Local space–time efficiency for inexpensive approximations of the heat and wave equations

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in collaboration with Vít Dolejší, Alexandre Ern, Nicolas Hugot, and Alexandre Imperiale

Inria Paris & Ecole des Ponts

CMAM, Vienna, July 22, 2026

Outline

The heat & wave equations

The heat equation

Find $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u &= 0 \quad \text{on } \Omega \times 0. \end{aligned}$$

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Continuous setting

- T: final time
- Ω: space domain (Lipschitz bounded polytope)
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Goals

Guaranteed a posteriori error estimate

$$|||u - u_{h\tau}|||^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2$$

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Guaranteed a posteriori error estimate **efficient**

$$|||u - u_{h\tau}|||^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2 \leq C_{\mathrm{eff}}^2 |||u - u_{h\tau}|||^2_{\Omega\times(0,T)},$$

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Guaranteed a posteriori error estimate **efficient** and **robust** with respect to the **final time**.

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Guaranteed a posteriori error estimate **locally space-time efficient** and **robust** with respect to the **final time**.

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$$\eta_K^n(u_{h\tau})^2 \leq C_{\text{eff}}^2 |||u - u_{h\tau}|||^2_{\omega_K\times I_n} \quad \text{for all } 1 \leq n \leq N \text{ and } K \in \mathcal{T}_h^n$$

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- finite elements of polynomial degree $p = 4$ in space
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- mass lumping
- equilibrated flux estimators

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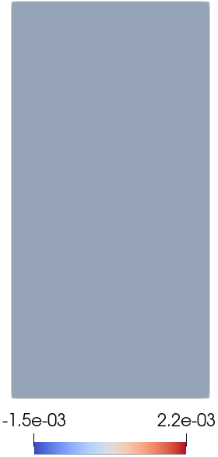
snapshots u_h^n



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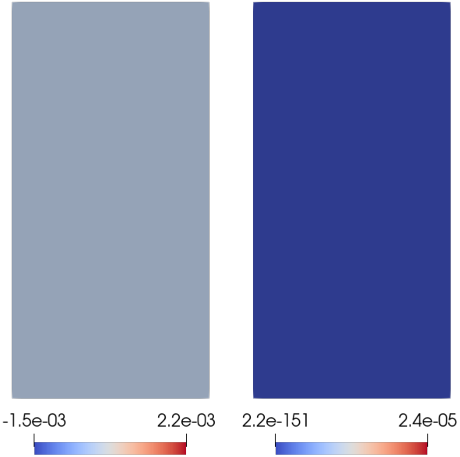

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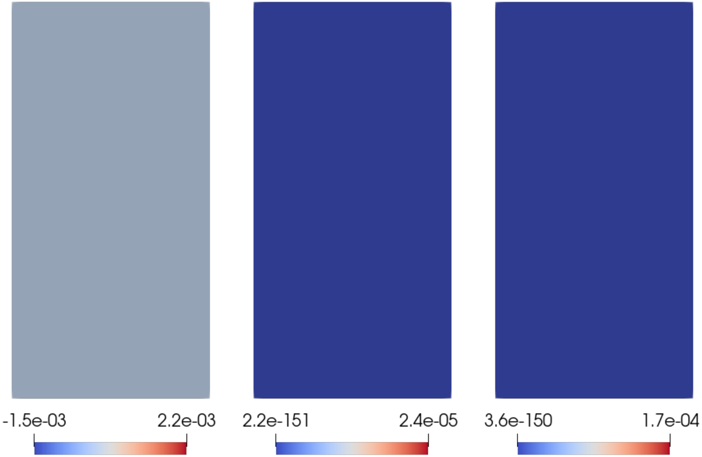
Goal: snapshots u_h^n $\to$ estimators $\eta_K^n(u_{h\tau})$



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Approximation u_h^n, error $|u - u_{h\tau}|_{H^1(K\times I_n)}$, estimators $\eta_K^n(u_{h\tau})$

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  - ✓ reliability $\|u - \mathcal{I}u_{h\tau}\|_Y^2 \le C^2 \sum_{n=1}^N \sum_{K \in \mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2$
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- Ern, Smears, & Vohralík (2017): local space-time efficiency in the Y norm
- Georgoulis & Makridakis (2023), Smears (2025): efficiency in the X norm

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Function spaces and their associated norms

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$$Y_T := \{v \in X \cap H^1(0,T;H^{-1}(\Omega));\ v = 0 \text{ on } \Omega \times T\},$$
$$\|v\|^2_{Y_T} := \int_0^T \{\|\partial_t v\|^2_{H^{-1}(\Omega)} + \|\nabla v\|^2\}\,\mathrm{d}t + \|v(\cdot,0)\|^2$$

Weak formulations

The heat equation

Definition (Weak solution)

$u \in X$ such that, for all $v \in H^1_{,T}(Q)$,

$$-(u, \partial_t v)_Q + (\nabla u, \nabla v)_Q = (f, v)_Q.$$

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Nonsymmetry

Trial space X or $H^1_{0,}(Q)$, test space $H^1_{,T}(Q)$.

Residuals and their dual norms

The heat equation

Definition (Residual)

For $u_{h\tau} \in X$, $\mathcal{R}(u_{h\tau}) \in (H^1_{,T}(Q))'$,

$$\langle \mathcal{R}(u_{h\tau}), v\rangle := (f, v)_Q + (u_{h\tau}, \partial_t v)_Q - (\nabla u_{h\tau}, \nabla v)_Q, \quad v \in H^1_{,T}(Q).$$

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$\mathcal{R}(u_{h\tau}) = 0$ if and only if $u_{h\tau} = u$.

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$$\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'} := \sup_{\substack{v \in H^1_{,T}(Q)\\ |v|_{H^1(Q)}=1}} \langle \mathcal{R}(u_{h\tau}), v\rangle.$$

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- induced by the weak formulation (problem-dependent $\times$ fixed space & norm)
- may be the only choice (sign-changing coefficients, implicit const. laws ...)

Inf-sup equalities, norms of the difference $u - u_{h\tau}$

The wave equation

- For the slightly bigger trial space $Y_0 \supset H^1_{0,}(Q)$,

$$\|u - u_{h\tau}\|_{Y_0} = \|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$$

by inf-sup of Steinbach & Zank (2022).

Inf-sup equalities, norms of the difference $u - u_{h\tau}$

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- For the slightly bigger test space $Y_T \supset H^1_{,T}(Q)$,

$$\|u - u_{h\tau}\|_X = \|\mathcal{R}(u_{h\tau})\|_{(Y_T)'}$$

by standard inf-sup theory.

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Inf-sup equalities, norms of the difference $u - u_{h\tau}$

- There holds

$$
\begin{aligned}
&\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'} \\
= &\sup_{\substack{v \in H^1_{,T}(Q) \\ |v|_{H^1(Q)}=1}} \left\{ (\partial_t(u_{h\tau} - u), \partial_t v)_Q - (\nabla(u_{h\tau} - u), \nabla v)_Q \right\} \\
\leq &\left(\|\partial_t(u - u_{h\tau})\|_Q^2 + \|\nabla(u - u_{h\tau})\|_Q^2 \right)^{1/2} \\
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Outline

Context

$$\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'} = \sup_{\substack{v \in H^1_{,T}(Q) \\ |v|_{H^1(Q)}=1}} \langle \mathcal{R}(u_{h\tau}), v \rangle$$

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- $\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$: dual, a priori global norm
- a priori no localization
- local space-time efficiency in a nonlocal norm?

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...**elliptic setting on** the space-time domain Q

Localization of the intrinsic dual residual norm

Theorem (Localization of the intrinsic dual residual norm)

Let $u_{h\tau} \in H^1_{0,}(Q)$ *(heat) or* $u_{h\tau} \in H^1_{0,}(Q) \cap H^2(0,T;L^2(\Omega))$ *(wave).*

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where the hidden constants only depend on the space dimension d and shape-regularity of the space and time meshes.

Outline

Reliable and locally space-time efficient a posteriori error estimates

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Let $u_{h\tau} \in H^1_{0,}(Q)$ (heat) or $u_{h\tau} \in H^1_{0,}(Q) \cap H^2(0, T; L^2(\Omega))$ (wave) be piecewise polynomials (order p in space, order q in time).

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Estimators

Residual-based estimators

$$\eta_K^n(u_{h\tau}) := \underbrace{h_{K\times I_n}\|f - \partial_{tt} u_{h\tau} + \Delta u_{h\tau}\|_{K\times I_n}}_{\text{volume residual}} + \left\{ \sum_{F\in\mathcal{F}_K^{\mathrm{int}}} \underbrace{h_{F\times I_n}\|[\![\nabla u_{h\tau}]\!]\cdot\boldsymbol{n}_F\|^2_{F\times I_n}}_{\text{face normal component jump}} \right\}^{1/2}$$

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Equilibrated fluxes estimators (reliability constant becomes 1)

$$\boldsymbol{\sigma}_{h\tau} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathrm{div},\Omega)) \text{ with } (f - \partial_{tt}u_{h\tau} - \nabla\cdot\boldsymbol{\sigma}_{h\tau}, 1)_{K\times I_n} = 0 \quad \forall 1 \le n \le N,\ \forall K \in \mathcal{T}_h,$$

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$$\eta_K^n(u_{h\tau})^2 := \underbrace{\frac{h^2_{K\times I_n}}{\pi^2}\|f - \partial_{tt}u_{h\tau} - \nabla\cdot\boldsymbol{\sigma}_{h\tau}\|^2_{K\times I_n}}_{\text{equilibrium (time)}} + \underbrace{\|\nabla u_{h\tau} + \boldsymbol{\sigma}_{h\tau}\|^2_{K\times I_n}}_{\text{constitutive law (space)}}$$

Discussion

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- in the elliptic case, since the problem is boundary value, to obtain orthogonality of the residual wrt $\psi^{\boldsymbol{a}}$, one needs to solve global a linear system
- in parabolic/hyperbolic cases, no condition is prescribed on the final time $\Longrightarrow$ **implicit and even explicit** time stepping is sufficient (no global linear space-time system needs to be solved, not even a global linear space system)

Outline

Crank–Nicolson method for the heat equation

Definition (Crank–Nicolson, heat)

Set $u_h^0 := 0$. Find u_h^n, $1 \leq n \leq N$, such that

$$\left(\frac{u_h^n - u_h^{n-1}}{\tau^n}, v_h\right)_\Omega + \left(\nabla \frac{u_h^n + u_h^{n-1}}{2}, \nabla v_h\right)_\Omega = \left(\frac{f(\cdot, t^n) + f(\cdot, t^{n-1})}{2}, v_h\right)_\Omega \qquad \forall v_h \in V_h.$$

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- $u_{h\tau}|_{I_n} := u_h^n \in X$: OK to define the heat residual $\mathcal{R}(u_{h\tau})$

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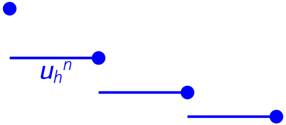

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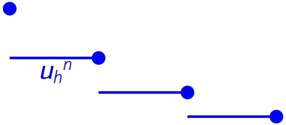

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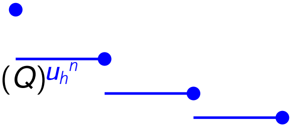

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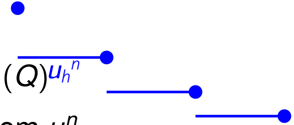

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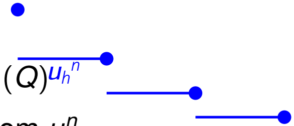

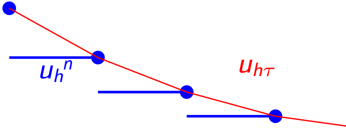

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- equivalently

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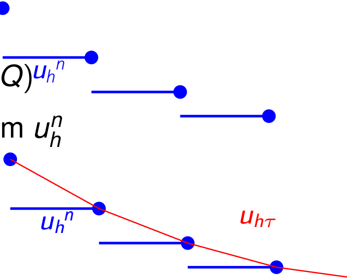

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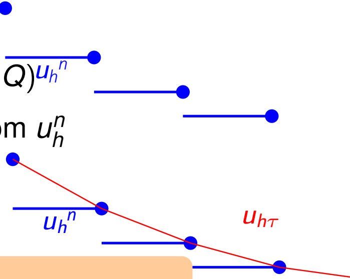

Orthogonality condition satisfied

$$(f, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}} \times I_n} - (\partial_t u_{h\tau}, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}} \times I_n} - (\nabla u_{h\tau}, \nabla \psi^{\mathbf{a}})_{\omega_{\mathbf{a}} \times I_n} = 0 \quad \forall 1 \le n \le N, \ \forall \mathbf{a} \in \mathcal{V}_h.$$

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$$\boxed{(u_h^n)_{0\le n\le N} \subset V_h} \xrightarrow[\text{reconstruction}]{\text{Temporal}} \boxed{u_{h\tau} \in \mathcal{P}_3(\mathcal{T}_\tau(0,T);V_h) \cap \mathcal{C}^1(0,T;V_h)}$$

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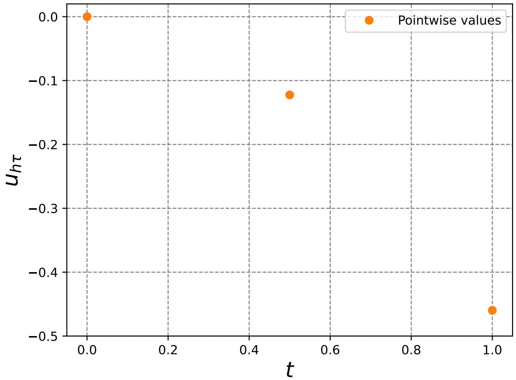

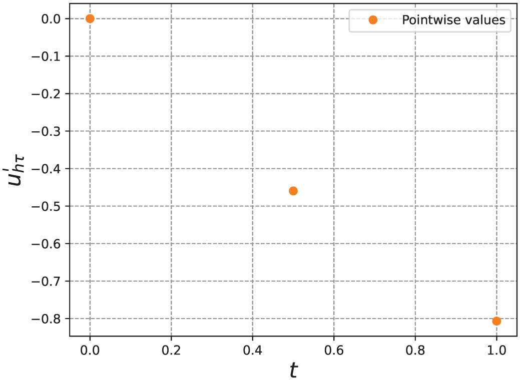

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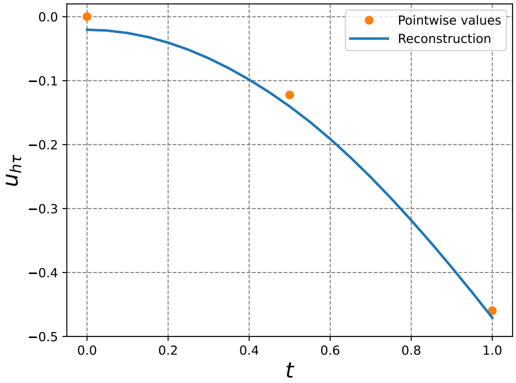

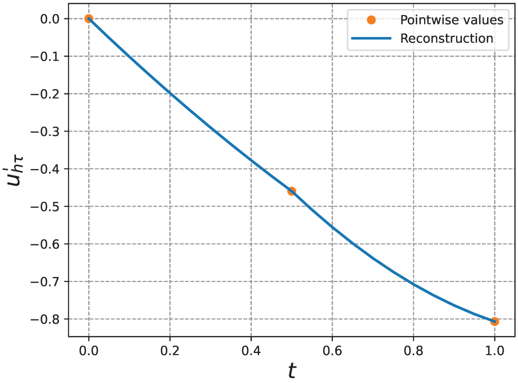

Outline

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- Crank–Nicolson in time
- space and time meshes both uniformly refined: $m = 1, 2, 3$

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Effectivity indices

- dual norm

$$i_{\mathrm{e}} := \frac{\eta}{\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'} + \text{jumps}} \geq 1$$

- (weighted) L^2 norm:

$$i_{\mathrm{e},H^1} := \frac{\eta}{\left(\tau^{-2}\|u - u_{h\tau}\|^2_Q + h^{-2}\|\nabla(u - u_{h\tau})\|^2_Q\right)^{1/2} + \text{jumps}} (< 1 \text{ possible})$$

Viscous Burgers equation

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$$\partial_t u - \nabla{\cdot}(\varepsilon \nabla u - \phi(u)) = 0 \quad \text{in } Q$$

- $\varepsilon = 10^{-2}$ or $\varepsilon = 10^{-4}$
- $\phi(u) = (u^2/2, u^2/2)^{\mathrm{T}}$
- $\Omega = (-1, 1) \times (-1, 1)$
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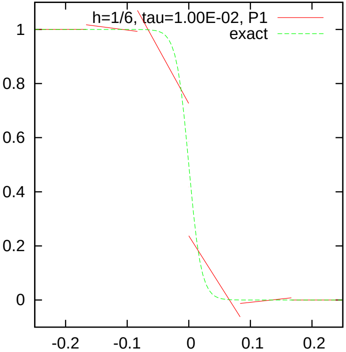
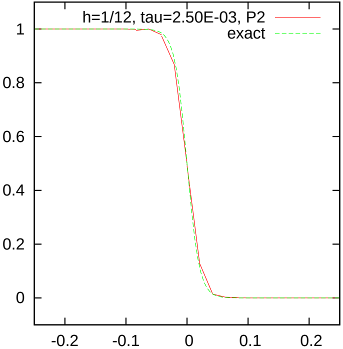
Exact solution

- $$u(x, y, t) = \left(1 + \exp\left(\frac{x + y + 1 - t}{2\varepsilon}\right)\right)^{-1}$$

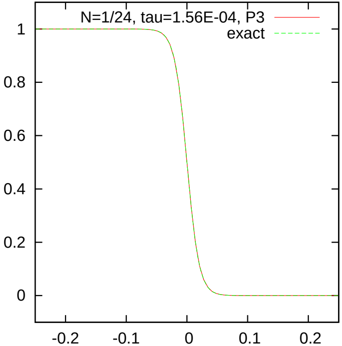
Exact and approximate solutions, $\varepsilon = 10^{-2}$



P_1 approximation on $\{h_1, \tau_1\}$



P_2 approximation on $\{h_2, \tau_2\}$



P_3 approximation on $\{h_3, \tau_3\}$

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Effectivity indices for varying ε and (h_0, τ_0)

ε		10^{-2}		10^{-2}		10^{-2}		10^{-4}	
(h_0, τ_0)		$(1/6, 0.05)$		$(1/6, 0.2)$		$(1/6, 0.0125)$		$(1/6, 0.05)$	
m	p	i_{e}	i_{e,H^1}	i_{e}	i_{e,H^1}	i_{e}	i_{e,H^1}	i_{e}	i_{e,H^1}
1	1	1.85	1.15	2.21	1.28	3.00	0.81	1.45	0.71
2	1	1.71	1.35	2.38	1.12	2.45	1.03	1.68	1.06
3	1	1.25	1.36	2.15	0.90	1.33	1.03	1.82	1.34
1	2	2.15	1.01	3.13	1.71	3.69	0.67	1.38	0.62
2	2	1.65	0.94	2.74	1.58	2.16	0.49	1.41	0.62
3	2	1.53	1.08	2.38	1.52	1.83	0.58	1.54	0.69
1	3	1.71	0.59	2.74	1.47	3.00	0.34	1.26	0.31
2	3	1.75	0.73	2.63	1.67	3.15	0.46	1.13	0.21
3	3	2.54	0.97	2.77	1.73	—	0.69	1.03	0.15

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Porous medium equation

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$$\partial_t u - \nabla\cdot(\boldsymbol{K}(u)\nabla u) = 0 \quad \text{in } Q$$

- $\boldsymbol{K}(u) = \kappa|u|^{\kappa-1}\underline{\boldsymbol{I}}$,
- $\kappa = 2$ or $\kappa = 4$
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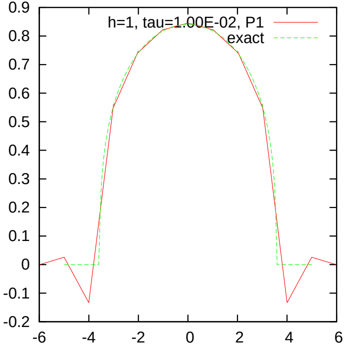
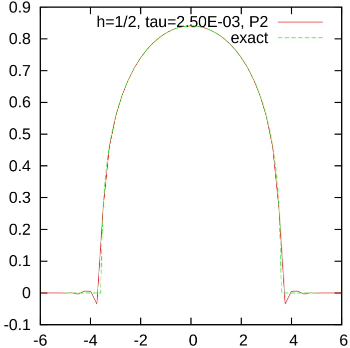
Barenblatt solution

- $$u(x,y,t) = \left\{\frac{1}{t+1}\left[1 - \frac{\kappa-1}{4\kappa^2}\frac{x^2+y^2}{(t+1)^{1/\kappa}}\right]_+^{\frac{\kappa}{\kappa-1}}\right\}^{\frac{1}{\kappa}}$$

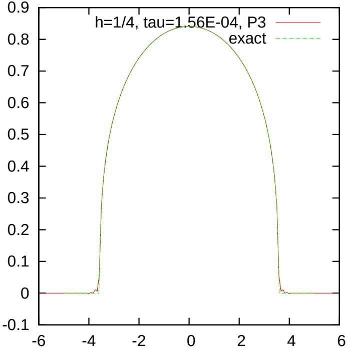
Exact and approximate solutions, $\kappa = 4$



P_1 approximation on $\{h_1, \tau_1\}$



P_2 approximation on $\{h_2, \tau_2\}$



P_3 approximation on $\{h_3, \tau_3\}$

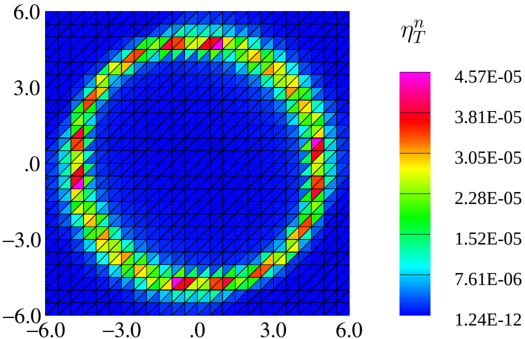
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Errors, estimators, and effectivity indices, $(h_0, \tau_0) = (0.5, 0.02)$

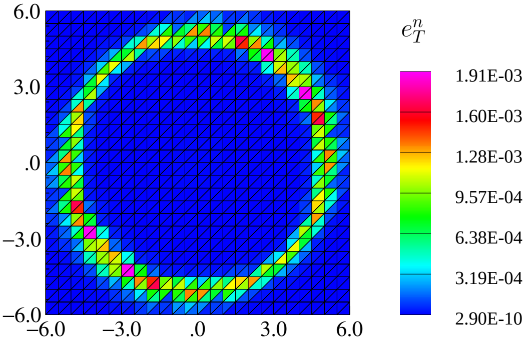
			$\kappa = 2$								$\kappa = 4$	
m	p	$\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$	$\eta_{\rm F}$	$\eta_{\rm R}$	$\eta_{\rm NC}$	$\eta_{\rm IC}$	$\eta_{\rm qd}$	η	$i_{\rm e}$	$i_{{\rm e},H^1}$	$i_{\rm e}$	$i_{{\rm e},H^1}$
1	1	7.90E-03	5.90E-03	1.32E-02	9.10E-03	3.23E-02	7.08E-05	5.88E-02	3.46	0.92	4.68	0.98
2	1	8.36E-03	4.64E-03	1.71E-02	8.46E-03	1.11E-02	3.99E-05	4.03E-02	2.40	1.46	3.72	1.62
		(-0.08)	(0.35)	(-0.38)	(0.10)	(1.54)	(0.83)	(0.54)				
3	1	8.91E-03	4.38E-03	2.18E-02	9.56E-03	3.44E-03	1.83E-05	3.87E-02	2.09	2.49	3.38	2.68
		(-0.09)	(0.08)	(-0.35)	(-0.18)	(1.69)	(1.13)	(0.06)				
1	2	1.09E-03	1.06E-02	1.06E-01	3.12E-02	1.35E-02	1.74E-04	1.61E-01	4.99	3.22	5.13	3.18
2	2	4.02E-04	8.04E-03	8.12E-02	2.37E-02	5.16E-03	6.40E-05	1.18E-01	4.90	3.89	5.05	3.84
		(1.43)	(0.40)	(0.39)	(0.40)	(1.39)	(1.45)	(0.45)				
3	2	1.28E-04	5.22E-03	5.33E-02	1.55E-02	1.69E-03	2.23E-05	7.57E-02	4.84	4.26	4.97	4.30
		(1.65)	(0.62)	(0.61)	(0.61)	(1.61)	(1.52)	(0.64)				
1	3	6.53E-04	2.26E-02	3.27E-01	7.58E-02	8.39E-03	1.36E-04	4.33E-01	5.67	5.01	5.67	4.88
2	3	1.78E-04	9.26E-03	1.38E-01	3.13E-02	3.14E-03	3.51E-05	1.82E-01	5.76	5.17	5.78	5.03
		(1.87)	(1.29)	(1.24)	(1.27)	(1.42)	(1.95)	(1.25)				
3	3	3.83E-05	3.41E-03	5.08E-02	1.15E-02	1.14E-03	8.89E-06	6.68E-02	5.80	5.21	5.85	5.10
		(2.22)	(1.44)	(1.44)	(1.45)	(1.46)	(1.98)	(1.44)				

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Exact and approximate error, $\kappa = 4$, $t = T$, $p = 2$, $m = 2$



Estimators η_K^n



Local error upper bounds

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

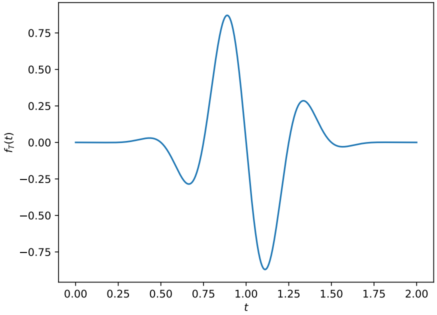
Outline

Data and solution

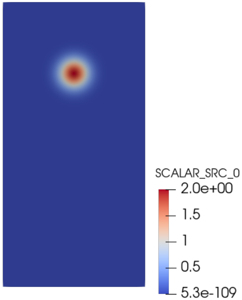
$$\Omega = (0,1) \times (0,2)$$

$$f_T(t) = -\sin(4\pi(t-1)) \times e^{-\left(\frac{t-1}{0.1}\right)^2}$$

$$f_X(x,y) = \exp\left(-\frac{(x-0.5)^2 + (y-1.5)^2}{0.1^2}\right)$$

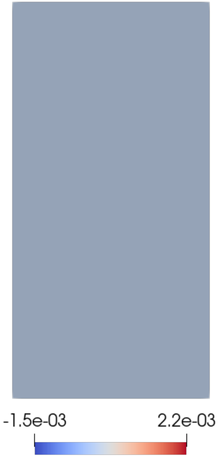


f_T



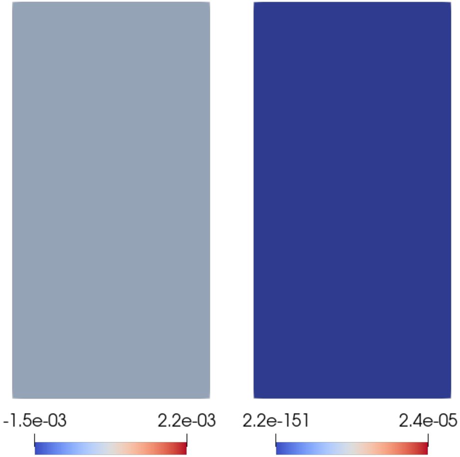
f_X

Approximation, error, and estimators



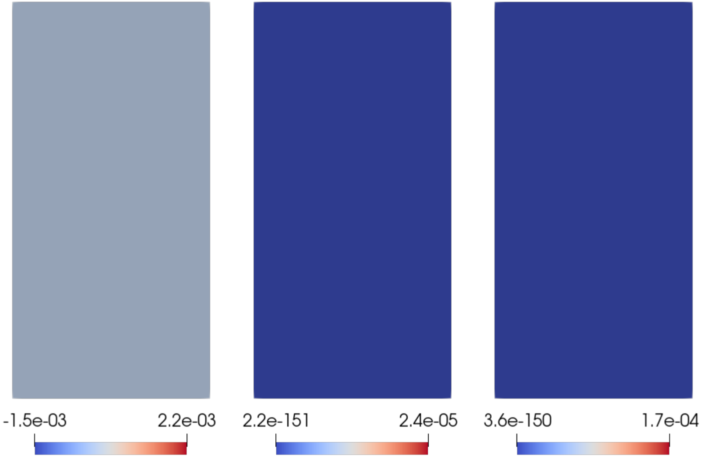
Approximation u_h^n

Approximation, error, and estimators



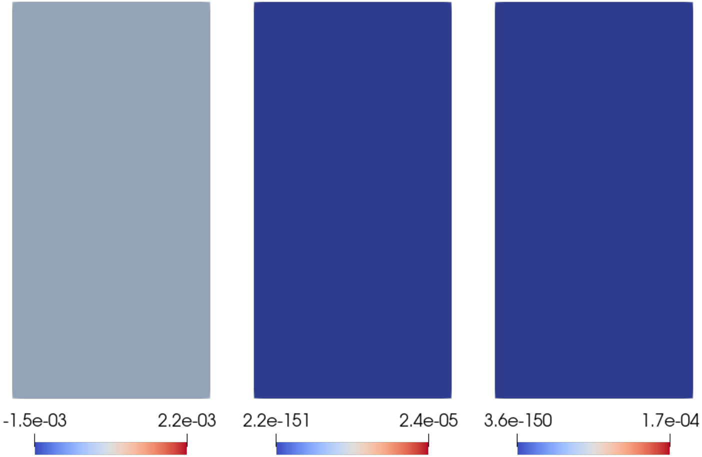
Approximation u_h^n, error $|u - u_{h\tau}|_{H^1(K \times I_n)}$

Approximation, error, and estimators



Approximation u_h^n, error $|u - u_{h\tau}|_{H^1(K \times I_n)}$, estimators $\eta_K^n(u_{h\tau})$

Approximation, error, and estimators



Remarks:

- equilibrated fluxes estimators
- local H^1 norms in place of the dual local norms:

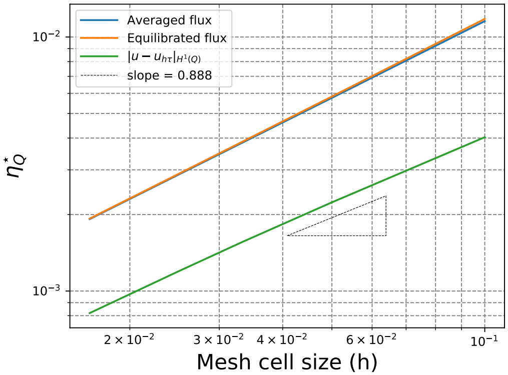
$$\|\mathcal{R}(u_{h\tau})\|_{(H_0^1(\omega_K \times I_n))'} \leq |v|^2_{H^1(\omega_K \times I_n)}$$

- quadrature error ignored

N. Hugot, A. Imperiale, M. Vohralík, preprint (2026).

Approximation u_h^n, error $|u - u_{h\tau}|_{H^1(K \times I_n)}$, estimators $\eta_K^n(u_{h\tau})$

Convergence rates and effectivity indices



Estimators and error

N. Hugot, A. Imperiale, M. Vohralík, preprint (2026).

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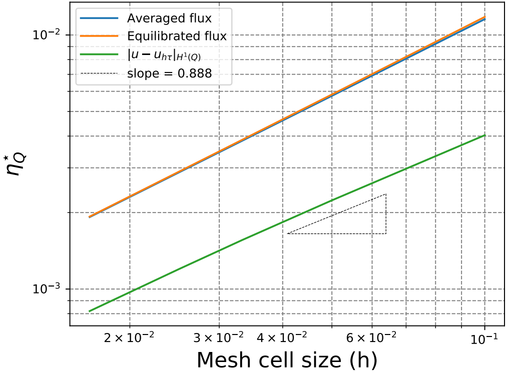


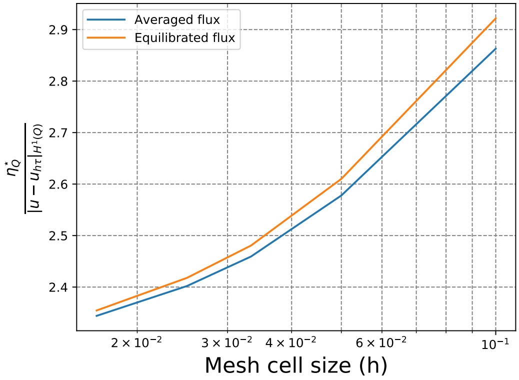
Estimators and error

Remarks:

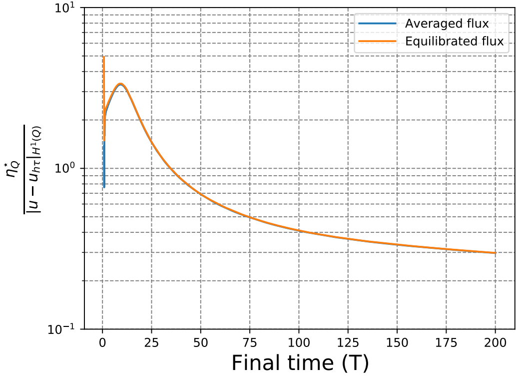
- estimators are **superior to the error**
- estimators and error are of **same order of magnitude**
- estimators and error have the **same order of convergence**

N. Hugot, A. Imperiale, M. Vohralík, preprint (2026).

Robustness wrt h and T



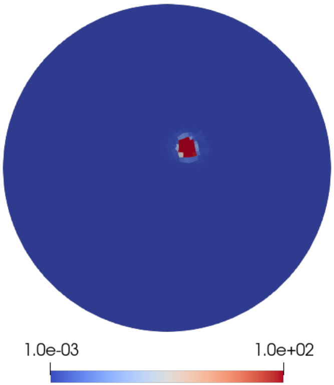
Dependence on h with $T = 5$



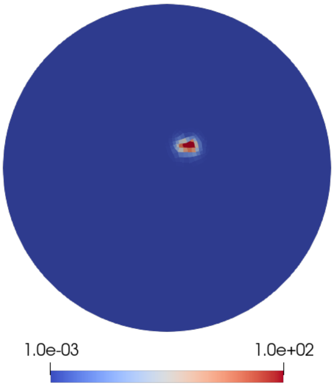
Dependence on T with $h = 0.1$

N. Hugot, A. Imperiale, M. Vohralík, preprint (2026).

Extension to elastic waves (stress tensor $\boldsymbol{\sigma} := 2\mu\varepsilon(u) + \lambda\,\mathrm{tr}(\varepsilon(u))\boldsymbol{I}$, Lamé parameters $\lambda = \mu = 0.4$)



$\eta_K^n(u_{h\tau})$



$|u - u_{h\tau}|_{H^1(K\times I_n)}$

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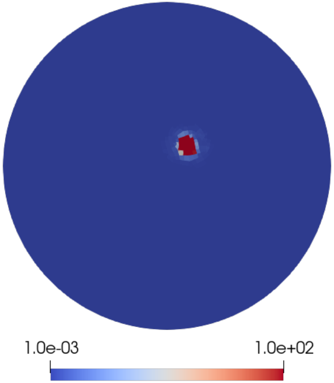

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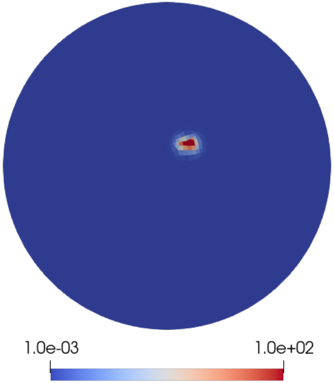

$|u - u_{h\tau}|_{H^1(K\times I_n)}$

Outline

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- **inexpensive** (explicit) **estimators** of the global error
- same methodology for both **parabolic** & **hyperbolic problems**
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- covers **implicit & explicit time-stepping** as well as time-evolving meshes

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- convergence, optimality?

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References

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Hugot N., Imperiale A., Vohralík M. Reliable and locally space-time efficient a posteriori estimates for the transient acoustic wave equation. HAL Preprint 05670166, 2026.

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HUGOT N., IMPERIALE A., VOHRALÍK M. Reliable and locally space-time efficient a posteriori estimates for the transient acoustic wave equation. HAL Preprint 05670166, 2026.

Thank you for your attention!

Outline

Localization of dual norms

Theorem (Localization of dual norms)

Let $\mathcal{R}(u_{h\tau}) \in (H^1_{,T}(Q))'$ be arbitrary. Let, for an index set $\mathcal{V}$,

$$\psi^{\boldsymbol{b}} \in W^{1,\infty}(Q) \subset H^1_{,T}(Q) \quad \begin{cases} \text{have local space–time supports, } \overline{\omega_{\boldsymbol{b}}} \\ \text{form a partition of unity } \sum_{\boldsymbol{b}\in\mathcal{V}} \psi^{\boldsymbol{b}} = 1. \end{cases}$$

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$$\langle \mathcal{R}(u_{h\tau}), \psi^{\boldsymbol{b}} \rangle = 0 \qquad \forall \boldsymbol{b} \in \mathcal{V} \text{ not on } \partial\Omega \times (0,T) \text{ and } \Omega \times 0.$$

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$$\|\mathcal{R}(u_{h\tau})\|^2_{(H^1_{,T}(Q))'} \lesssim \sum_{\boldsymbol{b}\in\mathcal{V}} \|\mathcal{R}(u_{h\tau})\|^2_{(H^1_0(\omega_{\boldsymbol{b}}))'} \lesssim \|\mathcal{R}(u_{h\tau})\|^2_{(H^1_{,T}(Q))'}.$$

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- known from elliptic a posteriori error analysis
- generalizes to $(W_0^{1,\alpha}(Q))'$, $1 \leq \alpha \leq \infty$, see Blechta, Málek, & Vohralík (2020) and the references therein

Outline

Inspection of the local efficiency proof (element residual)

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$$h_K \|\nabla(\psi_{K,n} v_{K,n})\|_{K \times I_n} \lesssim \|\psi_{K,n} v_{K,n}\|_{K \times I_n},$$
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- $\Longrightarrow$ space-time weighted mesh-dependent norm imposed on the test space

$$|v|^2_{H^1(Q)} := \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_h{}^n} \left\{ (\tau^n)^2 \|\partial_t v\|^2_{K \times I_n} + h_K^2 \|\nabla v\|^2_{K \times I_n} \right\}$$

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- $\Longrightarrow$ units ✓, space-time anisotropy ✓, time-evolving meshes ✓

Outline

Degenerate advection-diffusion equation

Degenerate advection-diffusion problem (Kačur 2001)

$$\partial_t u - \nabla\cdot(2\varepsilon u \nabla u - \phi(u)) = 0 \quad \text{in } Q$$

- $\varepsilon = 10^{-2}$
- $\phi(u) = 0.5(u^2, 0)^{\mathrm{T}}$
- $\Omega = (0,1) \times (0,1)$
- $T = 1$

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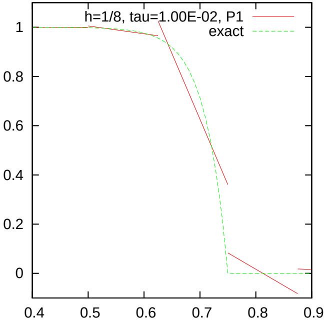
- $\Omega = (0,1) \times (0,1)$
- $T = 1$

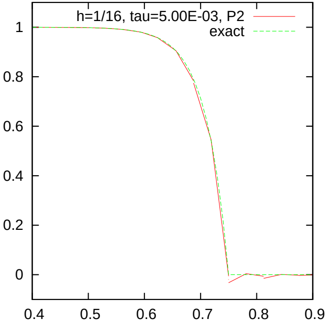
Exact solution

- $$u(x,y,t) = \begin{cases} 1 - \exp\left(\frac{v(x-vt-x_0)}{2\varepsilon}\right) & \text{for } x \le vt + x_0, \\ 0 & \text{for } x > vt + x_0 \end{cases}$$
- $x_0 = 1/4$ is the initial position of the front

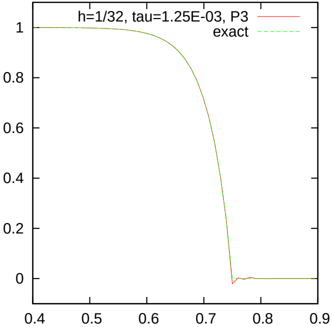
Exact and approximate solutions



P_1 approximation on $\{h_1, \tau_1\}$



P_2 approximation on $\{h_2, \tau_2\}$



P_3 approximation on $\{h_3, \tau_3\}$

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Errors, estimators, and effectivity indices, $(h_0, \tau_0) = (1/8, 0.05)$

m	p	$\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$	$\eta_{\rm F}$	$\eta_{\rm R}$	$\eta_{\rm NC}$	$\eta_{\rm IC}$	$\eta_{\rm qd}$	η	$i_{\rm e}$	$i_{{\rm e},H^1}$
1	1	9.91E-03	1.00E-02	6.02E-03	2.77E-02	2.31E-02	2.17E-03	6.62E-02	1.76	0.97
2	1	7.39E-03	7.71E-03	5.68E-03	1.62E-02	7.71E-03	1.23E-03	3.66E-02	1.55	1.02
		(0.42)	(0.37)	(0.08)	(0.78)	(1.59)	(0.82)	(0.86)		
3	1	4.58E-03	4.52E-03	4.95E-03	8.33E-03	1.86E-03	5.22E-04	1.89E-02	1.47	1.16
		(0.69)	(0.77)	(0.20)	(0.96)	(2.05)	(1.23)	(0.95)		
1	2	2.62E-03	3.30E-03	5.40E-03	9.33E-03	6.27E-03	6.74E-04	2.35E-02	1.97	0.73
2	2	1.11E-03	1.43E-03	1.93E-03	4.22E-03	1.09E-03	2.67E-04	8.34E-03	1.56	0.62
		(1.23)	(1.21)	(1.48)	(1.14)	(2.52)	(1.34)	(1.50)		
3	2	4.26E-04	5.63E-04	6.13E-04	1.84E-03	1.51E-04	1.00E-04	3.06E-03	1.35	0.57
		(1.38)	(1.34)	(1.65)	(1.20)	(2.85)	(1.42)	(1.45)		
1	3	6.48E-04	8.83E-04	1.03E-03	3.57E-03	1.19E-03	2.31E-04	6.47E-03	1.53	0.36
2	3	1.94E-04	2.63E-04	1.45E-04	1.21E-03	1.07E-04	6.39E-05	1.69E-03	1.21	0.25
		(1.74)	(1.74)	(2.84)	(1.56)	(3.48)	(1.85)	(1.93)		
3	3	4.42E-05	7.58E-05	2.58E-05	4.04E-04	7.47E-06	1.67E-05	5.07E-04	1.13	0.21
		(2.13)	(1.80)	(2.49)	(1.58)	(3.84)	(1.94)	(1.74)		

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)