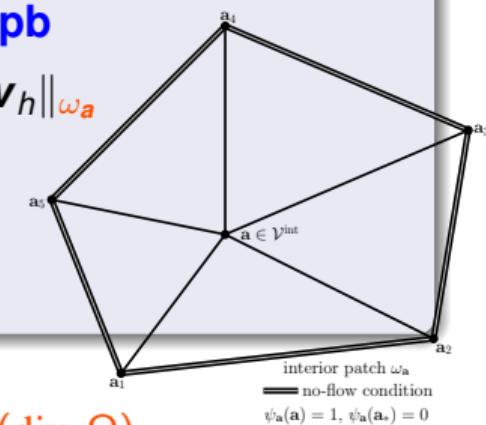
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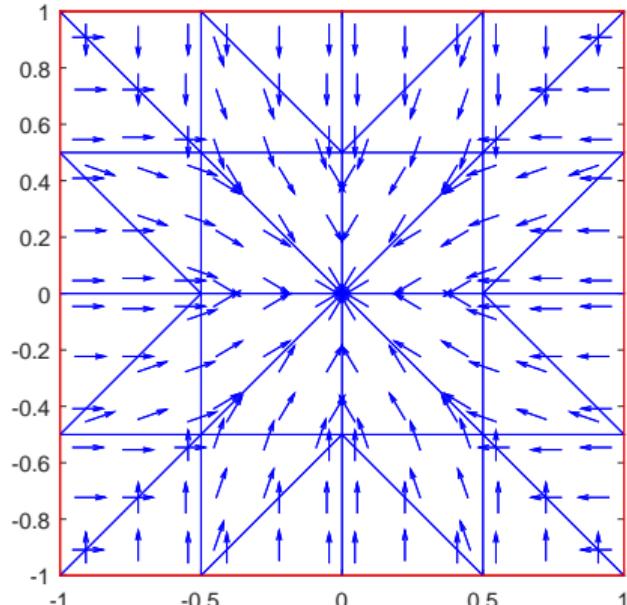
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Key points

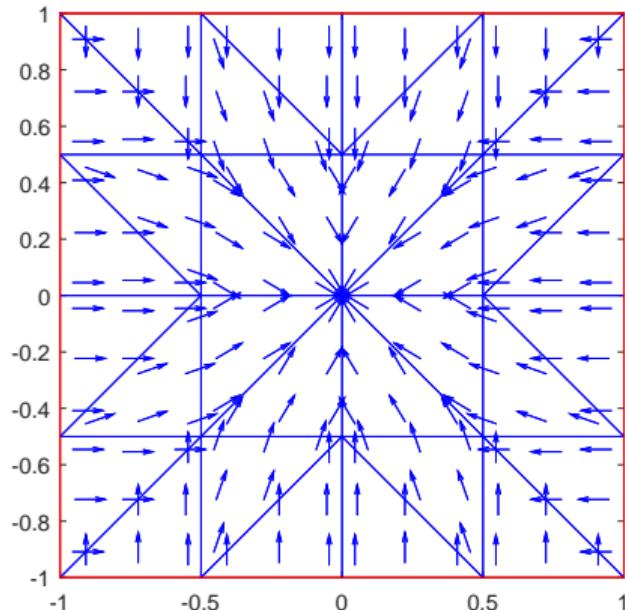
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Flux $-\nabla u_h \notin \mathbf{H}(\text{div}, \Omega)$, $\nabla \cdot (-\nabla u_h) \neq f$

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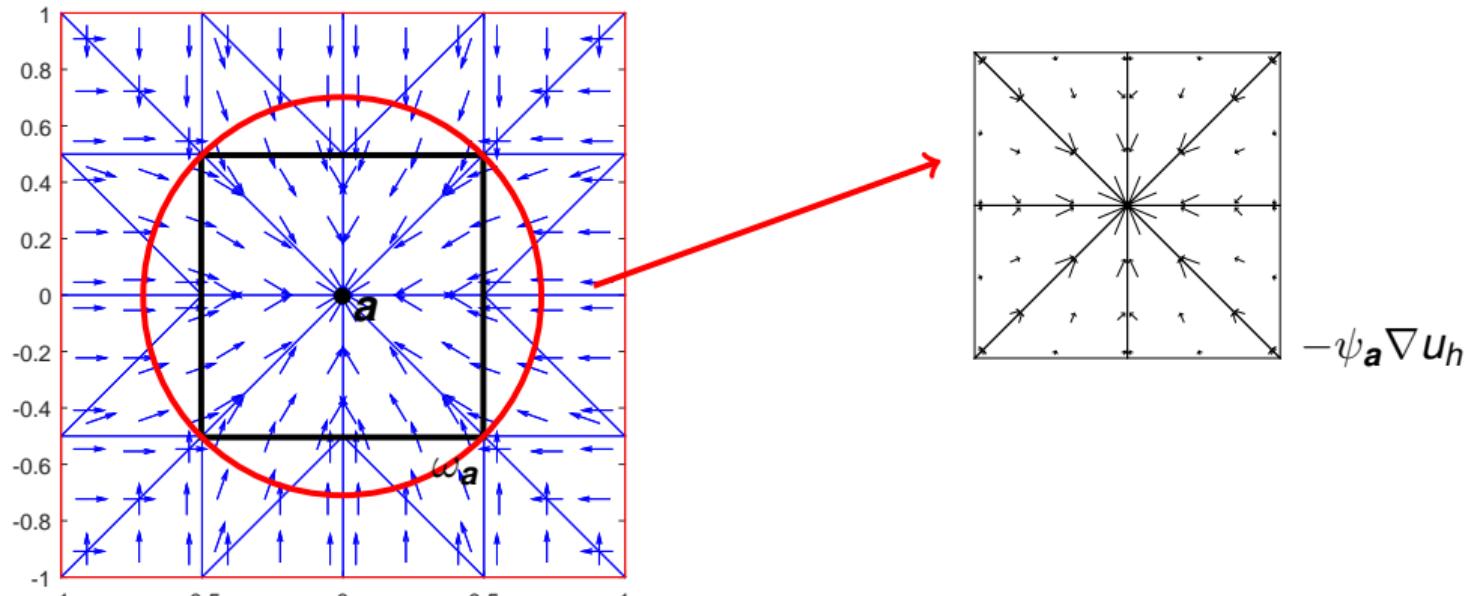


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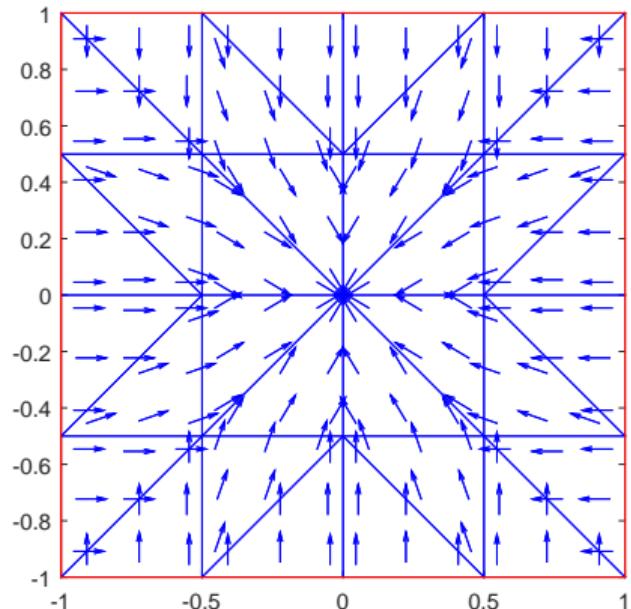


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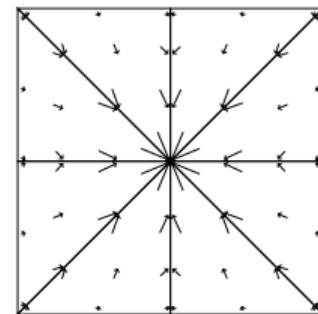
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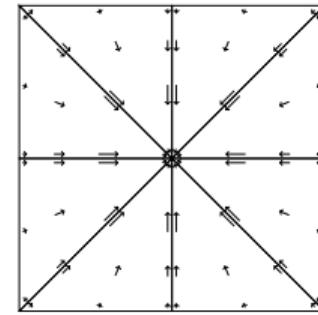
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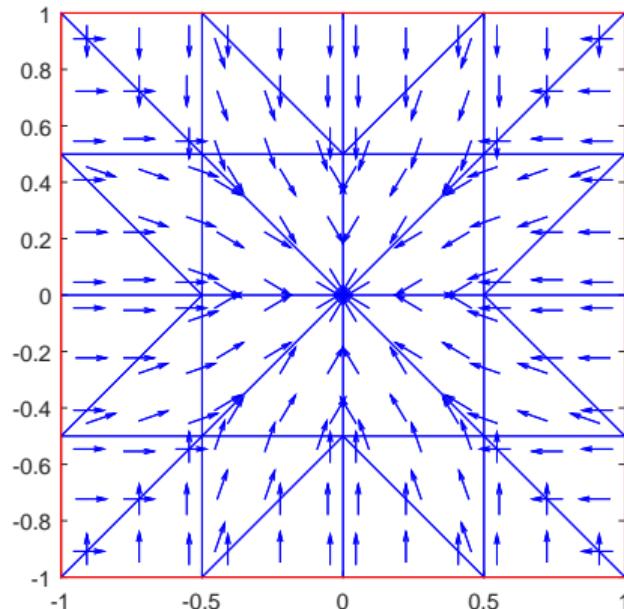


$-\psi_{\mathbf{a}} \nabla u_h$

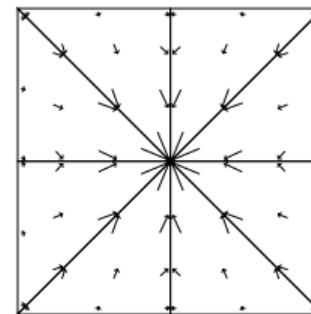


$\sigma_{\mathbf{h}}^{\mathbf{a}}$

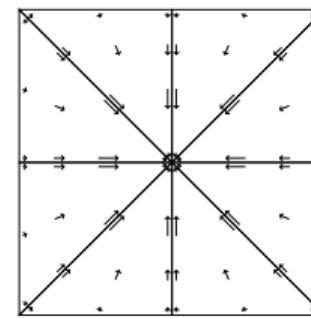
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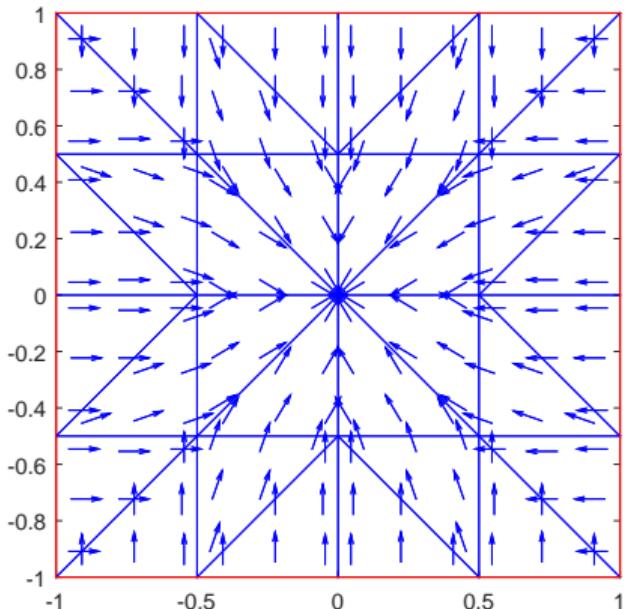
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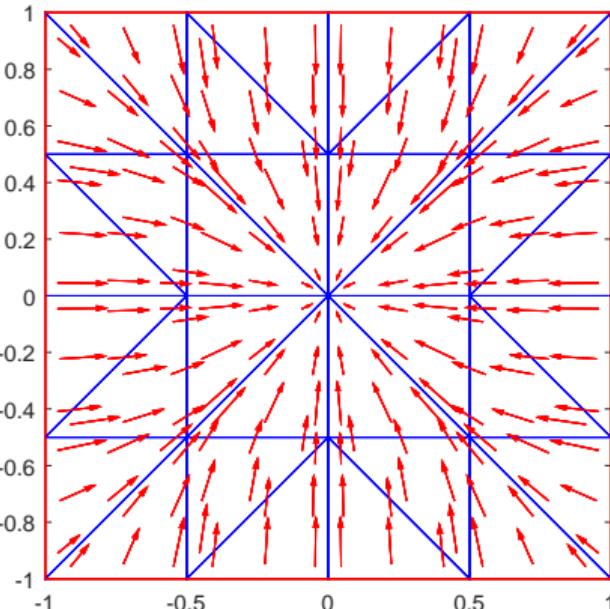
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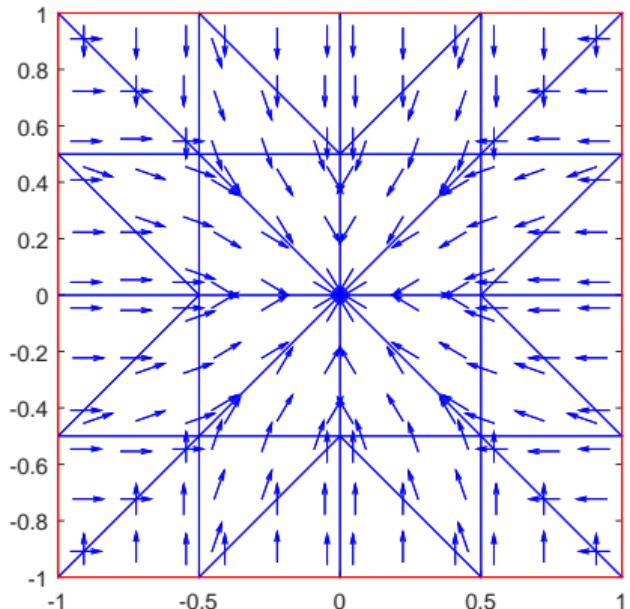
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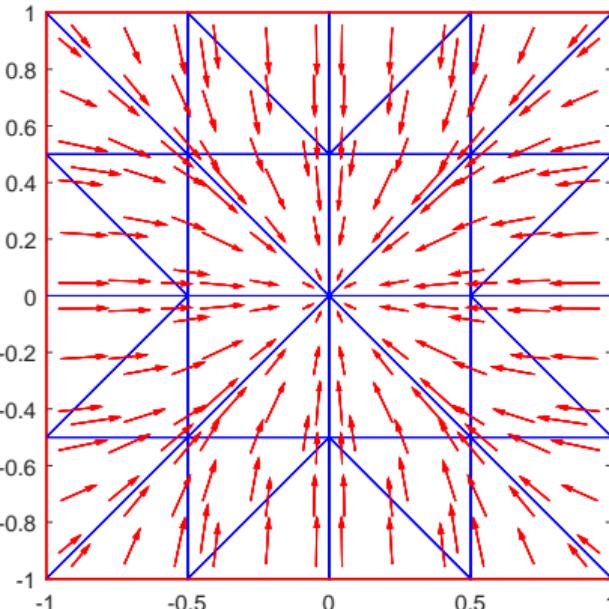
Equilibrated flux rec. σ_h

$$\underbrace{-\nabla u_h \in \mathcal{RT}_{p-1}(\mathcal{T}_h), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} - (\nabla u_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}_h^{\text{int}}} \rightarrow \sigma_h := \sum_{a \in \mathcal{V}_h} \sigma_h^a \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega), \nabla \cdot \sigma_h = \Pi_p f$$

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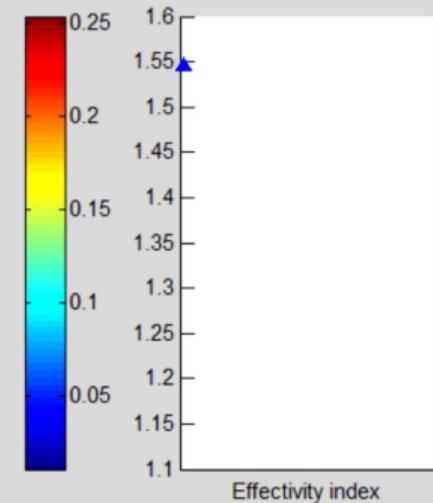
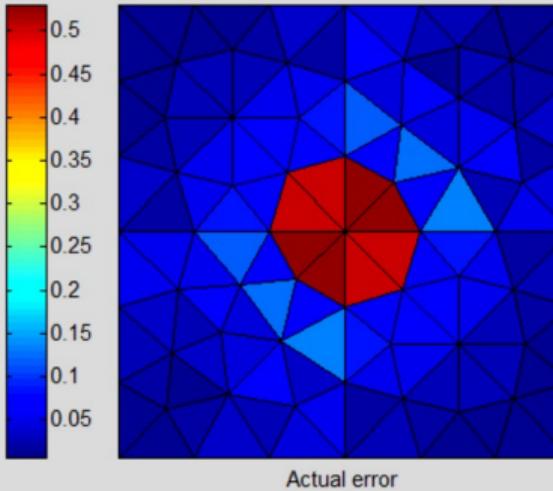
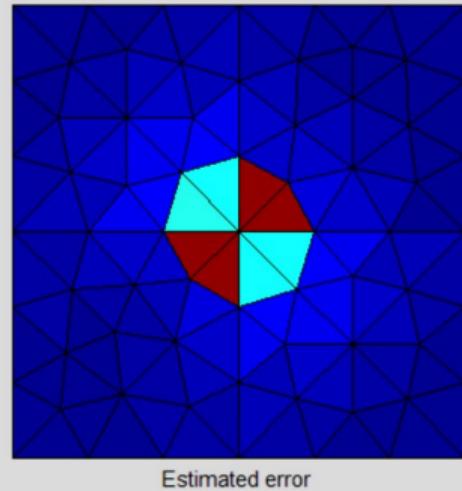
$$\underbrace{(-\nabla u_h, \psi_a)_{\omega_a} - (\nabla u_h, \nabla \psi_a)_{\omega_a}}_{(f, \psi_a)_{\omega_a} - (\nabla u_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}_h^{\text{int}}} = 0 \quad \forall a \in \mathcal{V}_h^{\text{int}}$$

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Outline

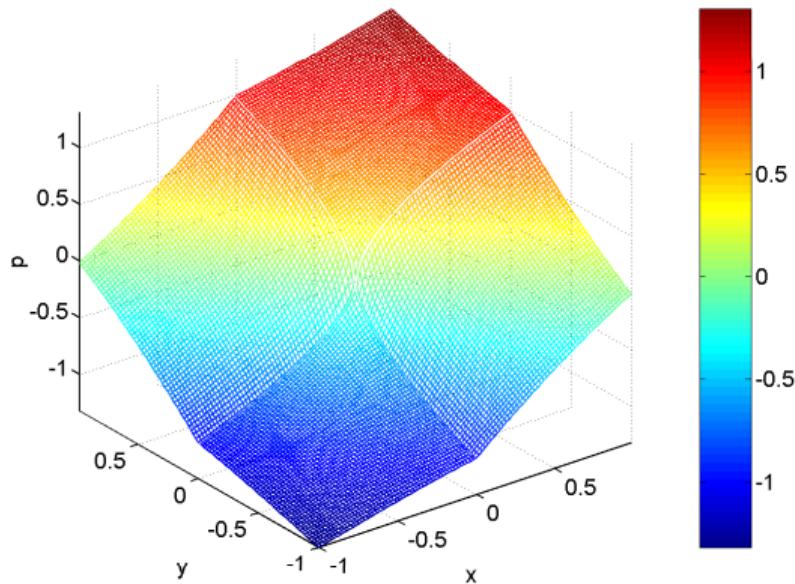
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Can we decrease the error efficiently? (adaptive mesh refinement)

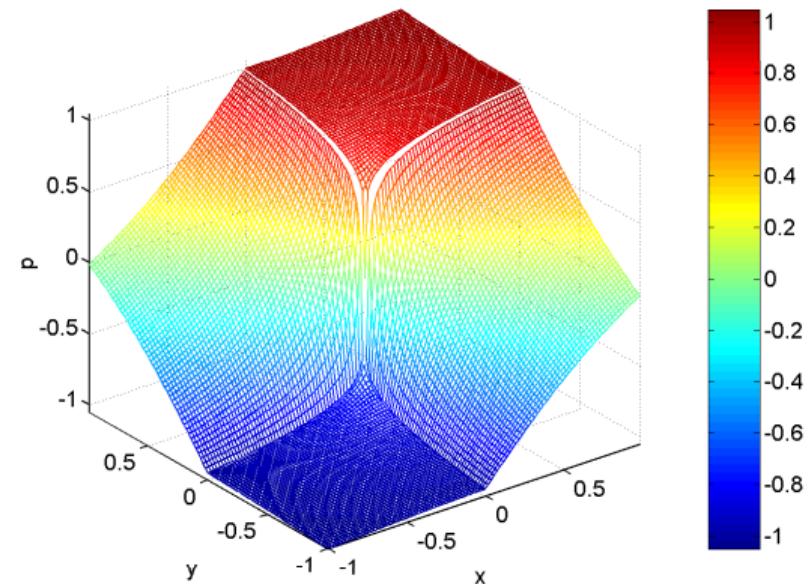


M. Vohralík, SIAM Journal on Numerical Analysis (2007)

Singular solutions

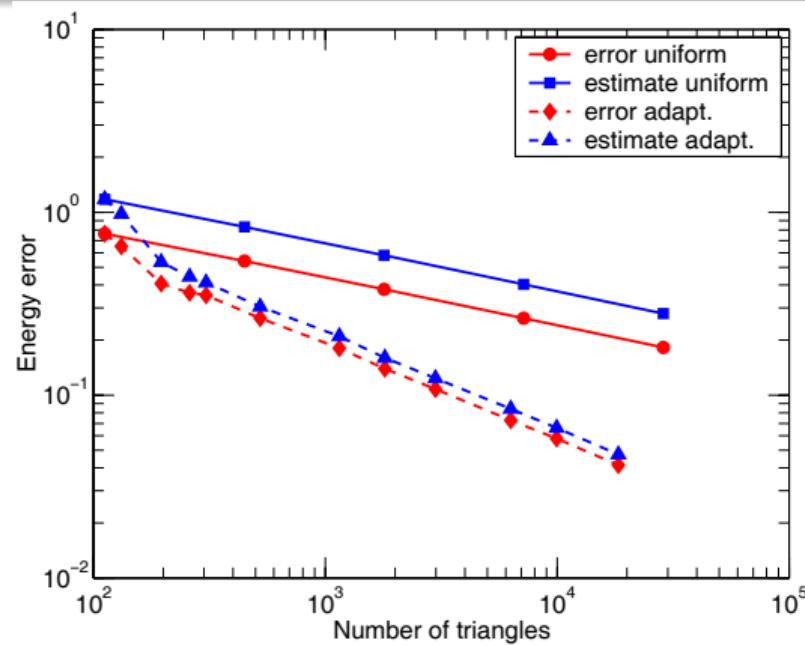


$H^{1.54}$ singularity

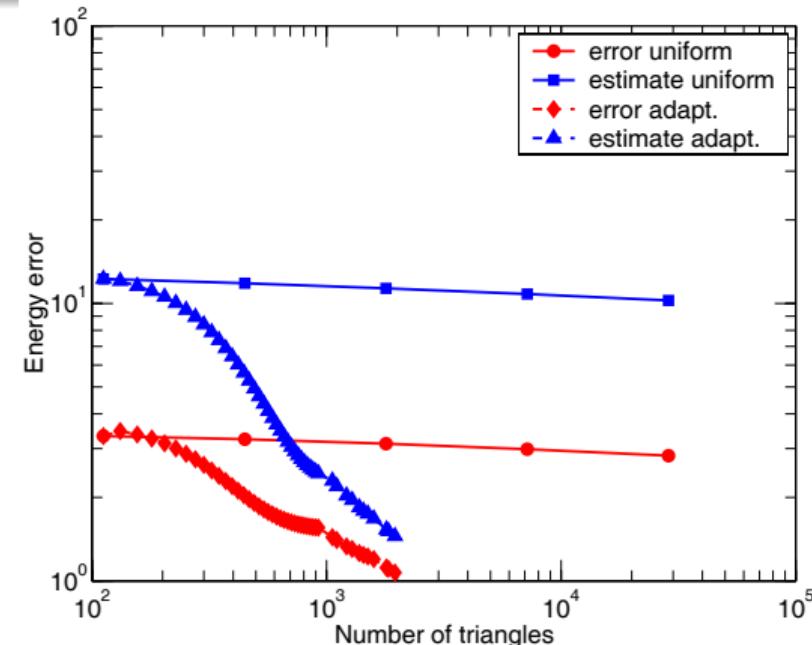


$H^{1.13}$ singularity

Estimated and actual error against the number of elements in uniformly/adaptively refined meshes (singular solutions)



$H^{1.54}$ singularity



$H^{1.13}$ singularity

Adaptive mesh refinement

Adaptive mesh refinement

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Adaptive mesh refinement

$$\sum_{K \in \mathcal{T}_\ell} \eta_K(u_\ell)^2 = \eta(u_\ell)^2$$

Adaptive mesh refinement

Adaptive mesh refinement

- Dörfler marking: subset \mathcal{M}_ℓ containing θ -fraction of the estimates

$$\sum_{K \in \mathcal{M}_\ell} \eta_K(u_\ell)^2 \geq \theta^2 \sum_{K \in \mathcal{T}_\ell} \eta_K(u_\ell)^2 = \theta^2 \eta(u_\ell)^2$$

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Convergence on a sequence of adaptively refined meshes

- $\|\nabla(u - u_\ell)\| \rightarrow 0$
- some mesh elements may not be refined at all: $h \searrow 0$
- Babuška & Miller (1987), Dörfler (1996)

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Optimal error decay rate wrt degrees of freedom

- $\|\nabla(u - u_\ell)\| \lesssim |\text{DoF}_\ell|^{-p/d}$ (replaces h^p)
- same for smooth & singular solutions: higher order only pay off for sm. sol.
- decays to zero as fast as on a best-possible sequence of meshes
- Morin, Nochetto, Siebert (2000), Stevenson (2005, 2007), Cascón, Kreuzer, Nochetto, Siebert (2008), Canuto, Nochetto, Stevenson, Verani (2017)

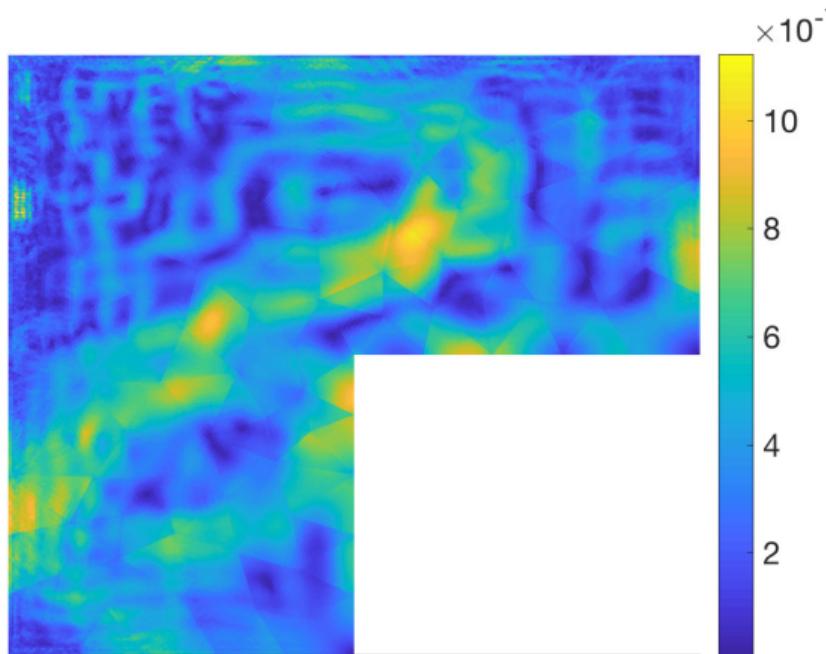
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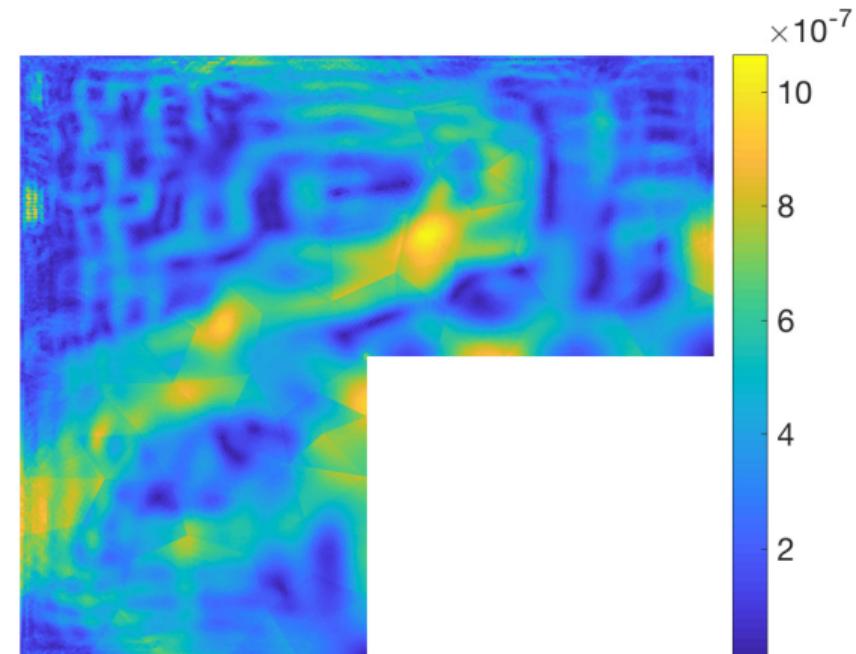
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Including algebraic error: $\mathbb{A}_\ell \mathbf{U}_\ell^i \neq \mathbf{F}_\ell$



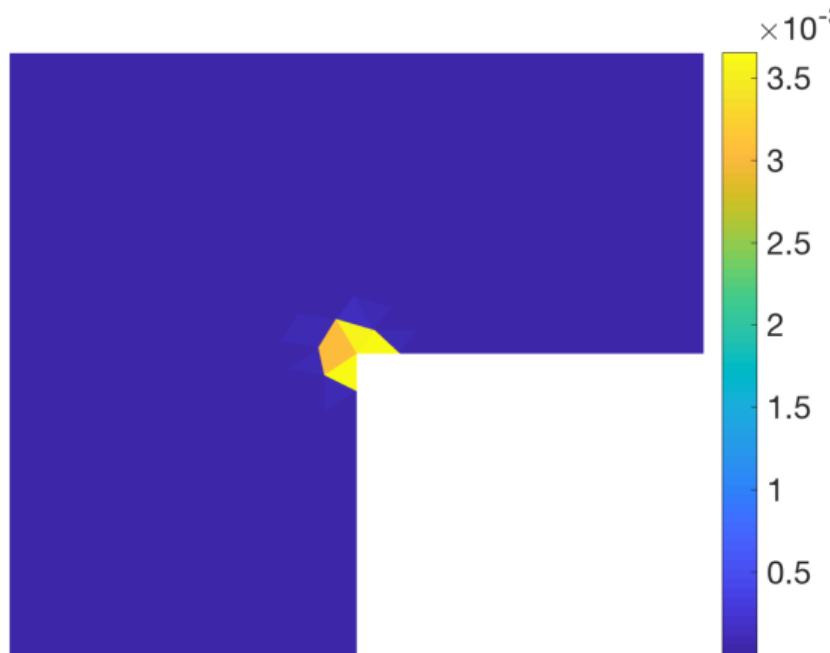
Estimated algebraic errors $\eta_{\text{alg}, \kappa}(u_\ell^i)$



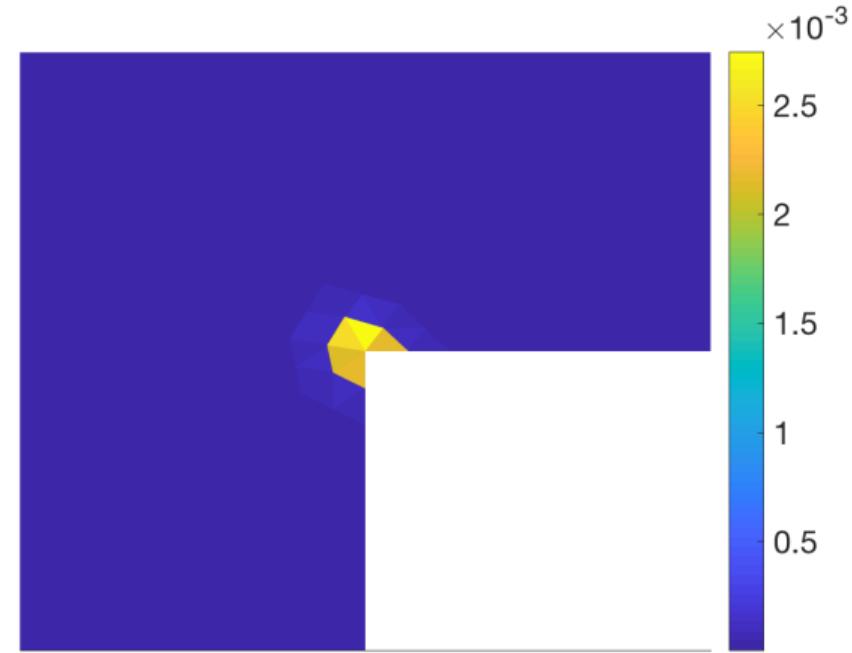
Exact algebraic errors $\|\nabla(u_\ell - u_\ell^i)\|_\kappa$

J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

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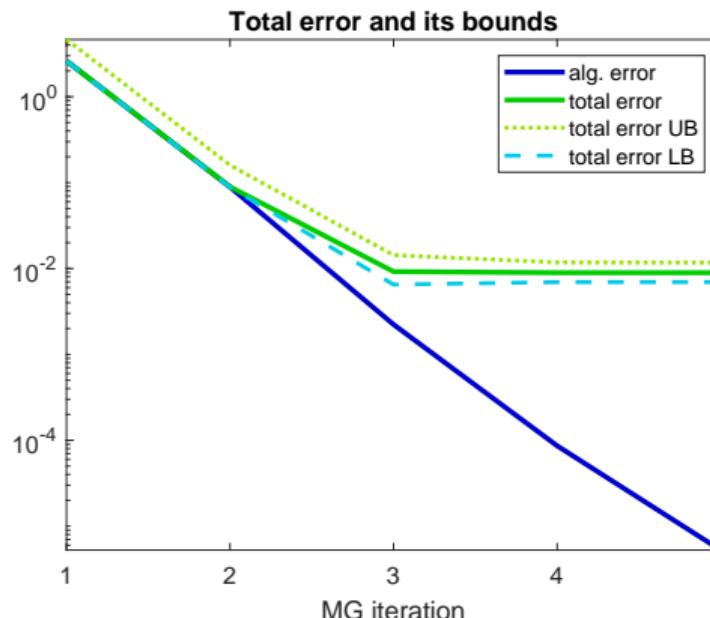
Estimated total errors $\eta_K(u_\ell^i)$



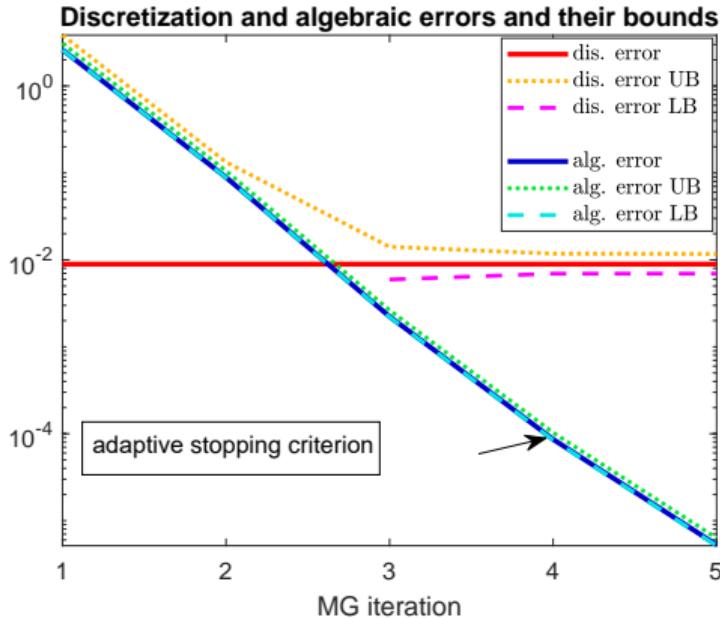
Exact total errors $\|\nabla(u - u_\ell^i)\|_K$

J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

Including algebraic error: $\mathbb{A}_\ell \mathbf{U}_\ell^i \neq \mathbf{F}_\ell$



Total error



Error components and adaptive st. crit.

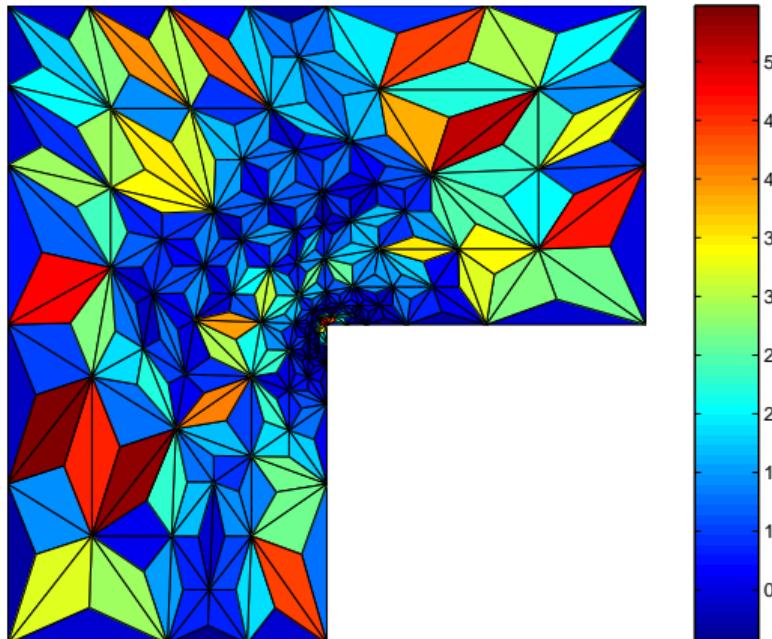
J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

Nonlinear pb $-\nabla \cdot \sigma(\nabla u) = f$: including linearization and algebraic error: $\mathcal{A}_\ell(\mathbf{U}_\ell^{k,r}) \neq \mathbf{F}_\ell$, $\mathbf{A}_\ell^{k-1} \mathbf{U}_\ell^{k,r} \neq \mathbf{F}_\ell^{k-1}$

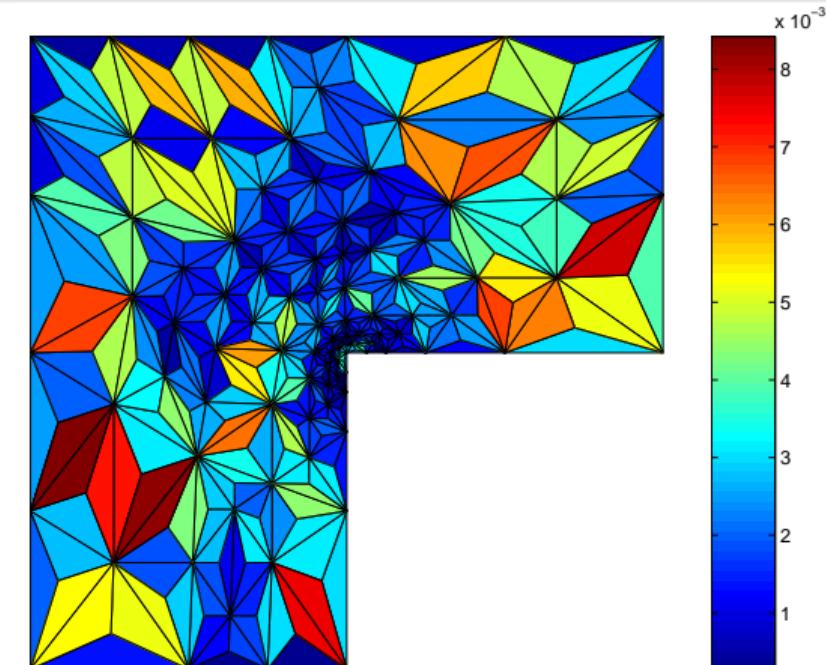
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Estimated errors $\eta_K(u_\ell^{k,i})$



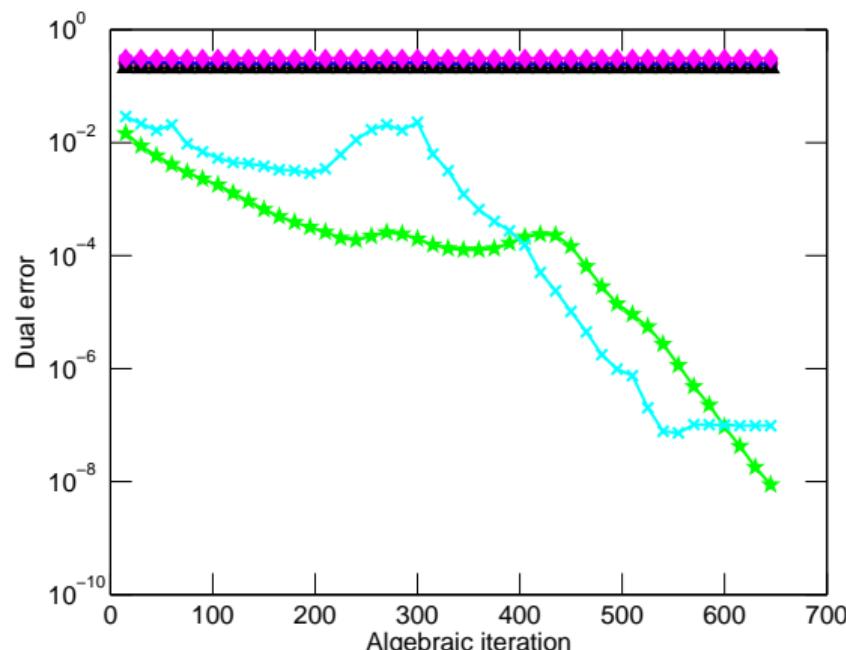
Exact errors $\|\sigma(\nabla u) - \sigma(\nabla u_\ell^{k,i})\|_{q,K}$

A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2013)

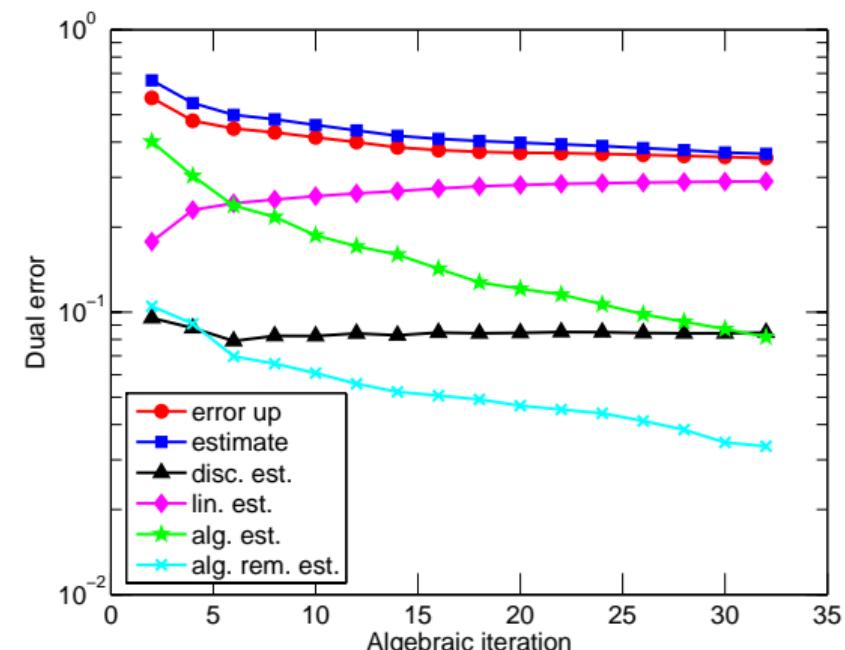
Estimation d'erreur a posteriori : principe et applications

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Nonlinear pb $-\nabla \cdot \sigma(\nabla u) = f$: including **linearization** and **algebraic error**: $\mathcal{A}_\ell(\mathbf{U}_\ell^{k,i}) \neq \mathbf{F}_\ell$, $\mathbb{A}_\ell^{k-1} \mathbf{U}_\ell^{k,i} \neq \mathbf{F}_\ell^{k-1}$

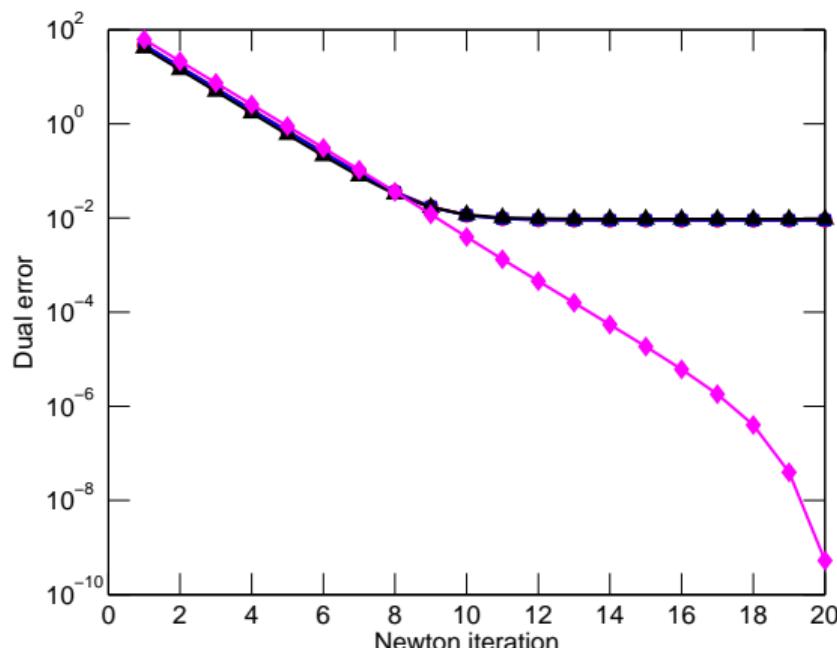


Newton

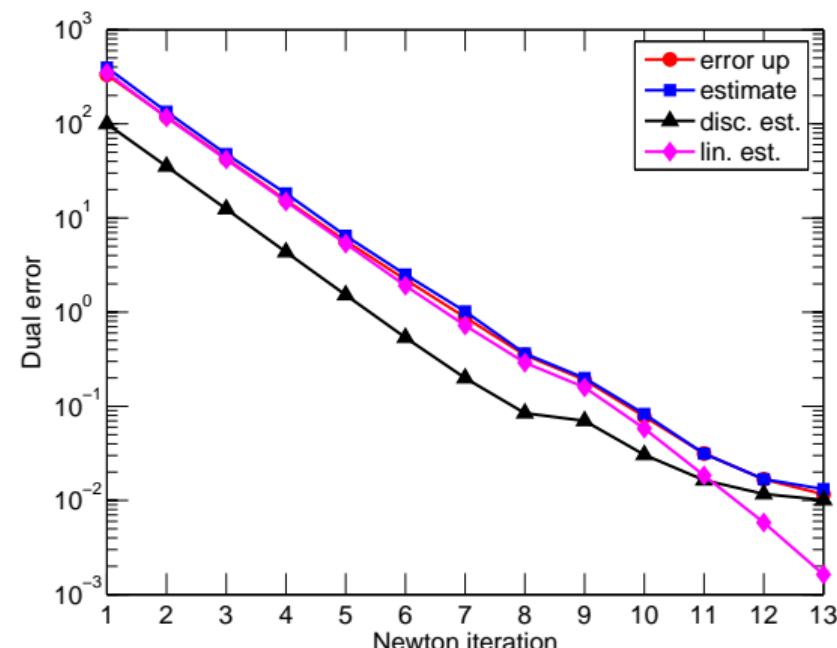


adaptive inexact Newton

Nonlinear pb $-\nabla \cdot \sigma(\nabla u) = f$: including linearization and algebraic error: $\mathcal{A}_\ell(U_\ell^{k,i}) \neq F_\ell$, $\mathbb{A}_\ell^{k-1} U_\ell^{k,i} \neq F_\ell^{k-1}$



Newton



adaptive inexact Newton

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Solver adaptivity (nonlinear problem, inexact solvers)

Fully adaptive algorithm

- total error estimate on mesh \mathcal{T}_ℓ , linearization step k , algebraic solver step i

$$\underbrace{\|u - u_\ell^{k,i}\|_*}_{\text{total error}} \leq \underbrace{\eta_{\ell,\text{disc}}^{k,i}}_{\text{discretization estimate}} + \underbrace{\eta_{\ell,\text{lin}}^{k,i}}_{\text{linearization estimate}} + \underbrace{\eta_{\ell,\text{alg}}^{k,i}}_{\text{algebraic estimate}}$$

- balancing error components: work where needed

$$\eta_{\ell,\text{alg}}^{k,i} \leq \gamma_{\text{alg}} \eta_{\ell,\text{lin}}^{k,i} \quad \text{stopping criterion linear solver}$$

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- link – inexact Newton method: Bank & Rose (1982), Hackbusch & Reusken (1989), Deuflhard (1991), Eisenstat & Walker (1994)

Convergence, optimal error decay rate wrt DoFs

- Gantner, Haberl, Praetorius, & Stiftner (2018), Heid & Wihler (2019)

Optimal error decay rate wrt overall computational cost

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Solver adaptivity (nonlinear problem, inexact solvers)

Fully adaptive algorithm (adaptive inexact Newton method)

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Outline

- 1 Introduction: a posteriori error control and adaptivity
- 2 Laplace equation: discretization error control and mesh adaptivity
 - A posteriori error control (discretization)
 - Potential reconstruction
 - Flux reconstruction
 - Balancing error components: mesh adaptivity
- 3 Nonlinear Laplace equation: overall error control and solver adaptivity
 - A posteriori error control (overall and components)
 - Balancing error components: solver adaptivity
- 4 Reaction–diffusion equation: robustness wrt parameters
- 5 Heat equation: robustness wrt final time and space–time localization
- 6 Helmholtz equation: asymptotic robustness
- 7 Conclusions

The reaction–diffusion equation: $f \in L^2(\Omega)$, $\varepsilon > 0$, $\kappa \geq 0$ parameters

Find $u : \Omega \rightarrow \mathbb{R}$ such that ($\varepsilon \ll \kappa$ **singular perturbation**)

$$\begin{aligned} -\varepsilon^2 \Delta u + \kappa^2 u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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$$\underbrace{\|u - u_h\|}_{\text{unknown error}} \quad \underbrace{\eta(u_h)}_{\text{computable estimator}}$$

error lower bound (efficiency, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

$$\eta(u_h) \leq C_{\text{eff}} \|u - u_h\|$$

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Equilibrated flux and potential reconstructions

Definition (Flux σ_h and potential ϕ_h)

For each vertex $a \in \mathcal{V}$, let

$$(\sigma_h^a, \phi_h^a) := \arg \min_{(v_h, q_h) \in \mathcal{RT}_p(\mathcal{T}^a) \times \mathcal{P}_p(\mathcal{T}^a)} \|v_h\|_{\omega_a} + \|q_h\|_{\omega_a}$$

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Comments

- **local discrete** constrained minimization problems
- choose the locally best possible estimators
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$$(\sigma_h^{\mathbf{a}}, \phi_h^{\mathbf{a}}) := \arg \min_{\substack{(\mathbf{v}_h, q_h) \in \mathcal{RT}_p(\mathcal{T}^{\mathbf{a}}) \times \mathcal{P}_p(\mathcal{T}^{\mathbf{a}}) \subset \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) \times L^2(\omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h + \kappa^2 q_h = \Pi_h(f\psi_{\mathbf{a}}) - \varepsilon^2 \nabla u_h \cdot \nabla \psi_{\mathbf{a}}}} J_{U_h}^{\mathbf{a}}(\mathbf{v}_h, q_h)$$

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with the weight $w_{\mathbf{a}} := \min \left\{ 1, C_* \sqrt{\frac{\varepsilon}{\kappa h_{\omega_{\mathbf{a}}}}} \right\}$. Combine

$$\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}} \sigma_h^{\mathbf{a}} \in \mathcal{RT}_p \cap \mathbf{H}(\text{div}, \Omega), \quad \phi_h := \sum_{\mathbf{a} \in \mathcal{V}} \phi_h^{\mathbf{a}} \in \mathcal{P}_p(\mathcal{T}_h).$$

Comments

- **local discrete** constrained minimization problems
- choose the locally **best-possible** estimators
- yields $\nabla \cdot \sigma_h + \kappa^2 \phi_h = \Pi_h f$

Equilibrated flux and potential reconstructions

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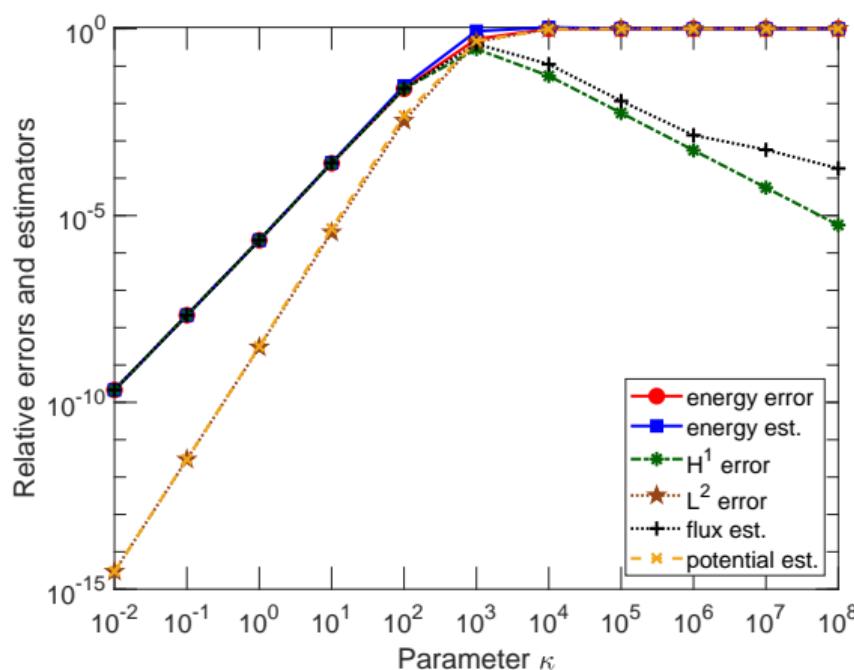
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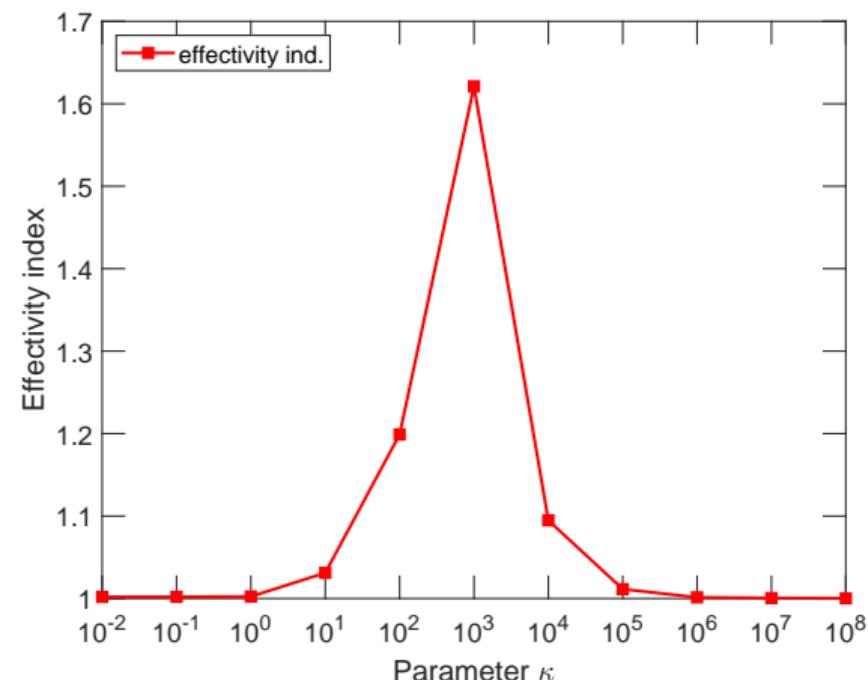
Comments

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Boundary layer, solution $u(x, y) = e^{-\frac{\kappa}{\varepsilon}x} + e^{-\frac{\kappa}{\varepsilon}y}$, $p = 2$

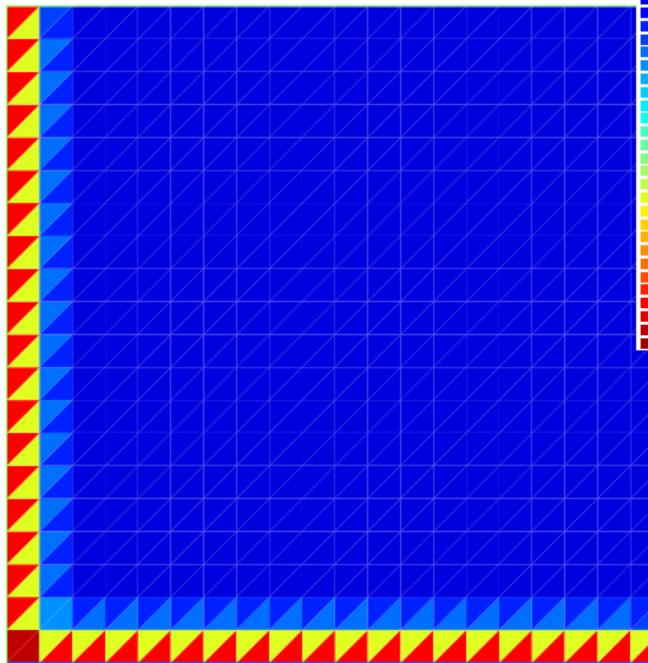


Relative energy errors and estimates

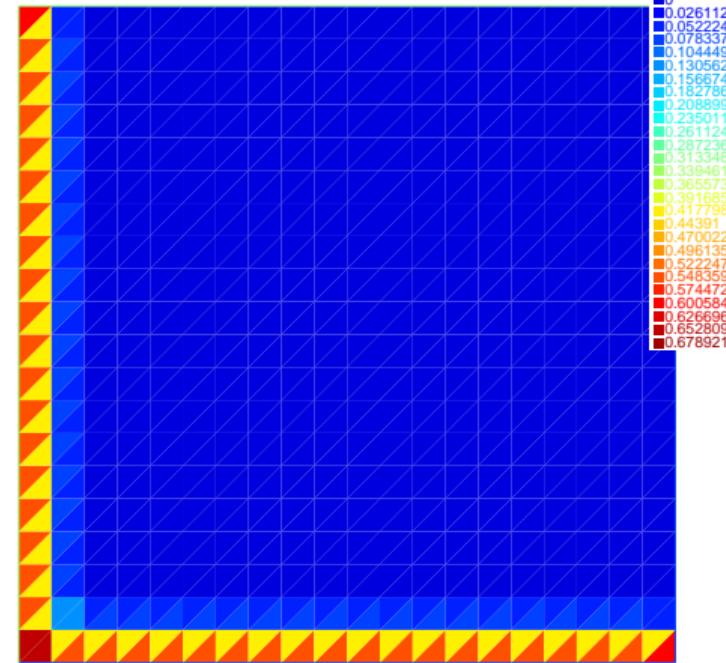
Effectivity indices $\eta(u_h)/\|u - u_h\|$

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estimators

Estimated error distribution $\eta_K(u_h)$

energy errors

Exact error distribution $\|u - u_h\|_K$

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The heat equation ($f \in L^2(0, T; L^2(\Omega))$, $u_0 \in L^2(\Omega)$)

The heat equation

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$$X := L^2(0, T; H_0^1(\Omega)), \|v\|_X^2 := \int_0^T \|\nabla v\|^2 dt,$$

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Y norm error is the dual X norm of the residual + initial condition error

$$\|u - u_{h\tau}\|_Y^2 = \sup_{v \in X, \|v\|_X=1} \left[\int_0^T (f, v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v) dt \right]^2 + \|u_0 - u_{h\tau}(0)\|^2$$

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- C_{eff} a generic constant independent of Ω , u , $u_{h\tau}$, h , p , τ , q ,

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Equilibrated flux reconstruction

Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\sigma_{h\tau}^{\mathbf{a},n} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_{h\tau}^{\mathbf{a},n}} \int_{I_n} \|\mathbf{v}_h + \psi_{\mathbf{a}} \nabla \mathbf{u}_{h\tau}\|_{\omega_{\mathbf{a}}}^2 dt.$$

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$$\sigma_{h\tau} := \sum_{n=1}^N \sum_{\mathbf{a} \in \mathcal{V}^n} \sigma_{h\tau}^{\mathbf{a},n}.$$

Comments

- satisfies $\sigma_{h\tau} \in L^2(0, T; \mathbf{H}(\text{div}, \Omega))$ with $\nabla \cdot \sigma_{h\tau} = f - \partial_t \mathcal{I} \mathbf{u}_{h\tau}$
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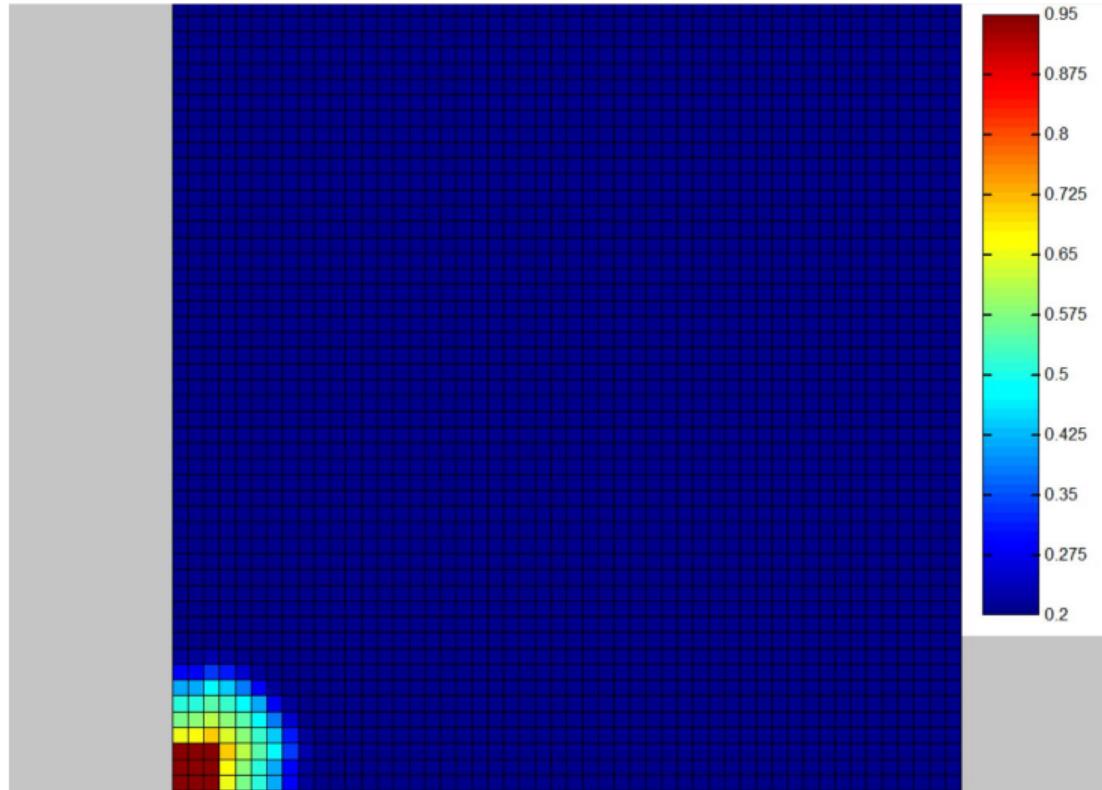
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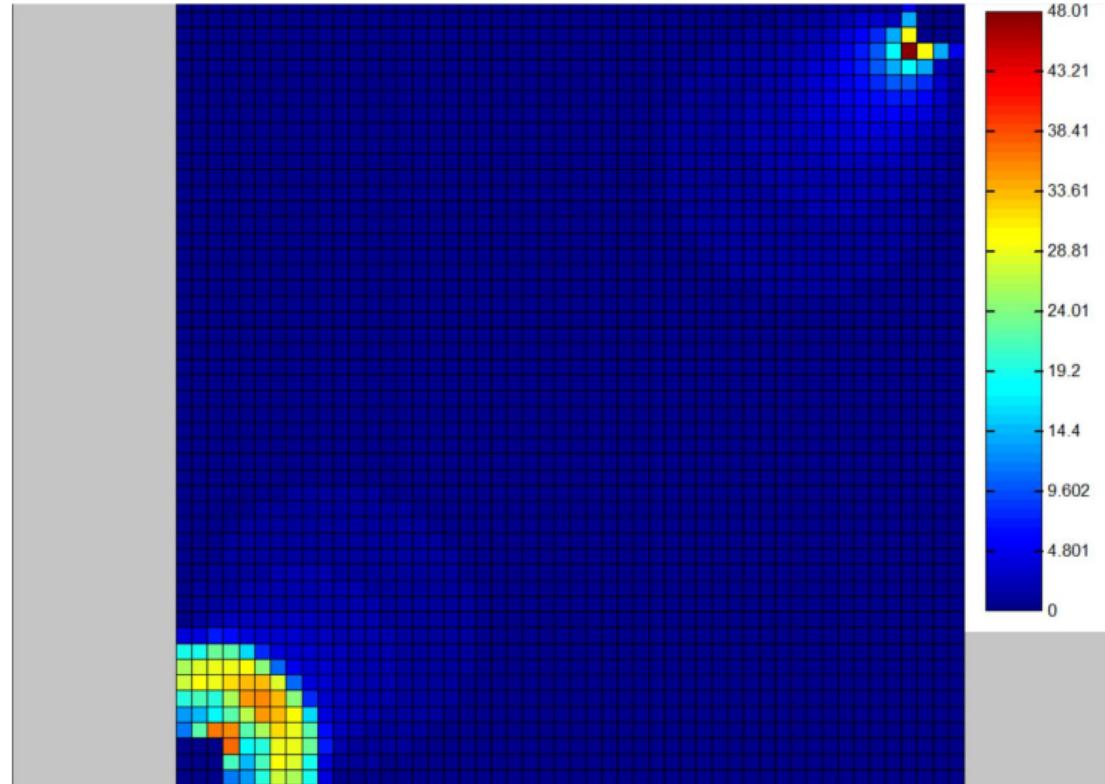
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Geological sequestration of CO₂, CO₂ saturation



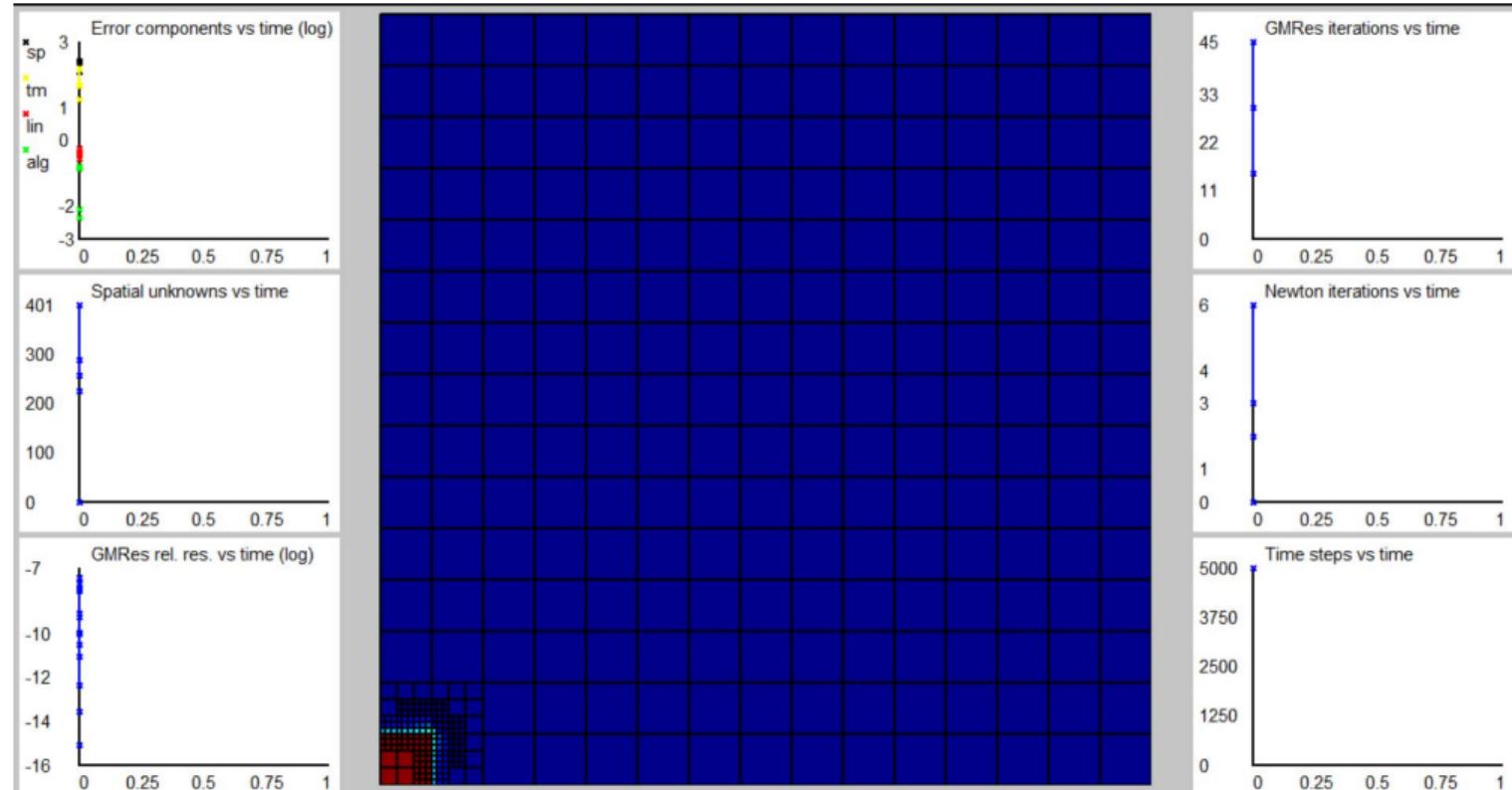
M. Vohralík, M. Wheeler, Computational Geosciences (2013)

Geological sequestration of CO₂, overall a posteriori estimate



M. Vohralík, M. Wheeler, Computational Geosciences (2013)

Geological sequestration of CO₂, full adaptivity



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The Helmholtz equation: $f \in L^2(\Omega)$, $\varepsilon > 0$, $\kappa \geq 0$ parameters

Find $u : \Omega \rightarrow \mathbb{C}$ such that ($\varepsilon \leq \kappa$)

$$\begin{aligned} -\varepsilon^2 \Delta u - \kappa^2 u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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Asymptotically robust local error lower bound (efficiency, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

$$\eta_K(u_h) \leq C_{\text{eff}} \|u - u_h\|_{\omega_K} \quad \forall K \in \mathcal{T}_h$$

- C_{eff} a generic constant independent of Ω , u , u_h , h

The Helmholtz equation: $f \in L^2(\Omega)$, $\varepsilon > 0$, $\kappa \geq 0$ parameters

Find $u : \Omega \rightarrow \mathbb{C}$ such that ($\varepsilon \leq \kappa$)

$$\begin{aligned} -\varepsilon^2 \Delta u - \kappa^2 u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Guaranteed error upper bound (reliability) ($u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, FEs)

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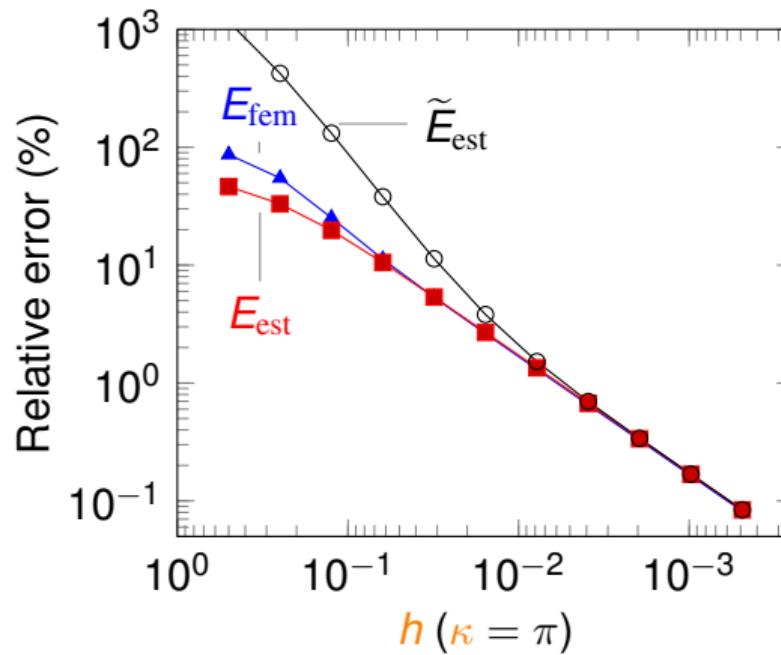
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Plane wave, $p = 1$ and $\kappa = \pi$

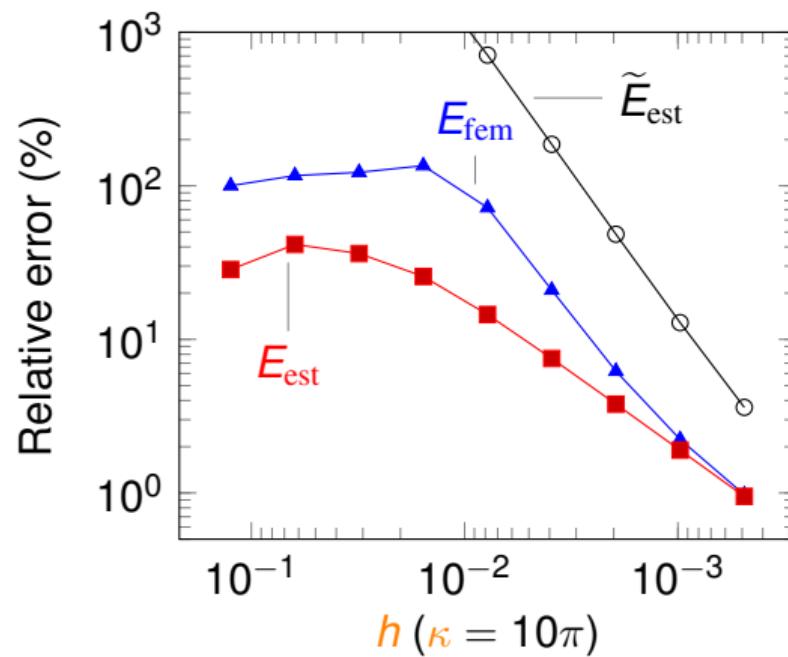


$$E_{\text{fem}} := \|e_h\|_{\kappa, \Omega}$$

$$E_{\text{est}} := \eta$$

$$\tilde{E}_{\text{est}} := (1 + C_{\text{ap}})\eta$$

Plane wave, $p = 1$ and $\kappa = 10\pi$

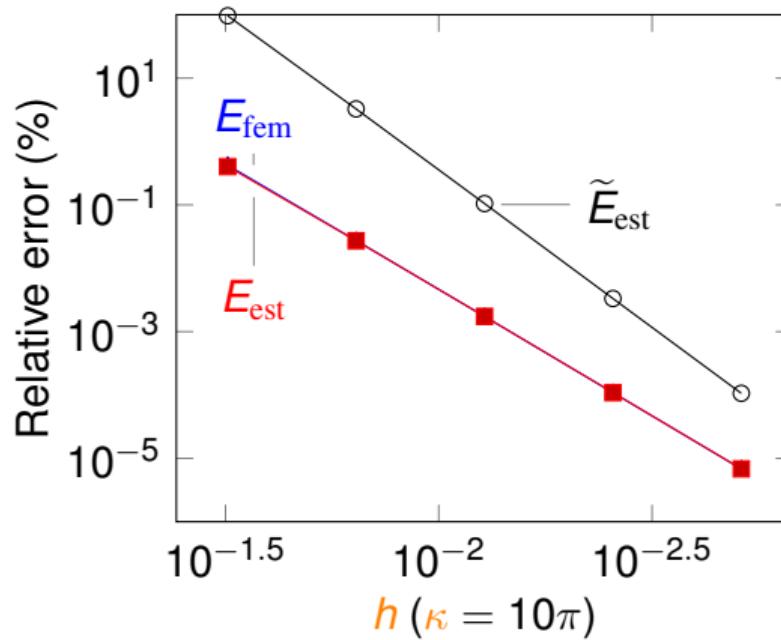


$$E_{\text{fem}} := \|\|e_h\|\|_{\kappa,\Omega}$$

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Plane wave, $p = 4$ and $\kappa = 10\pi$

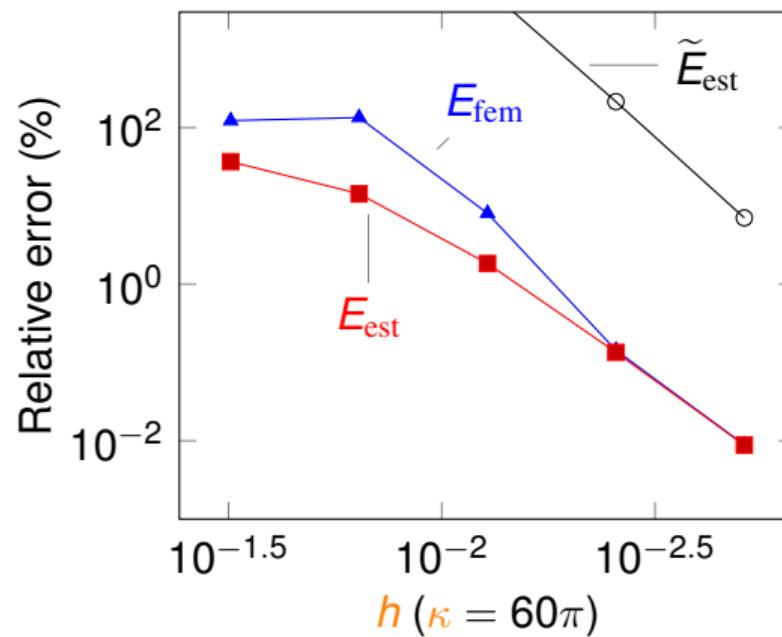


$$E_{\text{fem}} := \|\boldsymbol{e}_h\|_{\kappa, \Omega}$$

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Plane wave, $p = 4$ and $\kappa = 60\pi$

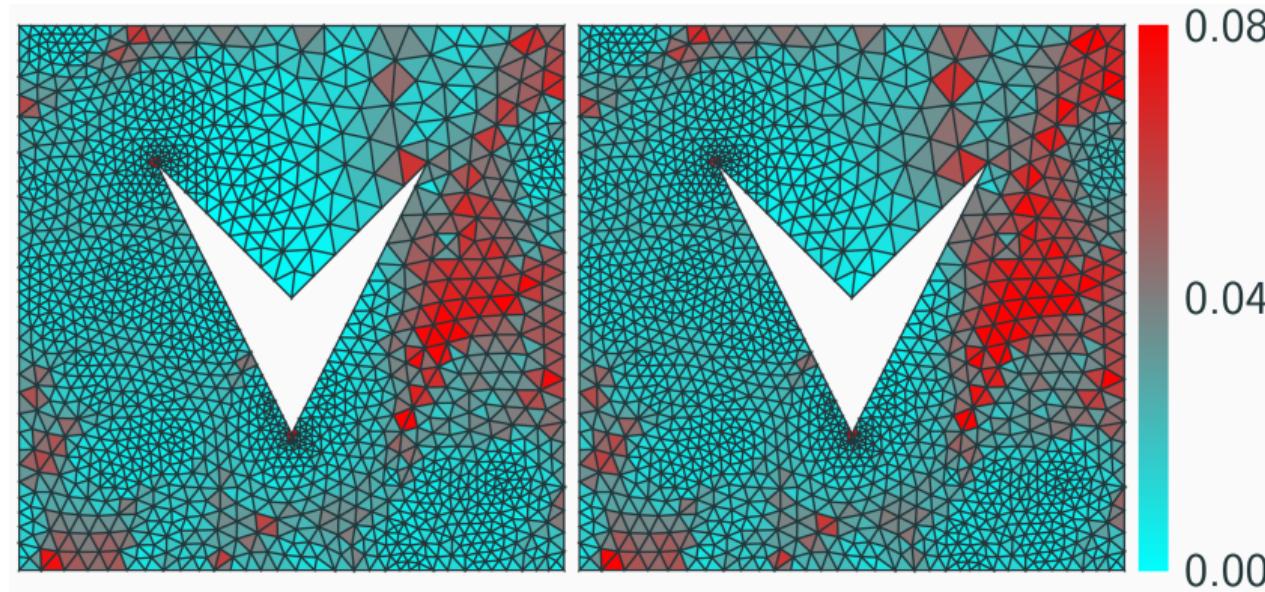


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Scattering by an non-trapping obstacle



Estimator η_K (left) and elementwise error $\|e_h\|_{\kappa,K}$ (right)

Outline

- 1 Introduction: a posteriori error control and adaptivity
- 2 Laplace equation: discretization error control and mesh adaptivity
 - A posteriori error control (discretization)
 - Potential reconstruction
 - Flux reconstruction
 - Balancing error components: mesh adaptivity
- 3 Nonlinear Laplace equation: overall error control and solver adaptivity
 - A posteriori error control (overall and components)
 - Balancing error components: solver adaptivity
- 4 Reaction–diffusion equation: robustness wrt parameters
- 5 Heat equation: robustness wrt final time and space–time localization
- 6 Helmholtz equation: asymptotic robustness
- 7 Conclusions

Conclusions

- a posteriori **error control**

Conclusions

- a posteriori **error control**

adaptivity: space mesh, time step,

Conclusions

- a posteriori **overall error control**
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- **full adaptivity**: space mesh, time step, linear solver, nonlinear solver, regularization, model,

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Thank you for your attention!

Outline

- Motivation
- Polynomial-degree (p) adaptivity

CDG Terminal 2E collapse in 2004 (opened in 2003)



- no earthquake, flooding, tsunami, heavy rain, extreme temperature
- deterministic, steady problem, PDE known, data known, implementation OK

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 Contents lists available at ScienceDirect
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Reliability study and simulation of the progressive collapse of Roissy Charles de Gaulle Airport

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^aÉcole Supérieure d'Ingénieurs de Beyrouth (ESIB), Université Saint-Joseph, CS2 Mar Roukhan, PO Box 11-534, Blvd El-Sabb Beirut 11072050,
^bUniversité de Poitiers, Institut Poisier, BP 10448, F-86360 Chasseneuil-Forez, France



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I believe **without error control**



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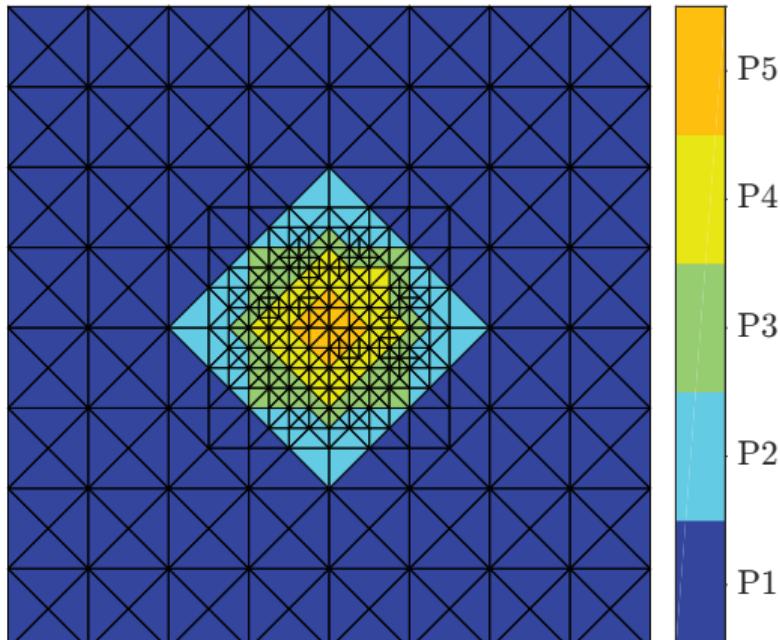
^bUniversité de Metz, Institut Navier, BP 70439, F-57043 Nancy Cedex, France



Outline

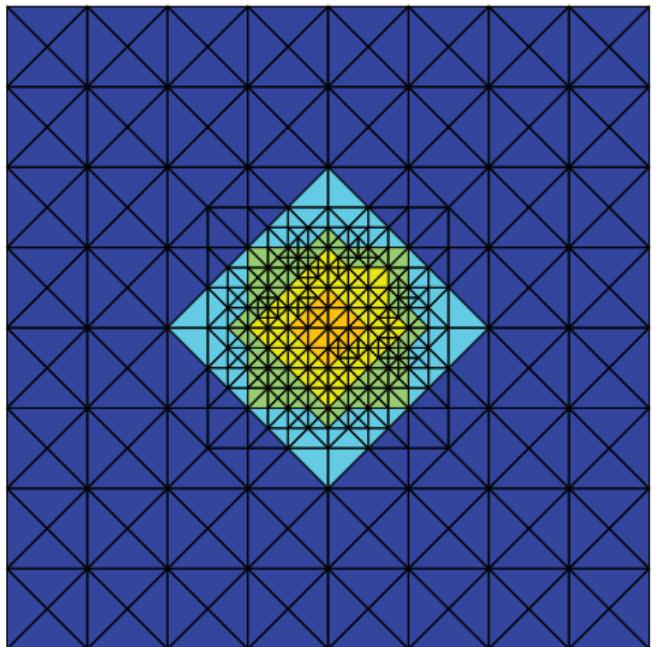
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Best-possible error decrease: ***hp*** adaptivity, (**smooth** solution)

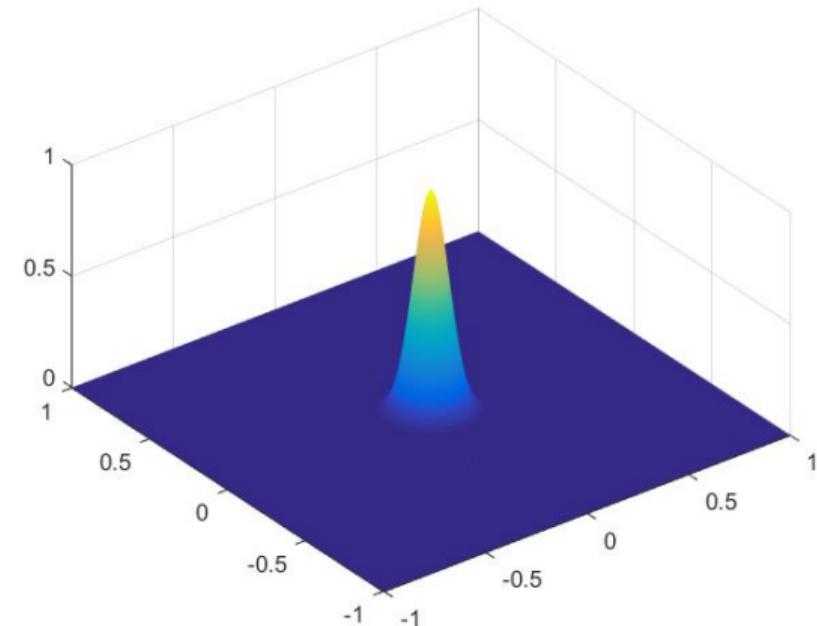


Mesh \mathcal{T}_ℓ and pol. degrees p_K

Best-possible error decrease: ***hp*** adaptivity, (**smooth** solution)

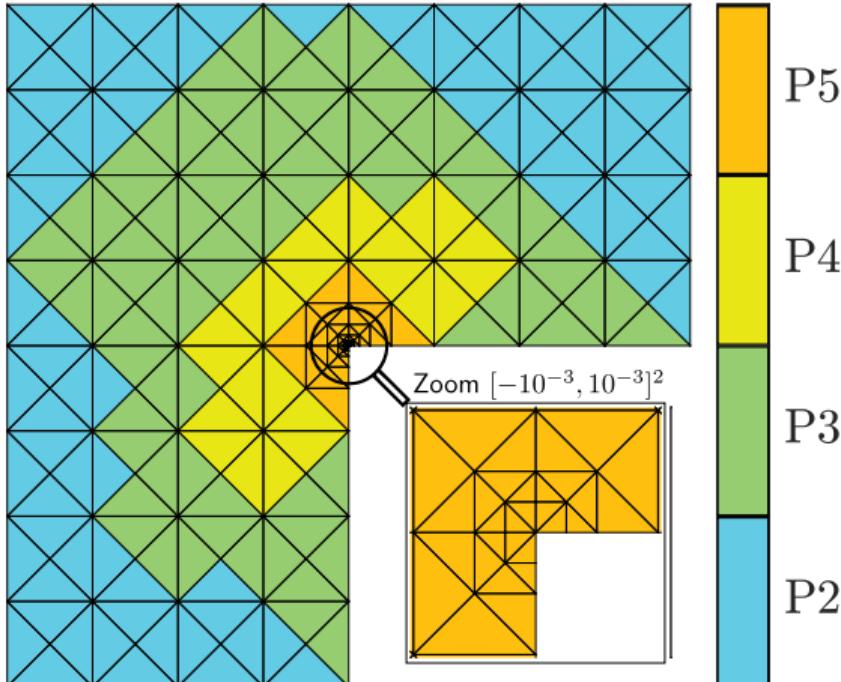


Mesh \mathcal{T}_ℓ and pol. degrees p_K



Exact solution

Best-possible error decrease: *hp* adaptivity, (singular) solution



Mesh \mathcal{T}_ℓ and polynomial degrees p_K

Best-possible error decrease: *hp* adaptivity, (singular) solution

