Guaranteed and robust a posteriori error estimates for the reaction–diffusion and heat equations

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École des Ponts ParisTech

Outline

An optimal a posteriori estimate for steady-state problems

Guaranteed upper bound

- $\|u - u_h\|^2_{?,\Omega} \leq \sum_{K \in \mathcal{T}} \eta_K(u_h)^2$
- no undetermined constant: **error control**

Global efficiency

- $\sum_{K \in \mathcal{T}} \eta_K(u_h)^2 \leq C_{\rm eff} \|u - u_h\|^2_{?,\Omega}$
- mathematical **equivalence** between the unknown **error** and known **estimate**

Robustness

- $C_{\rm eff}$ independent of the domain Ω, solutions u, u_h, **data**, meshes size and form

Small evaluation cost

- estimators $\eta_K(u_h)$ can be **evaluated cheaply** (locally) **from** u_h

Asymptotic exactness

- $\sum_{K \in \mathcal{T}} \eta_K(u_h)^2 / \|u - u_h\|^2_{?,\Omega} \searrow 1$
- overestimation factor goes to one with increasing effort

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Local efficiency

- $\eta_K(u_h) \leq C_{\mathrm{eff}} \|u - u_h\|_{?,\text{neighbors of } K}$
- **local** error lower bound (optimal space **mesh refinement**)

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Previous results (reaction–diffusion equation)

- Verfürth (1998) / Ainsworth and Babuška (1999): robustness wrt. singular perturbation
- Grosman (2006): robustness & anisotropic meshes, polynomial degree $p = 1$
- Cheddadi, Fučík, Prieto, Vohralík (2009): guaranteed upper bound & robustness, $p = 1$
- Ainsworth and Vejchodský (2011, 2014): guaranteed upper bound & robustness but requires submesh (complicated), (2019) without submesh (simple but with restrictions), $p = 1$
- Kopteva (2017): guaranteed upper bound, robustness, & anisotropic meshes, $p = 1$

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1. Introduction
2. The reaction–diffusion equation
   - Equivalence between error and dual norm of the residual
   - Guaranteed upper bound
   - Local efficiency and robustness
   - Numerical experiments
3. The heat equation
   - Equivalence between error and dual norm of the residual
   - High-order discretization & Radau reconstruction
   - Guaranteed upper bound
   - Local space-time efficiency and robustness
   - Numerical experiments
4. Conclusions and future directions

The reaction–diffusion equation

The reaction–diffusion equation

Find $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, such that

$$\begin{aligned} -\varepsilon^2 \Delta u + \kappa^2 u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- $f \in L^2(\Omega)$, $\varepsilon > 0$, $\kappa \geq 0$ fixed real parameters

Singular perturbation

- $\varepsilon \ll \kappa$

Weak solution

Find $u \in H_0^1(\Omega)$ such that

$$\varepsilon^2(\nabla u, \nabla v) + \kappa^2(u, v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Finite element approximation

Find $u_h \in V_h := \mathcal{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

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Outline

Equivalence between error and residual

Energy norm

$$|||\varphi|||^2 := \varepsilon^2\|\nabla\varphi\|^2 + \kappa^2\|\varphi\|^2 \qquad \varphi \in H_0^1(\Omega)$$

$$|||\varphi||| = \sup_{v\in H_0^1(\Omega);\ |||v|||=1} \{\varepsilon^2(\nabla\varphi,\nabla v) + \kappa^2(\varphi, v)\} \quad \varphi \in H_0^1(\Omega)$$

Residual of $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$, the misfit of u_h in the weak formulation:

$$\langle\mathcal{R}(u_h), v\rangle := (f, v) - \varepsilon^2(\nabla u_h, \nabla v) - \kappa^2(u_h, v) \qquad v \in H_0^1(\Omega)$$

- dual norm of the residual

$$|||\mathcal{R}(u_h)|||_{H^{-1}(\Omega)} := \sup_{v\in H_0^1(\Omega);\ |||v|||=1} \langle\mathcal{R}(u_h), v\rangle$$

Energy error is the dual norm of the residual

$$|||(u-u_h)||| = \sup_{v\in H_0^1(\Omega);\ |||v|||=1} \{(f,v) - \varepsilon^2(\nabla u_h,\nabla v) - \kappa^2(u_h,v)\} = |||\mathcal{R}(u_h)|||_{H^{-1}(\Omega)}$$

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Upper bound: motivation

Bound on the residual

- let $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $\phi_h \in L^2(\Omega)$ be such that $\boxed{\nabla\cdot\boldsymbol{\sigma}_h + \kappa^2 \phi_h = f}$
- $\boldsymbol{\sigma}_h$: **equilibrated flux reconstruction**, $\approx -\varepsilon^2 \nabla u$
- ϕ_h: **potential reconstruction**, $\approx u$
- Green theorem $(\nabla\cdot\boldsymbol{\sigma}_h, v) + (\boldsymbol{\sigma}_h, \nabla v) = 0$ for $v \in H_0^1(\Omega)$:

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\begin{aligned}
\langle \mathcal{R}(u_h), v\rangle &= (f, v) - \varepsilon^2(\nabla u_h, \nabla v) - \kappa^2(u_h, v)\\
&= -(\varepsilon \nabla u_h + \varepsilon^{-1}\boldsymbol{\sigma}_h, \varepsilon \nabla v) - (\kappa(u_h - \phi_h), \kappa v)\\
&\leq \left[\|\varepsilon \nabla u_h + \varepsilon^{-1}\boldsymbol{\sigma}_h\|^2 + \|\kappa(u_h - \phi_h)\|^2\right]^{\frac{1}{2}} |||v|||
\end{aligned}
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- then

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\boxed{|||(u - u_h)||| = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} \leq \left[\|\varepsilon \nabla u_h + \varepsilon^{-1}\boldsymbol{\sigma}_h\|^2 + \|\kappa(u_h - \phi_h)\|^2\right]^{\frac{1}{2}}}
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- how to obtain suitable practical (inexpensive) $\boldsymbol{\sigma}_h$ and ϕ_h?
- **counter-example** where $\|\varepsilon \nabla u_h + \varepsilon^{-1}\boldsymbol{\sigma}_h\|$ can largely overestimate the error

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$$\begin{aligned}\langle \mathcal{R}(u_h), v \rangle &= (f, v) - \varepsilon^2 (\nabla u_h, \nabla v) - \kappa^2 (u_h, v) \\ &= -(\varepsilon \nabla u_h + \varepsilon^{-1} \boldsymbol{\sigma}_h, \varepsilon \nabla v) - (\kappa(u_h - \phi_h), \kappa v) \\ &\leq \left[\|\varepsilon \nabla u_h + \varepsilon^{-1} \boldsymbol{\sigma}_h\|^2 + \|\kappa (u_h - \phi_h)\|^2 \right]^{\frac{1}{2}} |||v|||\end{aligned}$$

- then

$$\boxed{|||(u - u_h)||| = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} \leq \left[\|\varepsilon \nabla u_h + \varepsilon^{-1} \boldsymbol{\sigma}_h\|^2 + \|\kappa (u_h - \phi_h)\|^2 \right]^{\frac{1}{2}}}$$

- how to obtain suitable practical (inexpensive) $\boldsymbol{\sigma}_h$ and ϕ_h?
- **counter-example** where $\|\varepsilon \nabla u_h + \varepsilon^{-1} \boldsymbol{\sigma}_h\|$ can largely overestimate the error

Upper bound: motivation

Bound on the residual

- let $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $\phi_h \in L^2(\Omega)$ be such that $\boxed{\nabla\cdot\boldsymbol{\sigma}_h + \kappa^2\phi_h = f}$
- $\boldsymbol{\sigma}_h$: **equilibrated flux reconstruction**, $\approx -\varepsilon^2\nabla u$
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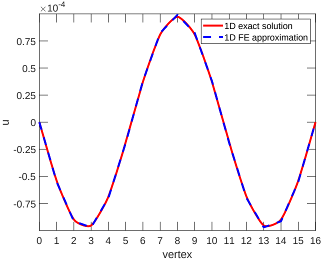
Counter-example ($-\varepsilon^2 \Delta u + \kappa^2 u = f$ in Ω, $u = 0$ on $\partial\Omega$)

Data

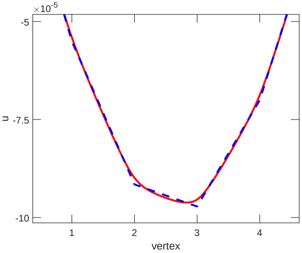
- $\Omega := (-1/2, 1/2)$, $d = 1$
- odd integer m
- f: piecewise affine Lagrange int. of $\cos(m\pi x)$
- uniform mesh $\mathcal{T}$ with $2N = (m+1)^2$ intervals
- $h = 1/(m+1)^2$

Finite element approximation

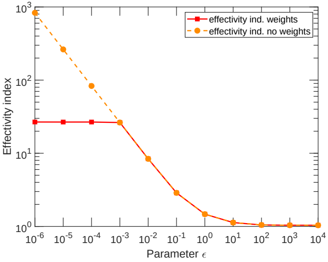
- $V_h := \mathcal{P}_1(\mathcal{T}) \cap H^1_0(\Omega)$
- $\mu_h := \frac{6}{2+\cos(m\pi h)} \frac{1-\cos(m\pi h)}{h^2}$
- $u_h = (\varepsilon^2 \mu_h + \kappa^2)^{-1} f$



$\varepsilon = 1$, $\kappa = 10^2$, $m = 3$



detailed view



$\kappa = 10^2$, $m = 3$, eff. ind.

Equilibrated flux and potential reconstructions

Definition (Flux $\boldsymbol{\sigma}_h$ and potential ϕ_h)

For each vertex $\mathbf{a} \in \mathcal{V}$, let

$$(\boldsymbol{\sigma}_h^{\mathbf{a}}, \phi_h^{\mathbf{a}})$$

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- **local discrete** constrained minimization problems
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with the weight $w_{\mathbf{a}} := \min\left\{1, C_* \sqrt{\frac{\varepsilon}{\kappa h_{\omega_{\mathbf{a}}}}}\right\}$. Combine

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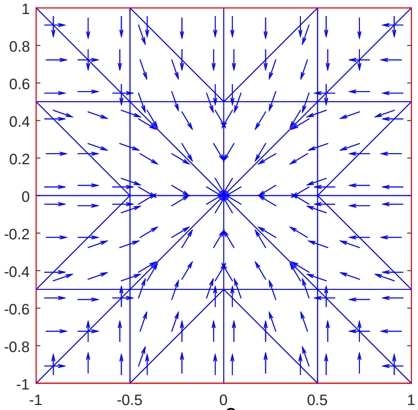
Equilibrated flux reconstruction



Flux $-\varepsilon^2 \nabla u_h$

$$\underbrace{-\varepsilon^2 \nabla u_h \in \mathcal{RTN}_p(\mathcal{T}),\, f \in L^2(\Omega)}_{(f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - \varepsilon^2(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - \kappa^2 (u_h, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \ \forall \mathbf{a} \in \mathcal{V}^{\mathrm{int}}}$$

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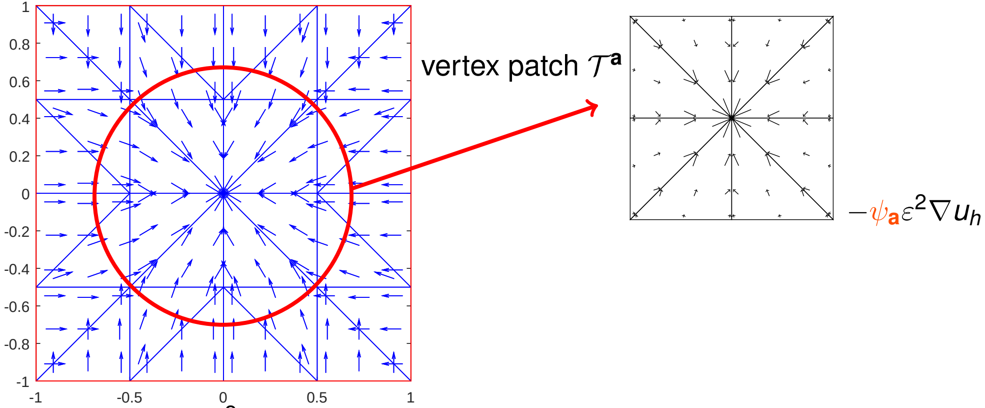

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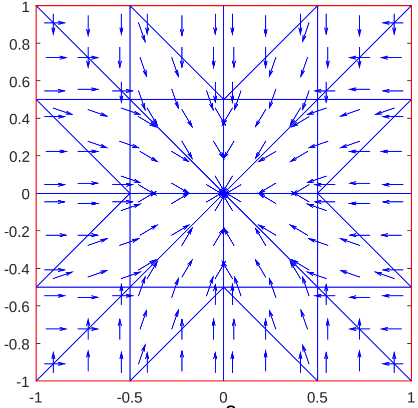

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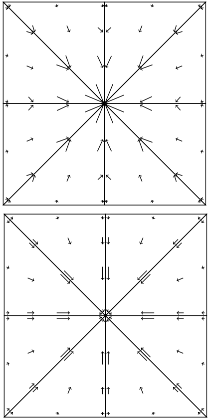

$-\psi_{\mathbf{a}} \varepsilon^2 \nabla u_h$

$\boldsymbol{\sigma}_h^{\mathbf{a}}$

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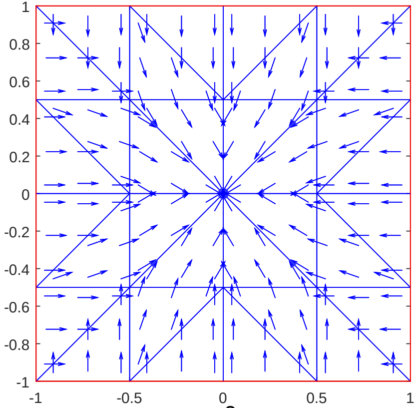
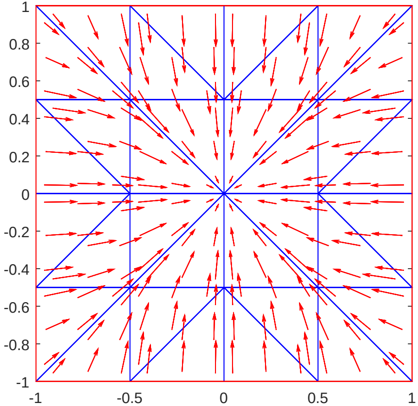


Flux $-\varepsilon^2 \nabla u_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

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Guaranteed a posteriori error estimate

Theorem (Guaranteed a posteriori error estimate)

Let u be the weak solution and let $u_h \in V_h$ be its finite element approximation. Let $\boldsymbol{\sigma}_h \in \mathcal{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ and $\phi_h \in \mathcal{P}_p(\mathcal{T})$ be the flux and potential reconstructions. Then

$$|||u - u_h|||^2 \leq \sum_{K \in \mathcal{T}} \left[w_K \|\varepsilon \nabla u_h + \varepsilon^{-1} \boldsymbol{\sigma}_h\|_K + \|\kappa(u_h - \phi_h)\|_K + \widetilde{w}_K \|f - \Pi_h f\|_K \right]^2$$

with

$$w_K := \min\left\{1, C_* \sqrt{\frac{\varepsilon}{\kappa h_K}}\right\}, \quad \widetilde{w}_K := \min\left\{\frac{h_K}{\pi\varepsilon}, \frac{1}{\kappa}\right\}, \qquad K \in \mathcal{T}.$$

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- **guaranteed upper bound** on the unknown error
- the weight w_K with min crucial for robustness

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Outline

Local efficiency and robustness

Theorem (Local efficiency and robustness)

Let f be a piecewise polynomial for simplicity. Then, for all $K \in \mathcal{T}$,

$$w_K \|\varepsilon \nabla u_h + \varepsilon^{-1} \boldsymbol{\sigma}_h\|_K + \|\kappa(u_h - \phi_h)\|_K \leq C_{\text{eff}} |||u - u_h|||_{\omega_K},$$

where the constant C_{eff} only depends on the space dimension d, the shape-regularity constant $\vartheta_{\mathcal{T}}$ of the mesh $\mathcal{T}$, and on the polynomial degree p of u_h.

Comments

- the computable elementwise estimators are **local** lower bounds for the unknown error
- C_{eff} independent of the parameters ε and $\kappa \Rightarrow$ **robustness**

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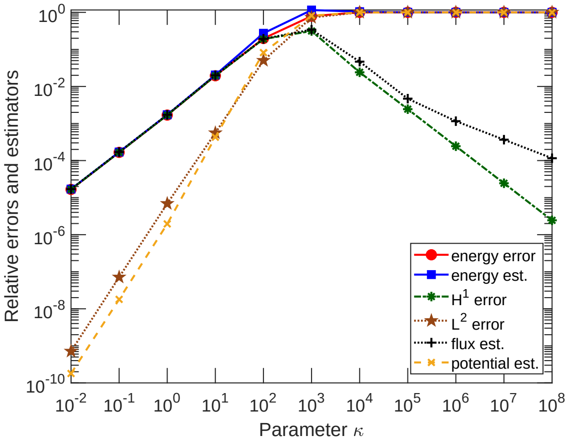
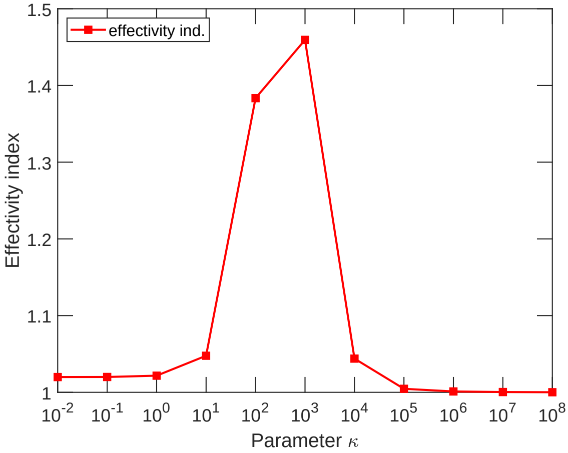
Boundary layer, solution $u(x,y) = e^{-\frac{\kappa}{\varepsilon}x} + e^{-\frac{\kappa}{\varepsilon}y}$, $p = 1$

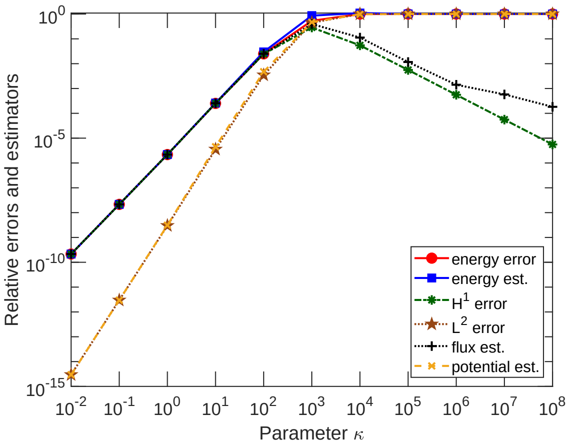


Relative energy errors and estimates



Effectivity indices $\eta(u_h)/|||u - u_h|||$

Boundary layer, solution $u(x, y) = e^{-\frac{\kappa}{\varepsilon}x} + e^{-\frac{\kappa}{\varepsilon}y}$, $p = 2$



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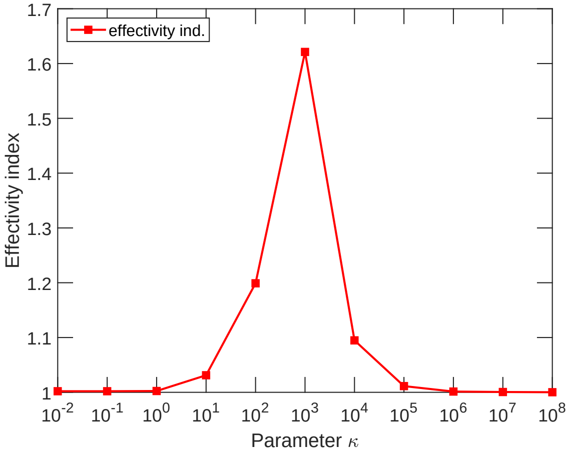

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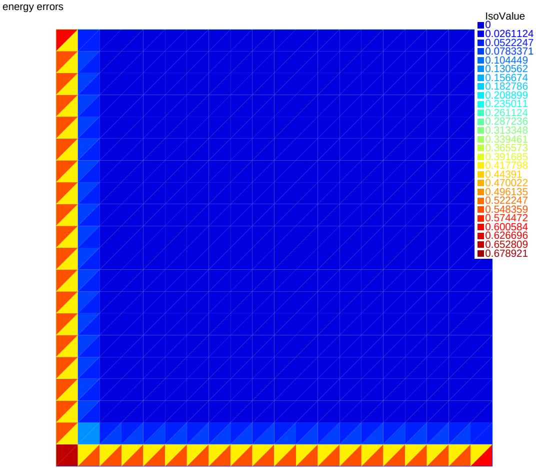
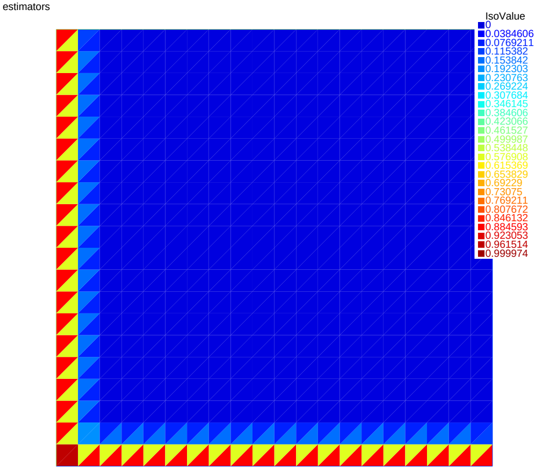
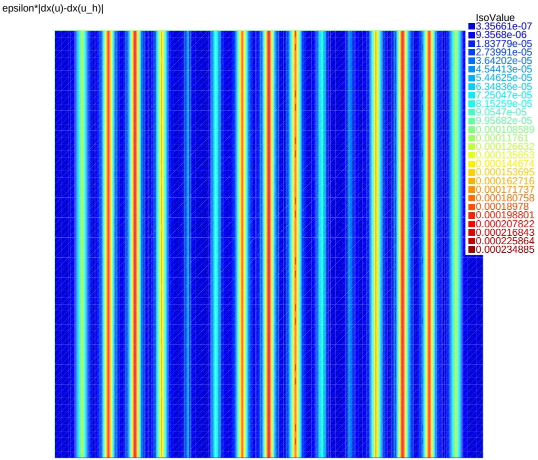


Exact en. error on each mesh element

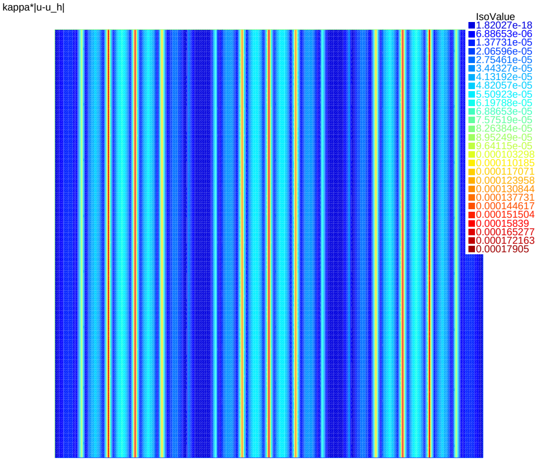


Estimated en. error on each mesh element

Counter-example with $\varepsilon = 1$, $\kappa = 10^2$, $m = 3$, $p = 1$

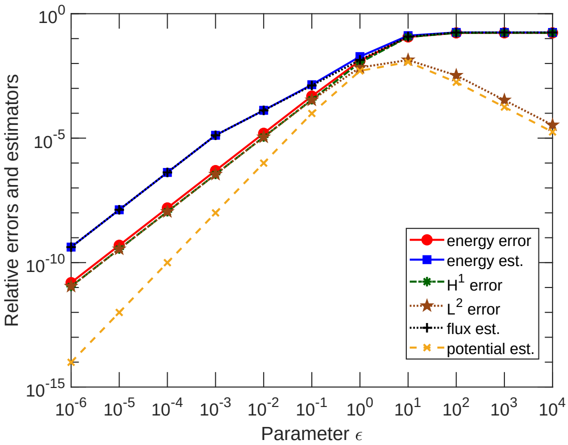


Pointwise H^1-seminorm errors



Pointwise $\kappa \times L^2$-norm errors

Counter-example with $\kappa = 10^2$, $m = 3$, $p = 1$



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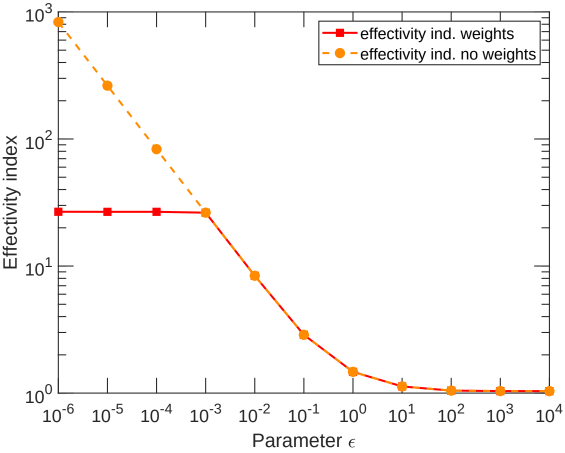

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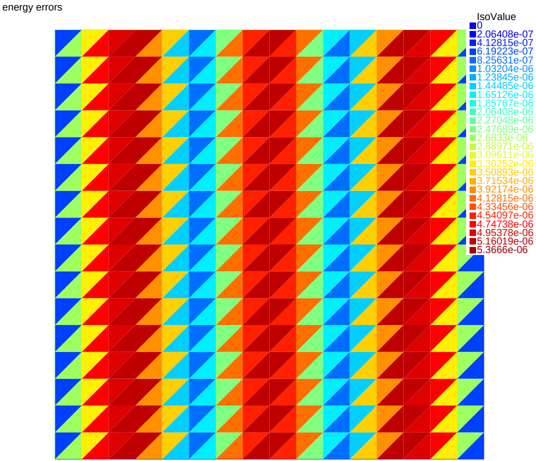

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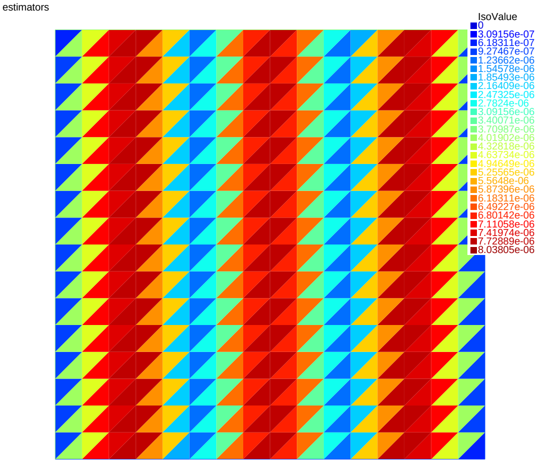

Estimated en. error on each mesh element

Outline

The heat equation ($f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$)

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

Spaces

$$\begin{aligned} X &:= L^2(0,T;H_0^1(\Omega)), \\ \|v\|_X^2 &:= \int_0^T \|\nabla v\|^2 \,\mathrm{d}t, \\ Y &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ \|v\|_Y^2 &:= \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2 \end{aligned}$$

Weak solution

Find $u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v \rangle + (\nabla u, \nabla v) \,\mathrm{d}t = \int_0^T (f, v) \,\mathrm{d}t \qquad \forall v \in X$$

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An optimal a posteriori estimate for evolutive problems

Guaranteed upper bound

- $\|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- no undetermined constant: **error control**

Local efficiency

- $\eta_K^n(u_{h\tau}) \leq C_{\text{eff}} \|u - u_{h\tau}\|_{?,\text{neighbors of } K\times(t^{n-1},t^n)}$
- optimal space-time mesh refinement
- **local** in **time** and in **space** error lower bound

Robustness

- C_{eff} independent of data, domain Ω, **final time** T, meshes, solutions u, $u_{h\tau}$, **polynomial degrees** of $u_{h\tau}$ in space p and in time q

Small evaluation cost

- estimators $\eta_K^n(u_{h\tau})$ can be **evaluated cheaply** (locally) **from** $u_{h\tau}$

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \searrow 1$
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Previous results

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Outline

Equivalence between error and residual

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

Spaces

$$\begin{aligned} X &:= L^2(0,T;H_0^1(\Omega)), \\ \|v\|_X^2 &:= \int_0^T \|\nabla v\|^2 \,\mathrm{d}t, \\ Y &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ \|v\|_Y^2 &:= \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2 \end{aligned}$$

Weak solution

Find $u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v\rangle + (\nabla u, \nabla v)\,\mathrm{d}t = \int_0^T (f,v)\,\mathrm{d}t \qquad \forall v \in X$$

Equivalence between error and residual

Theorem (Parabolic inf–sup identity)

For every $\varphi \in Y$, we have

$$\|\varphi\|_Y^2 = \left[\sup_{v \in X,\, \|v\|_X = 1} \int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v)\,\mathrm{d}t \right]^2 + \|\varphi(0)\|^2.$$

Residual of $u_{h\tau} \in Y$

- $\mathcal{R}(u_{h\tau}) \in X'$, the misfit of $u_{h\tau}$ in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v \rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v)\,\mathrm{d}t \qquad v \in X$$

- dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(u_{h\tau}), v \rangle$$

Y norm error is the dual X norm of the residual + initial condition error

$$\|u - u_{h\tau}\|_Y^2 = \|\mathcal{R}(u_{h\tau})\|_{X'}^2 + \|u_0 - u_{h\tau}(0)\|^2$$

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Proof of the parabolic inf–sup identity: $\varphi \in Y$

Proof.

- let $w_* \in X$ be defined by, a.e. in $(0, T)$,

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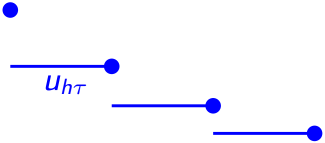
Outline

1. Introduction
2. The reaction–diffusion equation
   - Equivalence between error and dual norm of the residual
   - Guaranteed upper bound
   - Local efficiency and robustness
   - Numerical experiments
3. The heat equation
   - Equivalence between error and dual norm of the residual
   - High-order discretization & Radau reconstruction
   - Guaranteed upper bound
   - Local space-time efficiency and robustness
   - Numerical experiments
4. Conclusions and future directions

Approximate solution and Radau reconstruction



Approximate solution

- ✓ $u_{h\tau}(t)$, $t \in I_n$, is a piecewise continuous polynomial in space in $V_h^n := \{v_h \in H_0^1(\Omega),\ v_h|_K \in \mathcal{P}_{p_K}(K) \quad \forall K \in \mathcal{T}^n\}$
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Radau reconstruction

- ✓ $\mathcal{I}u_{h\tau} \in Y$, $\mathcal{I}u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n+1}(I_n; \widetilde{V_h^n})$ (Makridakis–Nochetto)

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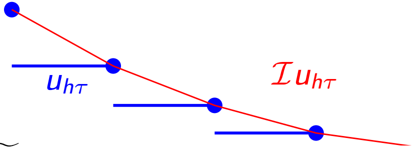

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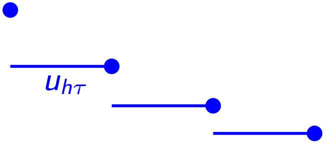

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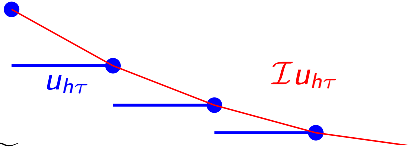

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Outline

Results in the Y norm

Theorem (Guaranteed upper bound in the Y norm)

Suppose no data oscillation (f and u_0 piecewise polynomial). Then, for any $\boldsymbol{\sigma}_{h\tau} \in L^2(0, T; \boldsymbol{H}(\mathrm{div}, \Omega))$ with $\nabla \cdot \boldsymbol{\sigma}_{h\tau} = f - \partial_t \mathcal{I} u_{h\tau}$, there holds

$$\|u - \mathcal{I} u_{h\tau}\|_Y^2 \leq \int_0^T \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I} u_{h\tau}\|^2 \, \mathrm{d}t.$$

Proof of the upper bound

Proof.

- equivalence error-residual (supposing no error in the initial condition):

$$\|u - \mathcal{I}u_{h\tau}\|_Y = \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(\mathcal{I}u_{h\tau}), v \rangle$$

- Green theorem

$$\int_0^T (\boldsymbol{\sigma}_{h\tau}, \nabla \mathcal{I}u_{h\tau}) + (\nabla\cdot\boldsymbol{\sigma}_{h\tau}, \mathcal{I}u_{h\tau})\,\mathrm{d}t = 0$$

- residual definition, Cauchy–Schwarz inequality:

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&\le \left\{ \int_0^T \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|^2\, \mathrm{d}t \right\}^{\frac{1}{2}} \|v\|_X
\end{aligned}$$

Equilibrated flux reconstruction

Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_{h\tau}^{\mathbf{a},n} \\ \nabla\cdot\boldsymbol{v}_h = \psi_{\mathbf{a}}(f - \partial_t \mathcal{I} u_{h\tau}) - \nabla\psi_{\mathbf{a}}\cdot\nabla u_{h\tau}}} \int_{I_n} \|\boldsymbol{v}_h + \psi_{\mathbf{a}} \nabla u_{h\tau}\|_{\omega_{\mathbf{a}}}^2 \,\mathrm{d}t.$$

Then set

$$\boldsymbol{\sigma}_{h\tau} := \sum_{n=1}^{N} \sum_{\mathbf{a} \in \mathcal{V}^n} \boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n}.$$

Comments

- ✓ satisfies $\boldsymbol{\sigma}_{h\tau} \in L^2(0,T;\boldsymbol{H}(\mathrm{div},\Omega))$ with $\nabla\cdot\boldsymbol{\sigma}_{h\tau} = f - \partial_t \mathcal{I} u_{h\tau}$
- ✗ a priori a local space-time problem, $\boldsymbol{V}_{h\tau}^{\mathbf{a},n} := \mathcal{Q}_{q_n}(I_n; \boldsymbol{V}_h^{\mathbf{a},n})$
- ✓ uncouples to q_n elliptic problems posed in $\boldsymbol{V}_h^{\mathbf{a},n}$

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Outline

1. Introduction
2. The reaction–diffusion equation
   - Equivalence between error and dual norm of the residual
   - Guaranteed upper bound
   - Local efficiency and robustness
   - Numerical experiments
3. The heat equation
   - Equivalence between error and dual norm of the residual
   - High-order discretization & Radau reconstruction
   - Guaranteed upper bound
   - Local space-time efficiency and robustness
   - Numerical experiments
4. Conclusions and future directions

Global efficiency ∼ missing Galerkin orthogonality

Efficiency

There holds

$$\int_{I_n} \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I} u_{h\tau}\|_{\Omega}^2 \,\mathrm{d}t \leq C_{\mathrm{eff}}^2 \|u - \mathcal{I} u_{h\tau}\|_{Y(I_n)} \qquad \forall 1 \leq n \leq N.$$

- ✗ local-in-time but **global-in-space** only (as in Verfürth & Bergam–Bernardi–Mghazli)

Reason

- ✗ $\mathcal{I} u_{h\tau}$ misses the Galerkin orthogonality:

$$\int_{I_n} (f, v_{h\tau}) - (\partial_t \mathcal{I} u_{h\tau}, v_{h\tau}) - (\nabla \mathcal{I} u_{h\tau}, \nabla v_{h\tau}) \mathrm{d}t$$

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Guaranteed upper bound

A decisive trick

Define the final norm as

$$\|u - u_{h\tau}\|^2_{\mathcal{E}_Y} := \|u - \mathcal{I}u_{h\tau}\|^2_Y + \underbrace{\|u_{h\tau} - \mathcal{I}u_{h\tau}\|^2_X}_{\text{known, computable}}$$

Theorem (Guaranteed upper bound)

Suppose no data oscillation (f and u_0 piecewise polynomial). Then there holds

$$\|u - u_{h\tau}\|^2_{\mathcal{E}_Y} \leq \sum_{n=1}^{N} \sum_{K \in \mathcal{T}^n} \int_{I_n} \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|^2_K + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2_K \,\mathrm{d}t.$$

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Local space-time efficiency and robustness

Local error contributions

$$|u - u_{h\tau}|^2_{\mathcal{E}_Y^{\mathbf{a},n}} = \int_{I_n} \|\partial_t(u - \mathcal{I}u_{h\tau})\|^2_{H^{-1}(\omega_{\mathbf{a}})} + \|\nabla(u - \mathcal{I}u_{h\tau})\|^2_{\omega_{\mathbf{a}}} \,\mathrm{d}t + \int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2_{\omega_{\mathbf{a}}} \,\mathrm{d}t$$

Theorem (Local space-time efficiency and robustness)

For each time-step interval I_n and for each element $K \in \mathcal{T}^n$, there holds, in the absence of data oscillation,

$$\int_{I_n} \|\sigma_{h\tau} + \nabla\mathcal{I}u_{h\tau}\|_K^2 + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|_K^2 \mathrm{d}t \le C_{\mathrm{eff}}^2 \sum_{\mathbf{a}\in\mathcal{V}_K} |u - u_{h\tau}|^2_{\mathcal{E}_Y^{\mathbf{a},n}}.$$

Comments

- ✓ **local** in **space** and in **time**
- ✓ C_{eff} only depends on shape regularity ⇒ **robustness** w.r.t the final time T and the **polynomial degrees** p and q

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recall

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For each time-step interval I_n and for each element $K \in \mathcal{T}^n$, there holds, in the absence of data oscillation,

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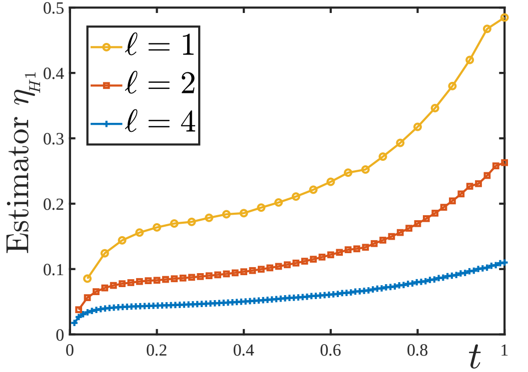
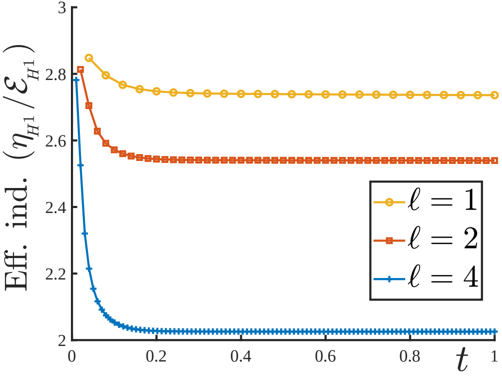
Outline

Richards equation $\partial_t S(u) - \nabla\cdot[\kappa(S(u))(\nabla u + g)] = f$ (results by K. Mitra, three levels of uniform space-time mesh refinement)

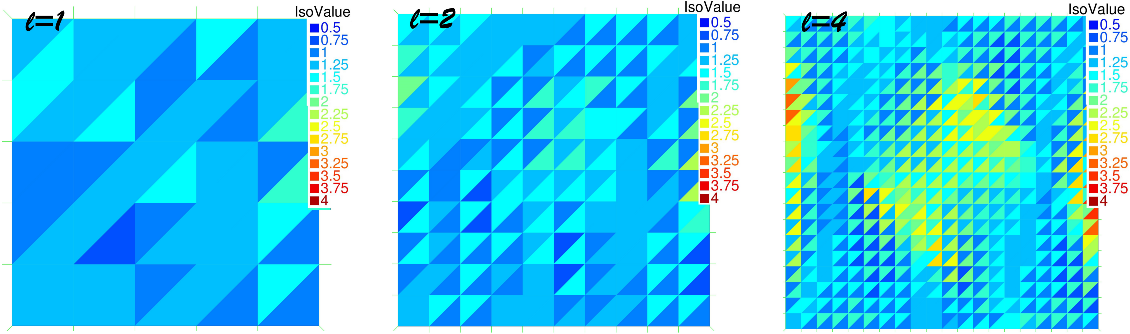


Estimators $\eta(u_{h\tau})$ as a function of T



Effectivity indices $\eta(u_{h\tau})/\|u - u_{h\tau}\|_{\mathcal{E}_Y}$

Richards equation, local space-time effectivity indices



Outline

Conclusions and future directions

Conclusions (reaction-diffusion equation)

- ✓ **guaranteed** upper bound
- ✓ local efficiency and **robustness** with respect to reaction and diffusion parameters
- ✓ simple form (no submesh), local estimators minimization, any polynomial degree

Conclusions (heat equation)

- ✓ **guaranteed** upper bound
- ✓ local **space-time** efficiency
- ✓ **robustness** with respect to both spatial and temporal polynomial degree
- ✓ arbitrarily large **coarsening** allowed

Future directions

- nonlinear and coupled problems

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Bibliography

- SMEARS, I., VOHRALÍK M., Simple and robust equilibrated flux a posteriori estimates for singularly perturbed reaction–diffusion problems, *ESAIM Math. Model. Numer. Anal.* **54**. (2020), 1951–1973.
- ERN A., SMEARS, I., VOHRALÍK M., Guaranteed, locally space-time efficient, and polynomial-degree robust a posteriori error estimates for high-order discretizations of parabolic problems, *SIAM J. Numer. Anal.* **55** (2017), 2811–2834.
- ERN A., SMEARS, I., VOHRALÍK M., Discrete p-robust $\boldsymbol{H}(\mathrm{div})$-liftings and a posteriori estimates for elliptic problems with H^{-1} source terms, *Calcolo* **54** (2017), 1009–1025.

Thank you for your attention!

Bibliography

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Fundamental results on a reference tetrahedron

Bounded right inverse of the divergence operator

- polynomial volume data
- Costabel & McIntosh (2010):

Let $K \in \mathcal{T}$ and $r \in \mathcal{P}_p(K)$. Then there exists $\xi_h \in \mathcal{RTN}_p(K)$ s.t. $\nabla\cdot\xi_h = r$ and
$$\|\xi_h\|_K \le C\|r\|_{H^{-1}(K)} = \sup_{v\in H_0^1(K),\, \|\nabla v\|_K=1} (r,v)_K.$$

Polynomial extensions in $\mathbf{H}(\mathrm{div})$

- polynomial boundary data
- Demkowicz, Gopalakrishnan, Schöberl (2012):

Let $K \in \mathcal{T}$ and $r \in \mathcal{P}_p(\mathcal{F}_K)$ satisfying $(r,1)_{\partial K} = 0$. Then there exists $\xi_h \in \mathcal{RTN}_p(K)$ s.t. $\xi_h\cdot\mathbf{n}_K = r$ on ∂K, $\nabla\cdot\xi_h = 0$ in K, and
$$\|\xi_h\|_K \le C\|r\|_{H^{-1/2}(\partial K)} = \sup_{v\in H^1(K),\, \|\nabla v\|_K=1} (r,v)_{\partial K}.$$

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General result on a physical tetrahedron

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then for $C = C(\kappa_K) > 0$,

$$\min_{\substack{\mathbf{v}_h \in \mathcal{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K \quad .$$

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Context

$$\begin{aligned} -\Delta\zeta_K &= r_K && \text{in } K,\\ -\nabla\zeta_K\cdot\mathbf{n}_K &= r_F && \text{on all } F\in\mathcal{F}_K^{\mathrm{N}},\\ \zeta_K &= 0 && \text{on all } F\in\mathcal{F}_K\setminus\mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\xi_K := -\nabla\zeta_K$.

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General result on a physical tetrahedron

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then for $C = C(\kappa_K) > 0$,

$$\|\boldsymbol{\xi}_{h,K}\|_K \overset{MFEs}{=} \min_{\substack{\mathbf{v}_h \in \mathcal{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K = C \|\boldsymbol{\xi}_K\|_K.$$

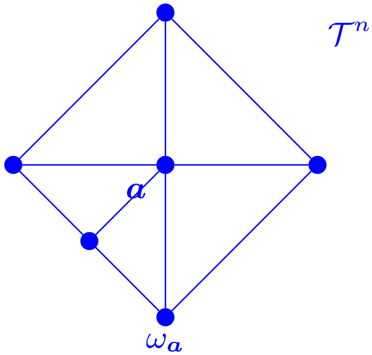
Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\boldsymbol{\xi}_K := -\nabla \zeta_K$.

Stable broken $\mathbf{H}(\mathrm{div})$ polynomial extension on a patch

- Braess, Pillwein, & Schöberl (2009), 2D
- Ern & V. (2016), 3D
- Ern, Smears, & V. (2017), 2-3D, patches with subrefinement



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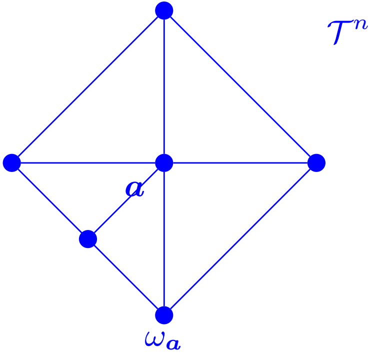

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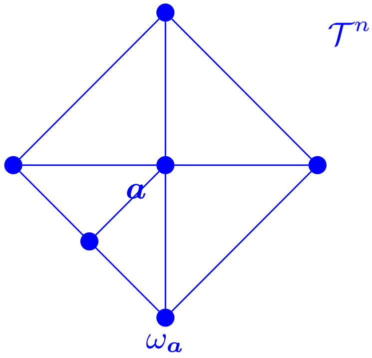

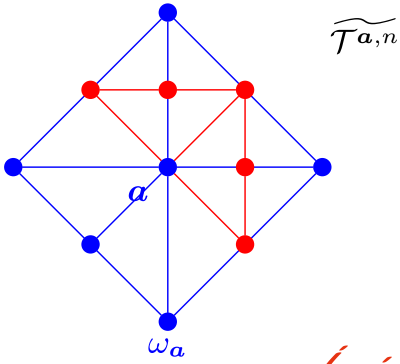

High-order space-time discretization

CG in space & DG in time

- ***p*-degree continuous piecewise polynomials in space**

$$V_h^n := \left\{ v_h \in H_0^1(\Omega),\ \ v_h|_K \in \mathcal{P}_{p_K}(K) \quad \forall K \in \mathcal{T}^n \right\}$$

- ***q*-degree discontinuous piecewise polynomials in time**

$$\mathcal{Q}_{q_n}(I_n; V) := \left\{ V\text{-valued pols of degree at most } q_n \text{ over } I_n \right\}$$

High-order discretization

Find $u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n}(I_n; V_h^n)$ with $u_{h\tau}(0) = \Pi_h u_0$ such that

$$\int_{I_n} (\partial_t u_{h\tau}, v_{h\tau}) + (\nabla u_{h\tau}, \nabla v_{h\tau})\,\mathrm{d}t - ((u_{h\tau}))_{n-1}, v_{h\tau}(t_{n-1}^+))$$
$$= \int_{I_n} (f, v_{h\tau})\,\mathrm{d}t \qquad \forall v_{h\tau} \in \mathcal{Q}_{q_n}(I_n; V_h^n) \qquad \forall 1 \le n \le N.$$

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Remedy

Augmented norm

- augment the norm: $\|v\|^2_{\mathcal{E}_Y} := \|\mathcal{I}v\|^2_Y + \|v - \mathcal{I}v\|^2_X$, $v \in Y + V_{h\tau}$
- $\mathcal{I}u = u \Rightarrow$

$$\|u - u_{h\tau}\|^2_{\mathcal{E}_Y} = \|u - \mathcal{I}u_{h\tau}\|^2_Y + \underbrace{\|u_{h\tau} - \mathcal{I}u_{h\tau}\|^2_X}_{\text{known, computable}}$$

- we are adding to Y norm the time jumps in X norm (Schötzau–Wihler):

$$\|u_{h\tau} - \mathcal{I}u_{h\tau}\|^2_{X(I_n)} = \int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2 \,\mathrm{d}t$$
$$= \frac{\tau_n(q_n+1)}{(2q_n+1)\,(2q_n+3)} \|\nabla(\!| u_{h\tau} |\!)_{n-1}\|^2$$

Equivalence between the Y and $\mathcal{E}_Y$ norms

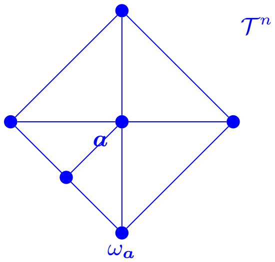
Theorem (Global equivalence)

Suppose no source term oscillation or no coarsening. Then there holds

$$\|u - \mathcal{I}u_{h\tau}\|_Y \leq \|u - u_{h\tau}\|_{\mathcal{E}_Y} \leq 3\|u - \mathcal{I}u_{h\tau}\|_Y$$

- the two norms $\|\cdot\|_Y$ and $\|\cdot\|_{\mathcal{E}_Y}$ still may differ locally
- in general, an additional source term oscillation or coarsening term appears

Handling mesh adaptivity



- refinement & coarsening can also involve changing polynomial degrees

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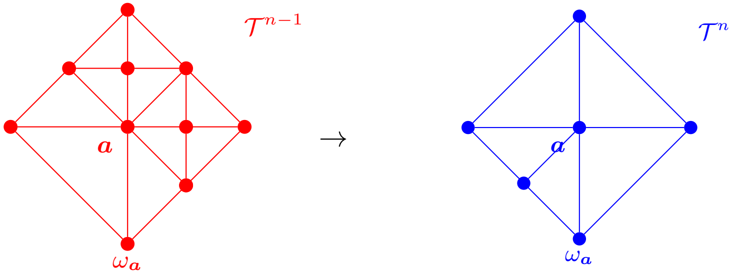

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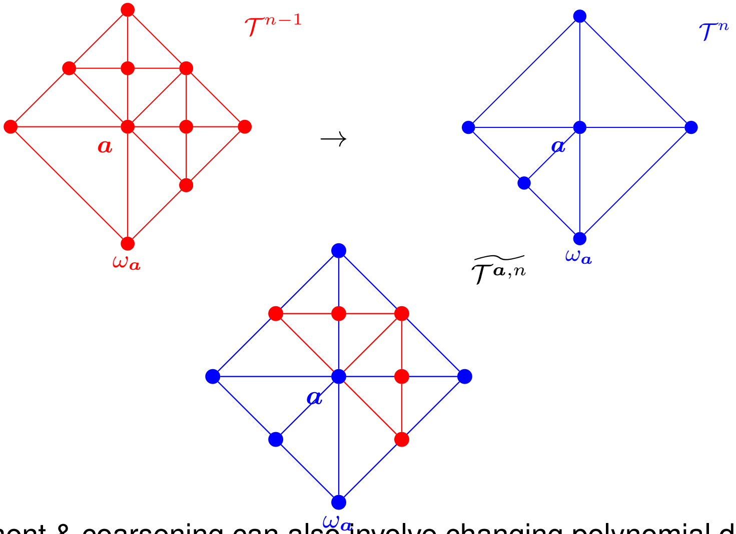

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