Stable local commuting projectors and global–local equivalences

Théophile Chaumont-Frelet, Leszek Demkowicz, and **Martin Vohralík**

Inria Paris & Ecole des Ponts

Prague, April 29, 2026

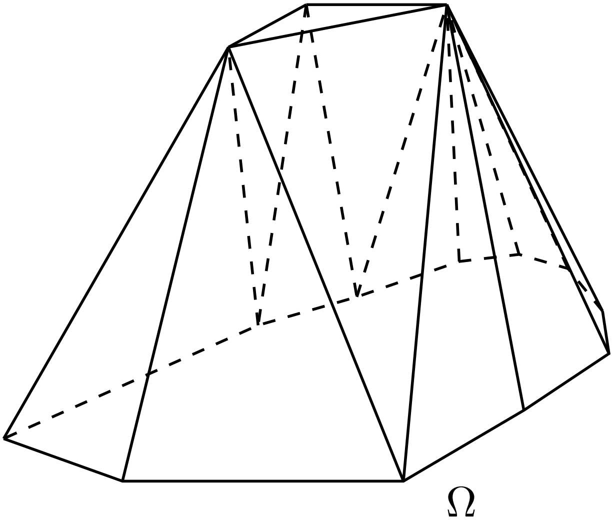
Outline

Outline

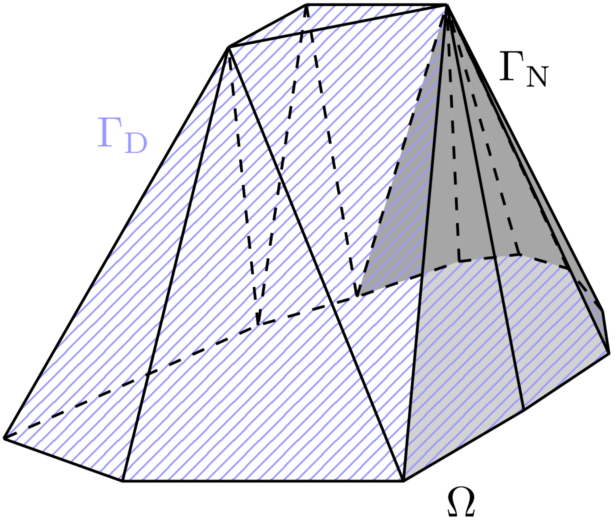
1. Domain, differential operators, Sobolev spaces, and de Rham sequences
2. Meshes and piecewise polynomial spaces
3. *p*-stable local commuting projectors
   - *p*-stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
4. *p*-robust global-best–local-best equivalence
   - *p*-robust global-best–local-best equivalence in $H^1(\Omega)$
   - *p*-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
5. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div}, \Omega)$
6. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - *p*-stable (broken) polynomial extensions
   - *p*-stable decompositions
7. Conclusions

Domain Ω

Domain Ω : polytope in $\mathbb{R}^3$



Domain Ω : polytope in $\mathbb{R}^3$ with boundary subsets Γ_D and Γ_N



Differential operators

Differential operators

Gradient

$$v : \Omega \to \mathbb{R}, \qquad \nabla v := \begin{pmatrix} \partial_{\boldsymbol{x}_1} v \\ \partial_{\boldsymbol{x}_2} v \\ \partial_{\boldsymbol{x}_3} v \end{pmatrix}$$

Differential operators

Gradient

$$v : \Omega \to \mathbb{R}, \qquad \nabla v := \begin{pmatrix} \partial_{\boldsymbol{x}_1} v \\ \partial_{\boldsymbol{x}_2} v \\ \partial_{\boldsymbol{x}_3} v \end{pmatrix}$$

Curl

$$\boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \nabla \times \boldsymbol{v} := \begin{pmatrix} \partial_{\boldsymbol{x}_2} \boldsymbol{v}_3 - \partial_{\boldsymbol{x}_3} \boldsymbol{v}_2 \\ \partial_{\boldsymbol{x}_3} \boldsymbol{v}_1 - \partial_{\boldsymbol{x}_1} \boldsymbol{v}_3 \\ \partial_{\boldsymbol{x}_1} \boldsymbol{v}_2 - \partial_{\boldsymbol{x}_2} \boldsymbol{v}_1 \end{pmatrix}$$

Differential operators

Gradient

$$v : \Omega \to \mathbb{R}, \qquad \nabla v := \begin{pmatrix} \partial_{\boldsymbol{x}_1} v \\ \partial_{\boldsymbol{x}_2} v \\ \partial_{\boldsymbol{x}_3} v \end{pmatrix}$$

Curl

$$\boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \nabla \times \boldsymbol{v} := \begin{pmatrix} \partial_{\boldsymbol{x}_2} \boldsymbol{v}_3 - \partial_{\boldsymbol{x}_3} \boldsymbol{v}_2 \\ \partial_{\boldsymbol{x}_3} \boldsymbol{v}_1 - \partial_{\boldsymbol{x}_1} \boldsymbol{v}_3 \\ \partial_{\boldsymbol{x}_1} \boldsymbol{v}_2 - \partial_{\boldsymbol{x}_2} \boldsymbol{v}_1 \end{pmatrix}$$

Divergence

$$\boldsymbol{v} : \Omega \to \mathbb{R}^3, \qquad \nabla \cdot \boldsymbol{v} := \partial_{\boldsymbol{x}_1} \boldsymbol{v}_1 + \partial_{\boldsymbol{x}_2} \boldsymbol{v}_2 + \partial_{\boldsymbol{x}_3} \boldsymbol{v}_3$$

Sobolev spaces

Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $\boldsymbol{L}^2(\Omega)$,
$H^1(\Omega) := \{v \in L^2(\Omega);\ \nabla v \in \boldsymbol{L}^2(\Omega)\}$

Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $\boldsymbol{L}^2(\Omega)$,
$H^1(\Omega) := \{v \in L^2(\Omega);\ \nabla v \in \boldsymbol{L}^2(\Omega)\}$

$\boldsymbol{H}(\mathrm{curl}, \Omega)$

vector-valued $\boldsymbol{L}^2(\Omega)$ functions with weak curls in $\boldsymbol{L}^2(\Omega)$,
$\boldsymbol{H}(\mathrm{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$

Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $\boldsymbol{L}^2(\Omega)$,
$H^1(\Omega) := \{v \in L^2(\Omega);\ \nabla v \in \boldsymbol{L}^2(\Omega)\}$

$\boldsymbol{H}(\mathrm{curl}, \Omega)$

vector-valued $\boldsymbol{L}^2(\Omega)$ functions with weak curls in $\boldsymbol{L}^2(\Omega)$,
$\boldsymbol{H}(\mathrm{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$

$\boldsymbol{H}(\mathrm{div}, \Omega)$

vector-valued $\boldsymbol{L}^2(\Omega)$ functions with weak divergences in $L^2(\Omega)$,
$\boldsymbol{H}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\}$

Sobolev spaces with BCs

$H^1_{0,\mathrm{D}}(\Omega)$

$H^1_{0,\mathrm{D}}(\Omega) := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_\mathrm{D} \text{ in the sense of traces}\}$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{N} \text{ in appropriate weak sense}\}$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{N} \text{ in appropriate weak sense}\}$

de Rham sequences

de Rham sequences

de Rham sequence

$$H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} \boldsymbol{H}(\mathrm{div}, \Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

de Rham sequences

de Rham sequence

$$H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

Sequence

$$\nabla\times(\nabla) = \mathbf{0} \quad \Leftrightarrow \quad \nabla H^1(\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \nabla\times\boldsymbol{v} = \mathbf{0}\}$$

$$\nabla\cdot(\nabla\times) = 0 \quad \Leftrightarrow \quad \nabla\times\boldsymbol{H}(\mathrm{curl},\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$$

de Rham sequences

de Rham sequence

$$H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

Sequence

$$\nabla\times(\nabla) = \mathbf{0} \quad \Leftrightarrow \quad \nabla H^1(\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \nabla\times\boldsymbol{v} = \mathbf{0}\}$$

$$\nabla\cdot(\nabla\times) = 0 \quad \Leftrightarrow \quad \nabla\times\boldsymbol{H}(\mathrm{curl},\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$$

Exact sequence on Ω with trivial topology

$$\nabla H^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \nabla\times\boldsymbol{v} = \mathbf{0}\}$$

$$\nabla\times\boldsymbol{H}(\mathrm{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$$

de Rham sequences

de Rham sequence

$$H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

Sequence

$$\nabla\times(\nabla) = \mathbf{0} \quad \Leftrightarrow \quad \nabla H^1(\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \nabla\times\boldsymbol{v} = \mathbf{0}\}$$

$$\nabla\cdot(\nabla\times) = 0 \quad \Leftrightarrow \quad \nabla\times\boldsymbol{H}(\mathrm{curl},\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$$

Exact sequence on Ω with trivial topology

$$\nabla H^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \nabla\times\boldsymbol{v} = \mathbf{0}\}$$

$$\nabla\times\boldsymbol{H}(\mathrm{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$$

de Rham sequence, with boundary conditions

$$H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

de Rham sequences

de Rham sequence

$$H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

Sequence

$$\nabla\times(\nabla) = \mathbf{0} \quad \Leftrightarrow \quad \nabla H^1(\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \nabla\times\boldsymbol{v} = \mathbf{0}\}$$
$$\nabla\cdot(\nabla\times) = 0 \quad \Leftrightarrow \quad \nabla\times\boldsymbol{H}(\mathrm{curl},\Omega) \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$$

Exact sequence on Ω with trivial topology

$$\nabla H^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \nabla\times\boldsymbol{v} = \mathbf{0}\}$$
$$\nabla\times\boldsymbol{H}(\mathrm{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$$

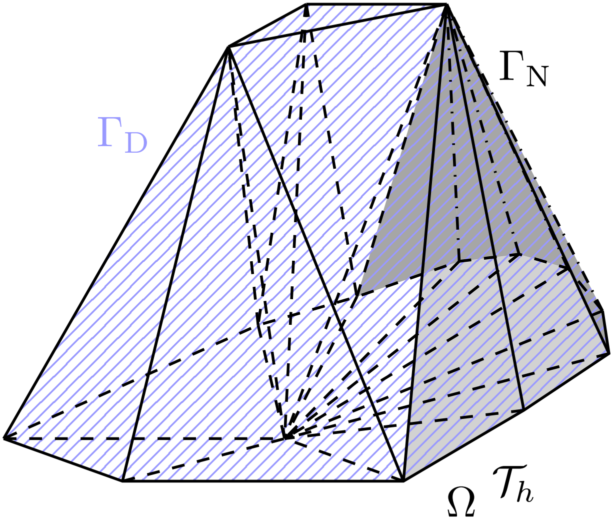
de Rham sequence, with boundary conditions

$$H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

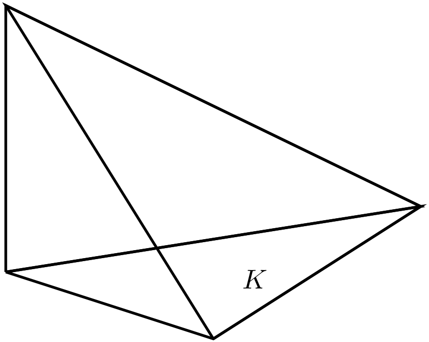
Exact sequence on Ω with trivial topology and boundary conditions

Outline

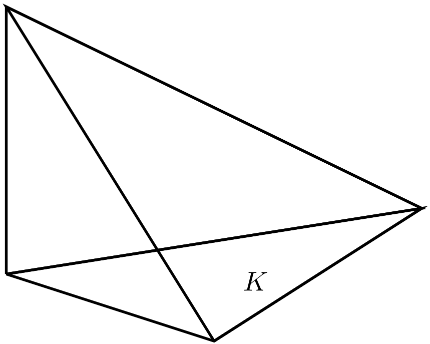
Meshes (mesh size *h*), elements, and patches

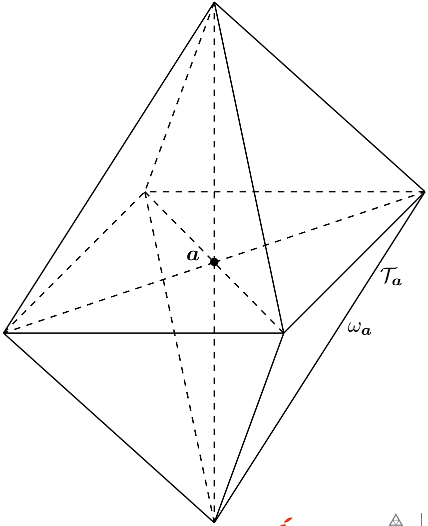


Meshes (mesh size h), elements, and patches



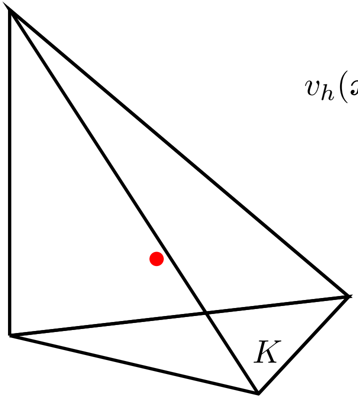
Meshes (mesh size h), elements, and patches





Polynomial space $\mathcal{P}_p(K)$, polynomial degree $p \geq 0$

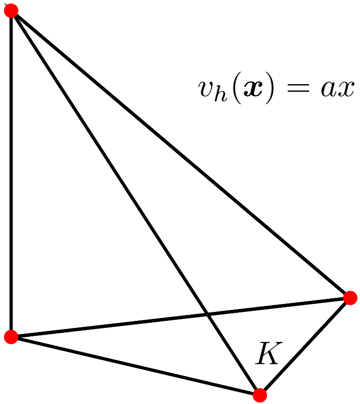
Polynomial space $\mathcal{P}_p(K)$, polynomial degree $p \geq 0$



$$v_h(\boldsymbol{x}) = a$$

$$p = 0$$

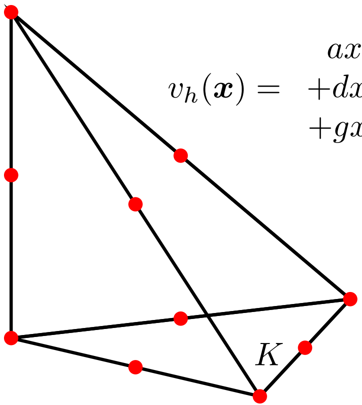
Polynomial space $\mathcal{P}_p(K)$, polynomial degree $p \geq 0$



$$v_h(\boldsymbol{x}) = ax + by + cz + d$$

$$p = 1$$

Polynomial space $\mathcal{P}_p(K)$, polynomial degree $p \geq 0$

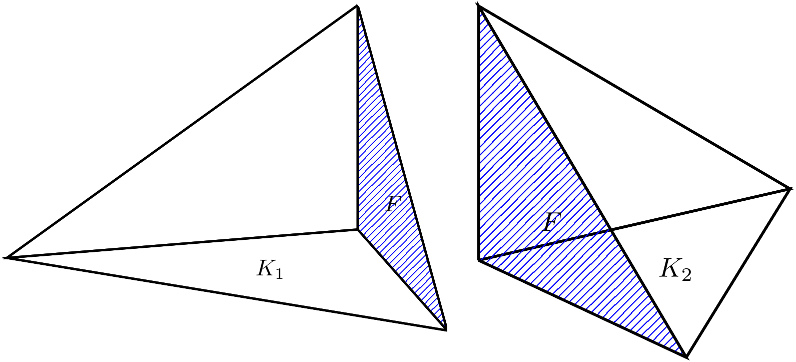


$$v_h(\boldsymbol{x}) = \begin{array}{c} ax^2 + by^2 + cz^2 \\ +dxy + eyz + fxz \\ +gx + hy + iz + j \end{array}$$

$p = 2$

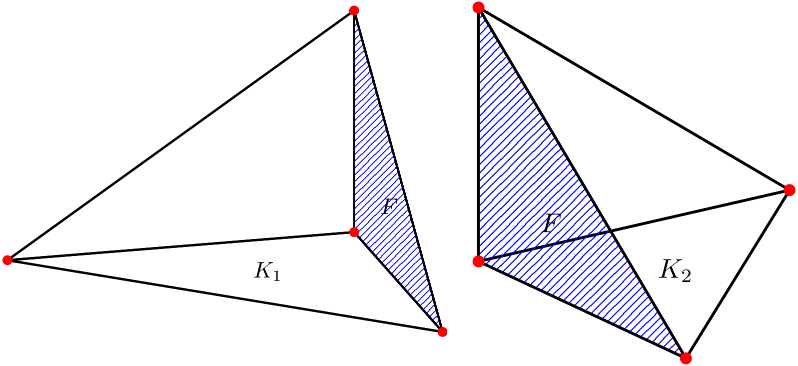
Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$

Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$



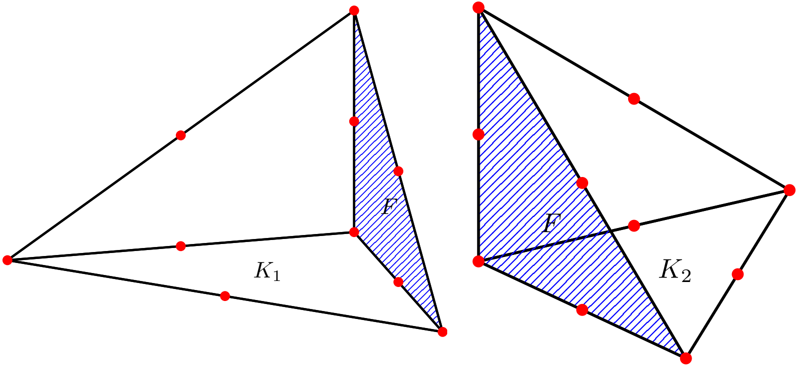
- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- $\Rightarrow$ ensure this by putting sufficient DoFs on the face F

Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$



- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- $\Rightarrow$ ensure this by putting sufficient DoFs on the face F

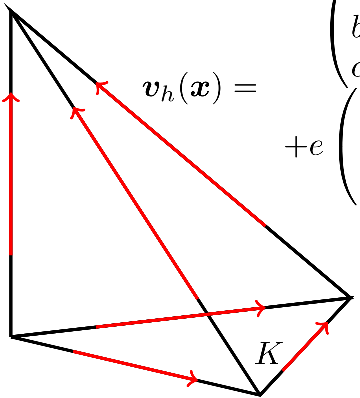
Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$



- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- $\Rightarrow$ ensure this by putting sufficient DoFs on the face F

Nédélec space $\mathcal{N}_p(K) := [\mathcal{P}_p(K)]^3 + \boldsymbol{x} \times [\mathcal{P}_p(K)]^3$, $p \geq 0$

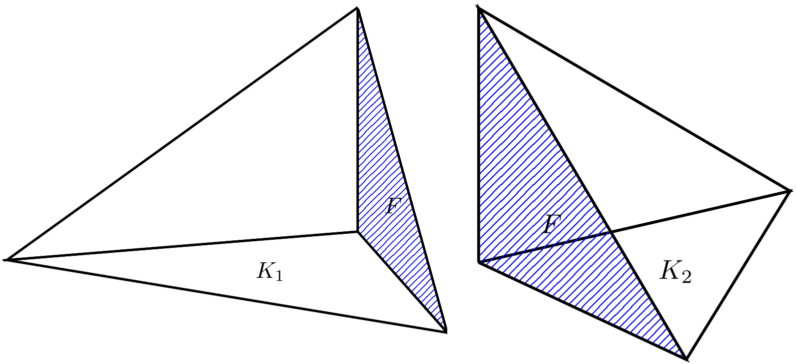
Nédélec space $\mathcal{N}_p(K) := [\mathcal{P}_p(K)]^3 + \boldsymbol{x} \times [\mathcal{P}_p(K)]^3$, $p \geq 0$



$$\boldsymbol{v}_h(\boldsymbol{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} + e \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} + f \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

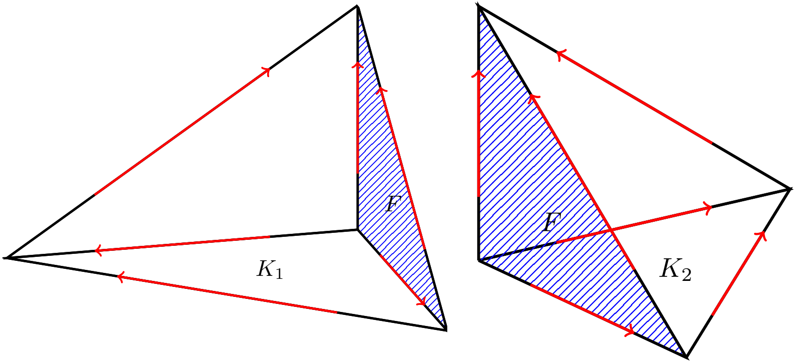
$p = 0$

Nédélec piecewise polynomial space $\mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$, $p \geq 0$



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_2)$, and $(\boldsymbol{v}|_{K_1} \times \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \times \boldsymbol{n}_F)|_F$ in appropriate sense
- $\Rightarrow$ ensure this by putting sufficient DoFs at the face F

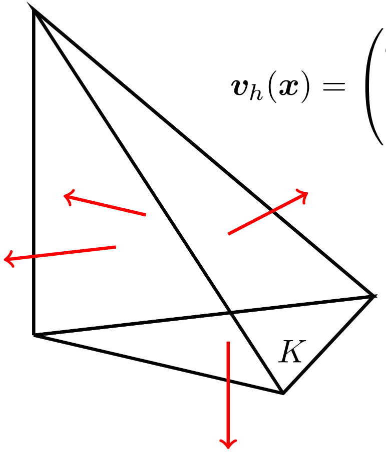
Nédélec piecewise polynomial space $\mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$, $p \geq 0$



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_2)$, and $(\boldsymbol{v}|_{K_1} \times \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \times \boldsymbol{n}_F)|_F$ in appropriate sense
- $\Rightarrow$ ensure this by putting sufficient DoFs at the face F

Raviart–Thomas space $\mathcal{RT}_p(K) := [\mathcal{P}_p(K)]^3 + \mathcal{P}_p(K)\boldsymbol{x}$, $p \geq 0$

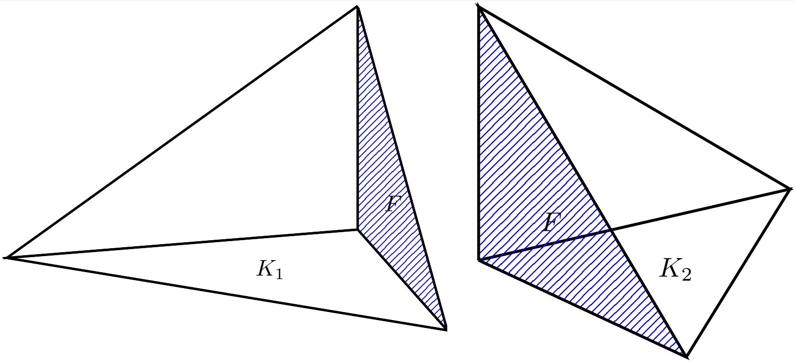
Raviart–Thomas space $\mathcal{RT}_p(K) := [\mathcal{P}_p(K)]^3 + \mathcal{P}_p(K)\boldsymbol{x}$, $p \geq 0$



$$\boldsymbol{v}_h(\boldsymbol{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

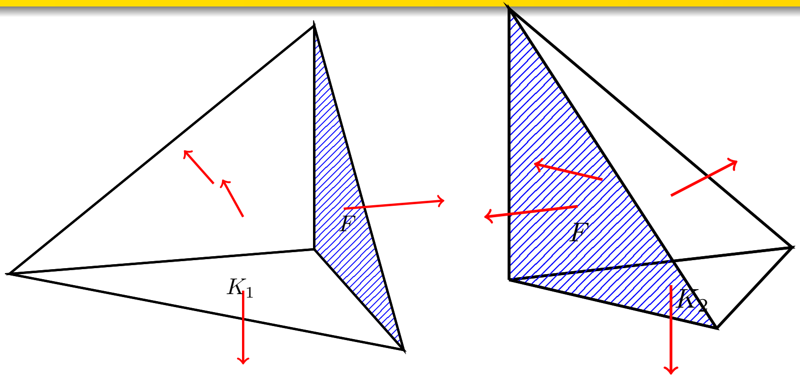
$p = 0$

Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $p \geq 0$



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_2)$, and $(\boldsymbol{v}|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \cdot \boldsymbol{n}_F)|_F$ in appropriate sense
- $\Rightarrow$ ensure this by putting sufficient DoFs on the face F

Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $p \geq 0$



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_2)$, and $(\boldsymbol{v}|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \cdot \boldsymbol{n}_F)|_F$ in appropriate sense
- $\Rightarrow$ ensure this by putting sufficient DoFs on the face F

Outline

Commuting de Rham diagram

Commuting de Rham diagram

$$H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

Commuting de Rham diagram

Commuting de Rham diagram

$$H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

$$\mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \xrightarrow{\nabla\cdot} \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega)$$

Commuting de Rham diagram

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\big\downarrow P^{\mathrm{grad}}_{h(p+1)} & & \big\downarrow \boldsymbol{P}^{\mathrm{curl}}_{hp} & & \big\downarrow \boldsymbol{P}^{\mathrm{div}}_{hp} & & \big\downarrow \Pi^p_h \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \Big\downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \Big\downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega) \end{array}$$

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ (**minimal regularity**, partial BCs)

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$** (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)
3. is defined simply, starting from the **elementwise L^2-orthogonal projection**

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$** (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)
3. is defined simply, starting from the **elementwise L^2-orthogonal projection**
4. has optimal *hp* **approximation property**, that of **elementwise div-unconstrained best approximation** (global–local equivalence)

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$** (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)
3. is defined simply, starting from the **elementwise L^2-orthogonal projection**
4. has optimal *hp* **approximation property**, that of **elementwise div-unconstrained best approximation** (global–local equivalence)
5. is ***p*-stable in $L^2(\Omega)$** (up to data oscillation)

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$** (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)
3. is defined simply, starting from the **elementwise L^2-orthogonal projection**
4. has optimal *hp* **approximation property**, that of **elementwise div-unconstrained best approximation** (global–local equivalence)
5. is ***p*-stable in $L^2(\Omega)$** (up to data oscillation)
6. satisfies the **commuting property** expressed by the arrows

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$** (**minimal regularity**, partial BCs)
2. is defined **locally** (in neighborhood of each mesh element)
3. is defined simply, starting from the **elementwise L^2-orthogonal projection**
4. has optimal *hp* **approximation property**, that of **elementwise div-unconstrained best approximation** (global–local equivalence)
5. is ***p*-stable in $L^2(\Omega)$** (up to data oscillation)
6. satisfies the **commuting property** expressed by the arrows
7. is **projector**, i.e., leaves intact piecewise polynomials

Outline

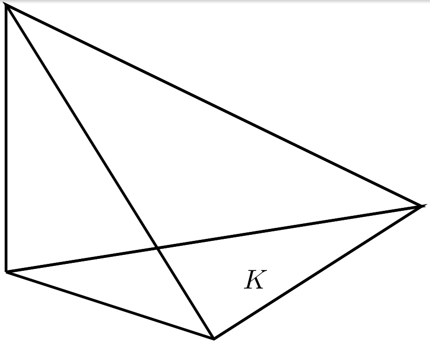
Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

- Whitney, Bossavit, Nédélec, Raviart, Thomas, Desbrun ...
- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$ regularity but not commuting
- Licht (2019): essential boundary conditions on a part of $\partial\Omega$
- Arnold and Guzmán (2021): $\boldsymbol{L}^2$-stable
- Ern, Gudi, Smears, and Vohralík (2022) & Chaumont-Frelet and Vohralík (2024): all the above properties
- none is p-robust

Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

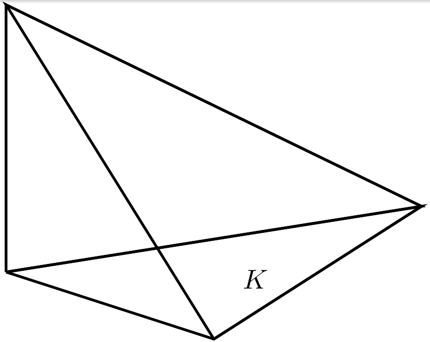
- Whitney, Bossavit, Nédélec, Raviart, Thomas, Desbrun ...
- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$ regularity but not commuting
- Licht (2019): essential boundary conditions on a part of $\partial\Omega$
- Arnold and Guzmán (2021): $\boldsymbol{L}^2$-stable
- Ern, Gudi, Smears, and Vohralík (2022) & Chaumont-Frelet and Vohralík (2024): all the above properties
- **none is p-robust**

Canonical elementwise interpolation



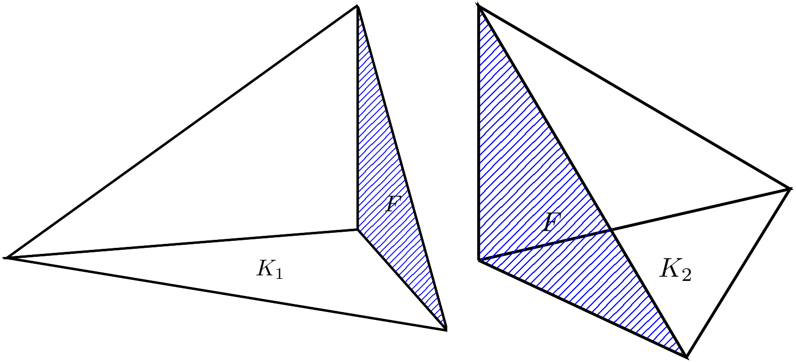
- $\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$
- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \Rightarrow \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{div}, K) \Rightarrow$ so interpolate $\boldsymbol{v}|_K$, elementwise interpolation

Canonical elementwise interpolation



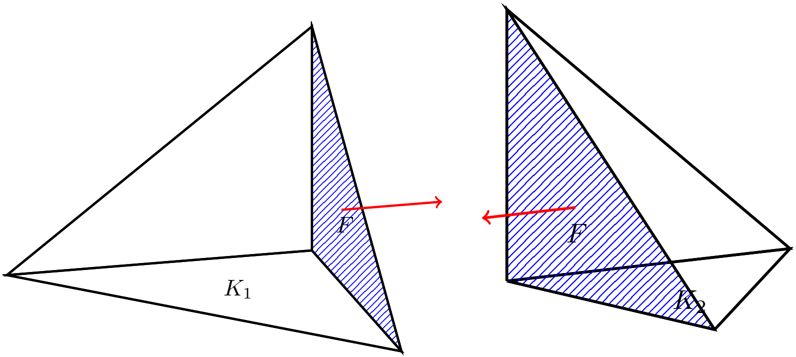
- $\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$
- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \Rightarrow \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{div}, K) \Rightarrow$ so interpolate $\boldsymbol{v}|_K$, elementwise interpolation

Canonical elementwise interpolation



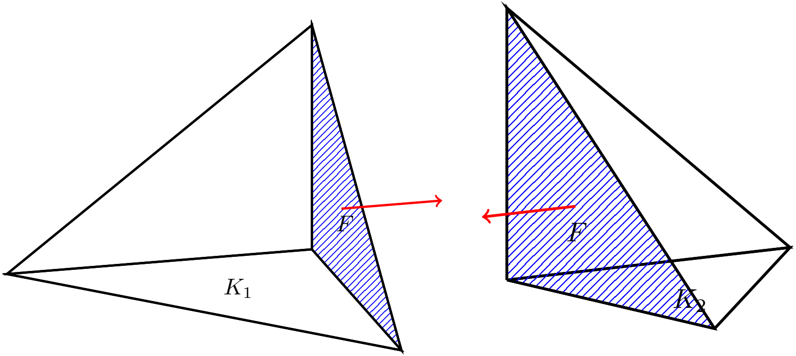
- $(\boldsymbol{v}_h|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}_h|_{K_2} \cdot \boldsymbol{n}_F)|_F$

Canonical elementwise interpolation



- $(\boldsymbol{v}_h|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}_h|_{K_2} \cdot \boldsymbol{n}_F)|_F$ $(= \langle \boldsymbol{v} \cdot \boldsymbol{n}_F, 1 \rangle_F / |F|$ for $p = 0)$

Canonical elementwise interpolation

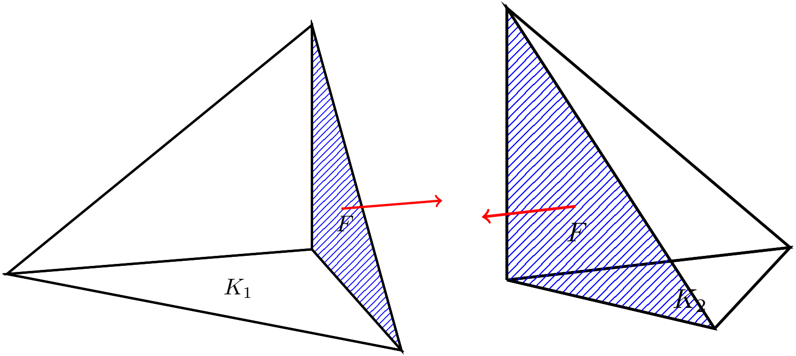


- $(\boldsymbol{v}_h|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}_h|_{K_2} \cdot \boldsymbol{n}_F)|_F$ $(= \langle \boldsymbol{v} \cdot \boldsymbol{n}_F, 1 \rangle_F / |F|$ for $p = 0)$

Clash

Face normal trace integrals $\langle \boldsymbol{v} \cdot \boldsymbol{n}_F, 1 \rangle_F / |F|$ not available in $\boldsymbol{H}(\mathrm{div})$.

Canonical elementwise interpolation

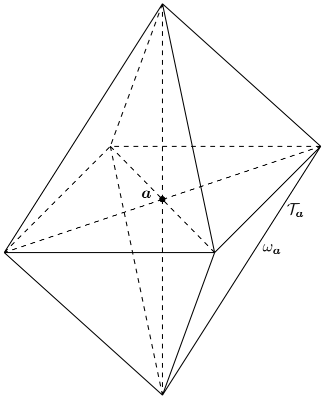


- $(\boldsymbol{v}_h|_{K_1}\cdot\boldsymbol{n}_F)|_F = (\boldsymbol{v}_h|_{K_2}\cdot\boldsymbol{n}_F)|_F$ $(= \langle \boldsymbol{v}\cdot\boldsymbol{n}_F, 1\rangle_F/|F|$ for $p = 0)$

Conclusion

Allows most of the properties but not the minimal regularity $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$.

Classical patchwise interpolation (Clément)

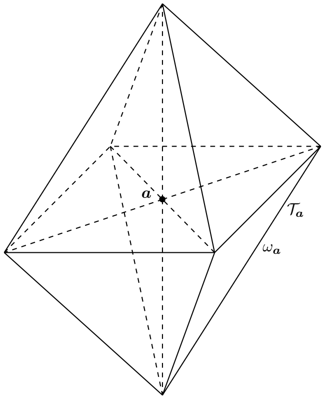


- some local-best polynomial approximation on $\omega_{\boldsymbol{a}}$
- values on $\omega_{\boldsymbol{a}}$ as coefficients for basis functions supported on $\omega_{\boldsymbol{a}}$

Conclusion

Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

Classical patchwise interpolation (Clément)



- some local-best polynomial approximation on $\omega_{\boldsymbol{a}}$
- values on $\omega_{\boldsymbol{a}}$ as coefficients for basis functions supported on $\omega_{\boldsymbol{a}}$

Conclusion

Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

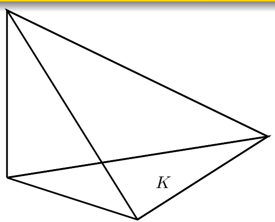
Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ be given.

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ be given.

1. Each $K \in \mathcal{T}_h$: **elementwise L^2-orthogonal projection**

$$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$



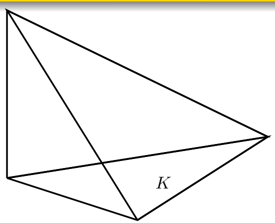
A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ be given.

1. Each $K \in \mathcal{T}_h$: **elementwise L^2-orthogonal projection**

$$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

(discrete but normal-trace discontinuous).



A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

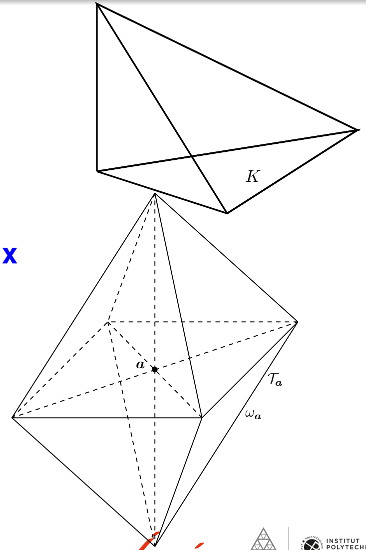
Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ be given.

1. Each $K \in \mathcal{T}_h$: **elementwise L^2-orthogonal projection**

$$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

(discrete but normal-trace discontinuous).

2. Obtain $\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ by applying the **flux equilibration** to $\boldsymbol{\tau}_h$ (local energy minimizations on the vertex patch subdomains $\omega_{\boldsymbol{a}}$).



A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

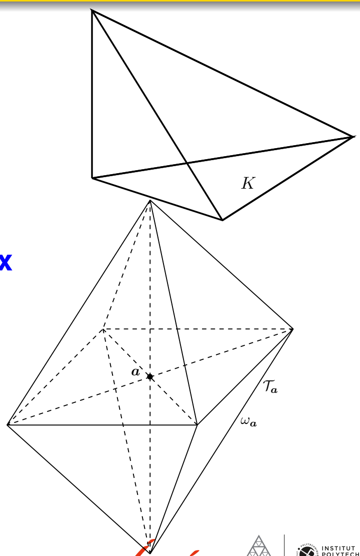
Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ be given.

1. Each $K \in \mathcal{T}_h$: **elementwise L^2-orthogonal projection**

$$\boldsymbol{\tau}_h|_K := \arg\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

(discrete but normal-trace discontinuous).

2. Obtain $\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ by applying the **flux equilibration** with an **additional orthogonality constraint** to $\boldsymbol{\tau}_h$ (local energy minimizations on the vertex patch subdomains $\omega_{\boldsymbol{a}}$).



A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

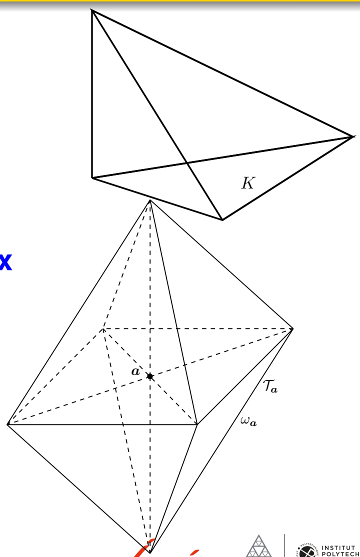
Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ be given.

1. Each $K \in \mathcal{T}_h$: **elementwise L^2-orthogonal projection**

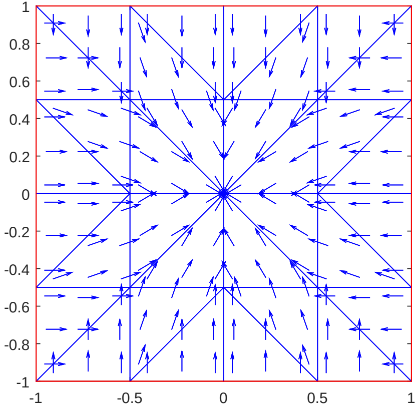
$$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

(discrete but normal-trace discontinuous).

2. Obtain $\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ by applying the **flux equilibration** with an **additional orthogonality constraint** to $\boldsymbol{\tau}_h$ (local energy minimizations on the vertex patch subdomains $\omega_{\boldsymbol{a}}$).
3. Apply a ***p*-stable decomposition** on extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$ to conforming projections of the reminder $\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h \Rightarrow$ correction $\boldsymbol{\zeta}_h$; $\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) := \boldsymbol{\sigma}_h + \boldsymbol{\zeta}_h$.

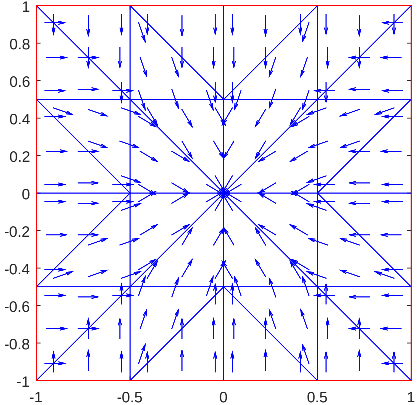


Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$

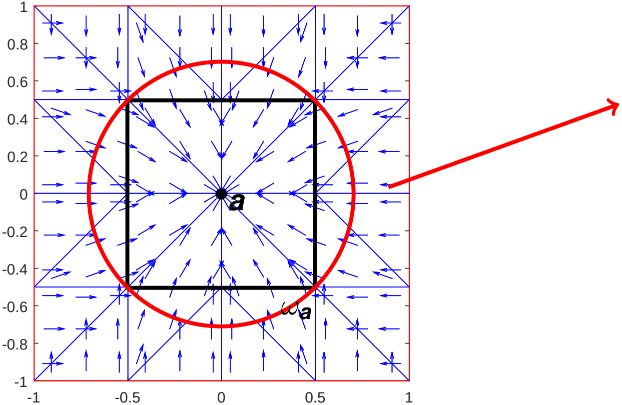
Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)

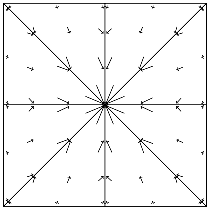


Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



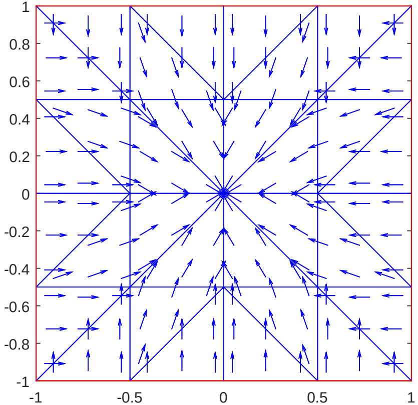


$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$

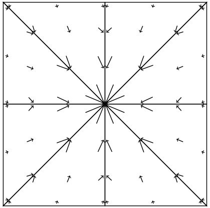
Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

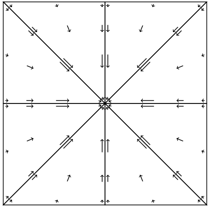
Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



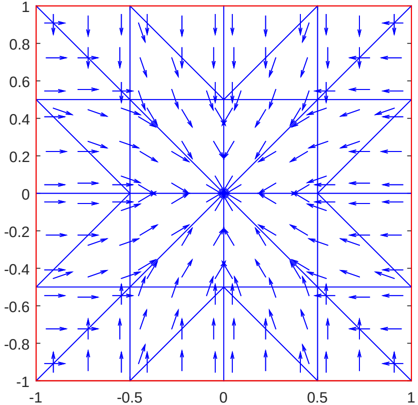
$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$



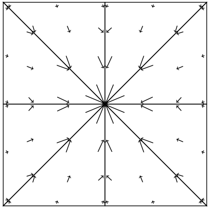
$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

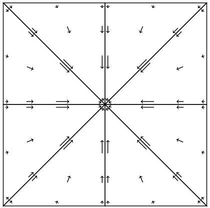
Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$

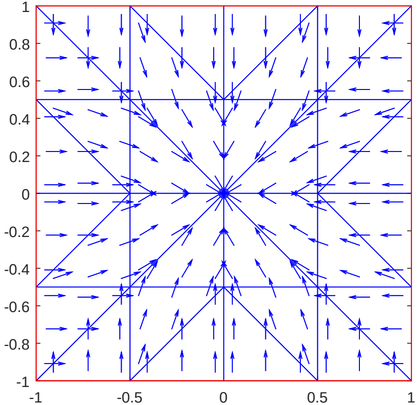


$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

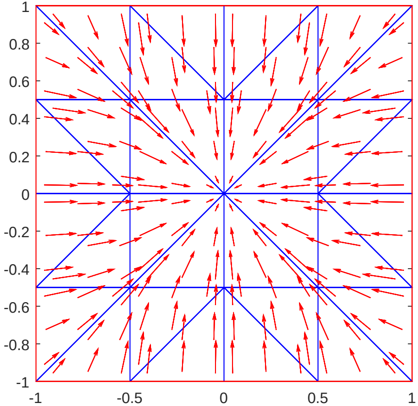
$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \qquad \boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=f\psi^{\boldsymbol{a}}+\boldsymbol{\tau}_h\cdot\nabla\psi^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$

Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



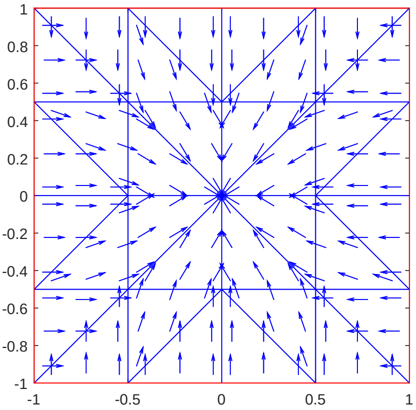
Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



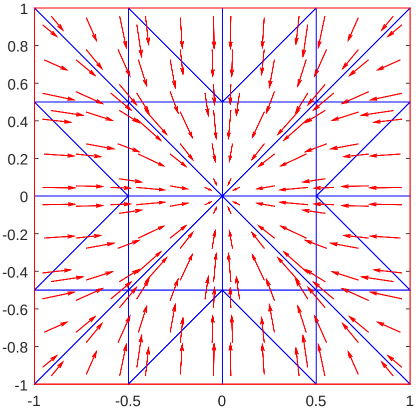
Equilibrated flux rec. $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}),\, \nabla\cdot\boldsymbol{\sigma}_h = f$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



Equilibrated flux rec. $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_h = f$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$

   (patchwise **flux equilibration**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \qquad \text{(gluing patchwise contributions)}$$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)
2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \nabla \cdot \boldsymbol{v} + \nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{v}}} \|\psi^{\boldsymbol{a}} \boldsymbol{\tau}_h - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)
2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v})}} \|\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

(patchwise **flux equilibration**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)
2. $\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}} \nabla \cdot \boldsymbol{v} + \nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{v})}} \left\| \boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi^{\boldsymbol{a}} \boldsymbol{\tau}_h) - \boldsymbol{v}_h \right\|_{\omega_{\boldsymbol{a}}}$

(patchwise **flux equilibration**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}} \nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h), \boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p \geq 1}} \left\| \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{v}_h \right\|_{\omega_{\boldsymbol{a}}}$$

(patchwise **flux equilibration** with an **additional orthogonality constraint**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}} \nabla \cdot \boldsymbol{v} + \nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{v}) \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi^{\boldsymbol{a}} \boldsymbol{\tau}_h), \boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p \geq 1}} \|\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi^{\boldsymbol{a}} \boldsymbol{\tau}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

(patchwise **flux equilibration** with an **additional orthogonality constraint**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h, \boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

(patchwise divergence-free remainder equilibration with an additional constraint)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p\geq 1}} \left\|\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \qquad \text{(gluing patchwise contributions)}$$

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

   (patchwise divergence-free remainder equilibration with an additional constraint)

   $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{b}\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{b}} \text{ with in particular } \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$$

   (patchwise ***p*-stable** remainder **decomposition**)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p \geq 1}} \left\|\boldsymbol{I}_{\mathcal{RT}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

   (patchwise divergence-free remainder equilibration with an additional constraint)

   $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{b}\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{b}}$$ with in particular $\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, $\nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$

   (patchwise ***p*-stable** remainder **decomposition**)

   $$\boldsymbol{\zeta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}$$ (gluing patchwise correction contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}} := \boldsymbol{\sigma}_h + \boldsymbol{\zeta}_h$

1. $$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p \geq 1}} \|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

   (patchwise divergence-free remainder equilibration with an additional constraint)

   $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{b}\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{b}}$$ with in particular $\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, $\nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$

   (patchwise ***p*-stable** remainder **decomposition**)

   $$\boldsymbol{\zeta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}$$ (gluing patchwise correction contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_{hp}^{\mathrm{div}}$ *is* ***commuting*** *since*

$$\nabla\cdot\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$)

$\boldsymbol{P}_{hp}^{\mathrm{div}}$ *is* ***commuting*** *and* ***projector*** *since*

$$\nabla\cdot\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega),$$

$$\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega).$$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}}: \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_{hp}^{\mathrm{div}}$ *is* ***commuting*** *and* ***projector*** *since*

$$\nabla\cdot\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$
$$\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega).$$

It has ***p-robust local-best approximation property***

since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $K \in \mathcal{T}_h$,

$$\|\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\cdot(\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}))\|_K\right)^2$$
$$\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \left\{ \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(L)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2 \right\},$$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

*$\boldsymbol{P}_{hp}^{\mathrm{div}}$ is **commuting** and **projector** since*

$$\nabla{\cdot}\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla{\cdot}\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$

$$\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega).$$

*It has **p-robust local-best approximation property** and is **p-robustly L^2 stable** up to data oscillation, since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $K \in \mathcal{T}_h$,*

$$\|\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla{\cdot}(\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}))\|_K\right)^2$$

$$\lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \left\{ \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(L)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla{\cdot}\boldsymbol{v} - \Pi_h^p(\nabla{\cdot}\boldsymbol{v})\|_L\right)^2 \right\},$$

$$\|\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v})\|_K^2 \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \left\{ \|\boldsymbol{v}\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla{\cdot}\boldsymbol{v} - \Pi_h^p(\nabla{\cdot}\boldsymbol{v})\|_L\right)^2 \right\}.$$

Outline

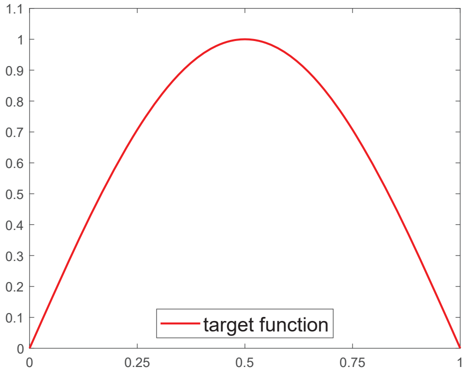
Global-best approximation ≈ local-best approximation

Previous contributions

- Carstensen, Peterseim, and Schedensack (2012): H^1 (lowest-order case $p = 1$)
- Aurada, Feischl, Kemetmüller, Page, and Praetorius (2013): H^1 (boundary approximation context)
- Veeser (2016): H^1 (any p, p-dependent constant)
- Canuto, Nochetto, Stevenson, and Verani (2017): H^1 (improvement of the p dependence of the equivalence constant in 2D)
- Ern, Gudi, Smears, and Vohralík (2022): $\boldsymbol{H}(\mathrm{div})$ (any p, p-dependent constant)
- Chaumont-Frelet and Vohralík (2021, 2022): $\boldsymbol{H}(\mathrm{curl})$ (any p, p-dependent constant)
- Gawlik, Holst, and Licht (2021): finite element exterior calculus context (any p, p-dependent constant)

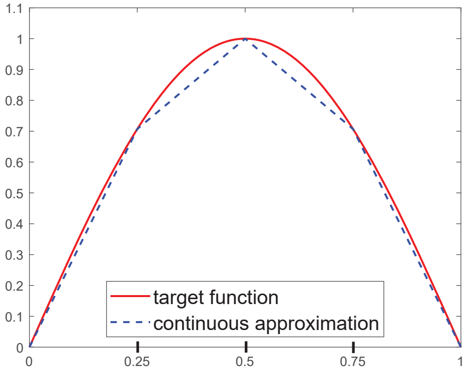
Outline

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



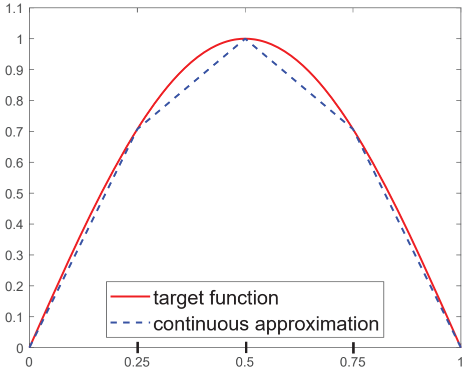
Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

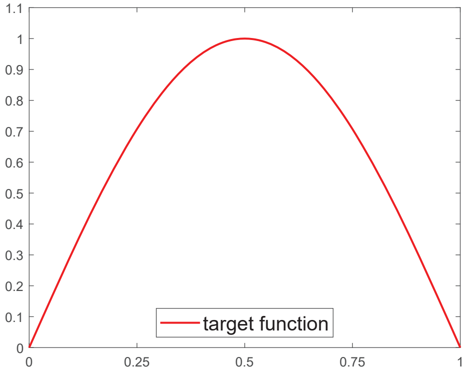


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

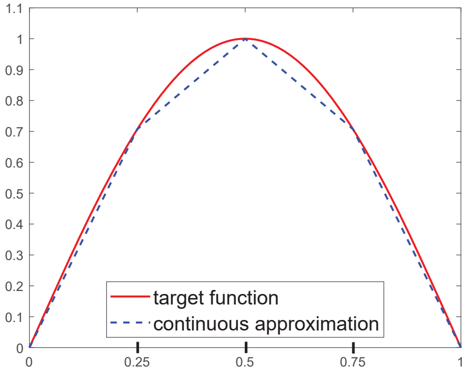


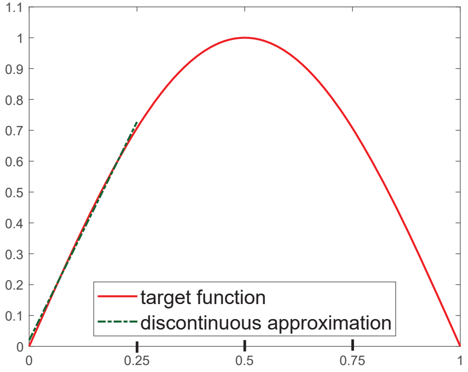
Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space



Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

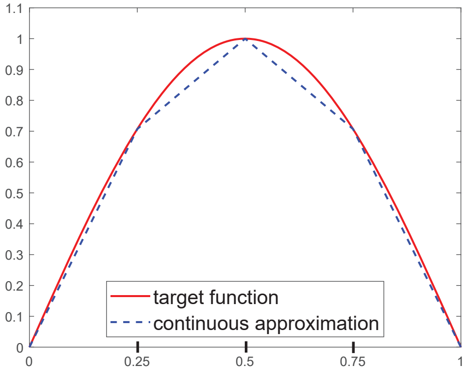




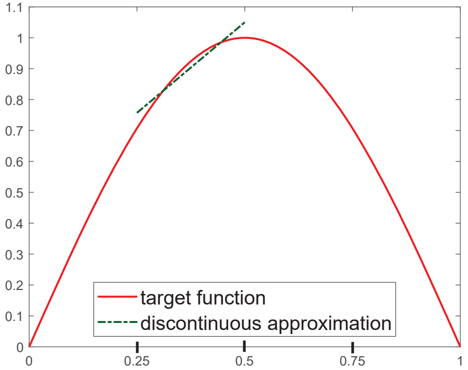
Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **bigger** space

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D

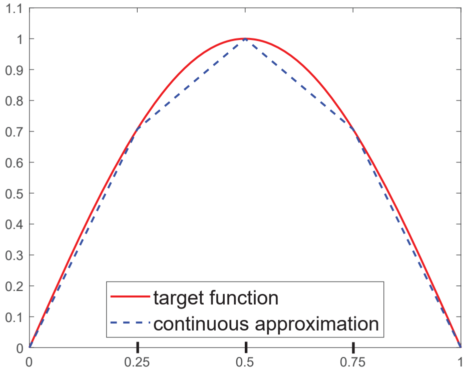


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

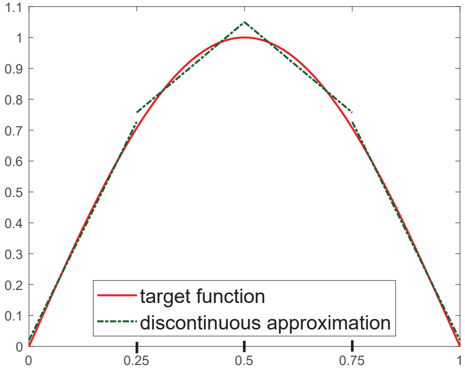


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **bigger** space

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

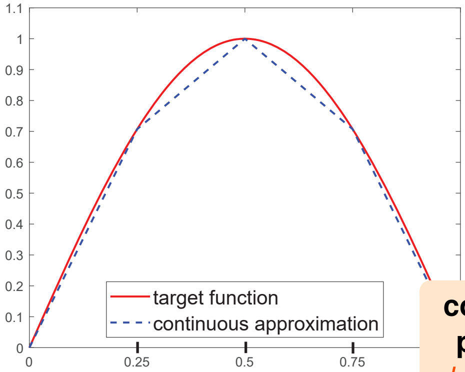


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

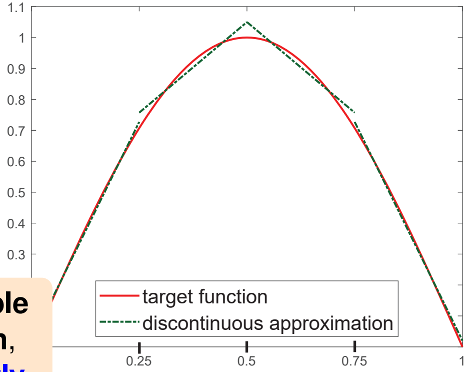


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **bigger** space

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



comparable precision, *hp*-robustly



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **bigger** space

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Global–local equivalence in H_0^1, Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (Global–local equivalence in H^1_0, Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

$$\min_{\textit{smaller space}} \approx_p \min_{\textit{bigger space}}$$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Global–local equivalence in H_0^1, Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

$$\min_{\textit{CG space}} \approx_p \min_{\textit{DG space}}$$

- $\approx_p$: up to a generic constant that only depends on space dimension d, shape-regularity of the mesh $\mathcal{T}_h$, and polynomial degree p

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|^2_K}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|^2_K}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (p-robust global–local equivalence in $H_{0,\mathrm{D}}^1(\Omega)$)

Let $v \in H_{0,\mathrm{D}}^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Outline

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Global–local equivalence in $\boldsymbol{H}$(div), Ern, Gudi, Smears, & V. (2022))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Global–local equivalence in $\boldsymbol{H}$(div), Ern, Gudi, Smears, & V. (2022))

$$\min_{\textit{smaller space with constraints}} \approx_p \min_{\textit{bigger space without constraints}}$$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Global–local equivalence in $\boldsymbol{H}$(div), Ern, Gudi, Smears, & V. (2022))

$$\min_{\textit{MFE space with constraints}} \approx_p \min_{\textit{broken MFE space without constraints}}$$

- $\approx_p$: only depends on space dimension d, shape-regularity of $\mathcal{T}_h$, and polynomial degree p

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (p-robust global–local equivalence in $\boldsymbol{H}(\mathrm{div})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RTN}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RTN}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx$: only depends on d and shape-regularity of $\mathcal{T}_h$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (p-robust global–local equivalence in $\boldsymbol{H}$(div))

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RTN}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RTN}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx$: only depends on d and shape-regularity of $\mathcal{T}_h$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (p-robust global–local equivalence in $\boldsymbol{H}$(div))

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RTN}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2}_{\text{global-best on } \Omega}$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RTN}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\text{local-best on each } K}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- $\approx$: only depends on d and shape-regularity of $\mathcal{T}_h$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div},\Omega)$

Theorem (p-robust global–local equivalence in $\boldsymbol{H}$(div))

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RTN}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2}$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RTN}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- $\approx$: only depends on d and shape-regularity of $\mathcal{T}_h$

Outline

1. Domain, differential operators, Sobolev spaces, and de Rham sequences
2. Meshes and piecewise polynomial spaces
3. p-stable local commuting projectors
   - p-stable local commuting projector in $\boldsymbol{H}(\mathrm{div},\Omega)$
4. p-robust global-best–local-best equivalence
   - p-robust global-best–local-best equivalence in $H^1(\Omega)$
   - p-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div},\Omega)$
5. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div},\Omega)$
6. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - p-stable (broken) polynomial extensions
   - p-stable decompositions
7. Conclusions

The Laplace equation (source term $f \in L^2(\Omega)$)

The Laplace equation

Find $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_\mathrm{D}, \\ -\nabla u \cdot \boldsymbol{n}_\Omega &= 0 && \text{on } \Gamma_\mathrm{N}. \end{aligned}$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \, \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \, \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Dual weak formulation

$\boldsymbol{\sigma} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with* $\nabla{\cdot}\boldsymbol{\sigma} = f$ *such that*

$$(\boldsymbol{\sigma}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \textit{ with } \nabla{\cdot}\boldsymbol{v} = 0.$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \; \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \, \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

Dual weak formulation

$\boldsymbol{\sigma} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with* $\nabla \cdot \boldsymbol{\sigma} = f$ *such that*

$$(\boldsymbol{\sigma}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \text{ with } \nabla \cdot \boldsymbol{v} = 0.$$

Dual finite element approximation

$\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with* $\nabla \cdot \boldsymbol{\sigma}_h = \Pi_h^p f$ *such that*

$$(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \text{ with } \nabla \cdot \boldsymbol{v}_h = 0.$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \, \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

Dual weak formulation

$\boldsymbol{\sigma} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with* $\nabla\cdot\boldsymbol{\sigma} = f$ *such that*

$$(\boldsymbol{\sigma}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \text{ with } \nabla\cdot\boldsymbol{v} = 0.$$

Dual finite element approximation

$\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with* $\nabla\cdot\boldsymbol{\sigma}_h = \Pi_h^p f$ *such that*

$$(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \text{ with } \nabla\cdot\boldsymbol{v}_h = 0.$$

Error characterisation

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|$$

The Laplace equation (source term $f \in L^2(\Omega)$): **global-best approximations**

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

Error characterisation

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_h^p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|$$

The Laplace equation (source term $f \in L^2(\Omega)$): **global-best approximations (constrained)**

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

Error characterisation

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_h^p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|$$

Outline

Approximation error estimates in the $H^1(\Omega)$ context

h approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p, s-1\}} |v|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

Approximation error estimates in the $H^1(\Omega)$ context

h approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p, s-1\}} |v|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

Approximation error estimates in the $H^1(\Omega)$ context

hp approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, \not{p}) \frac{h^{\min\{p, s-1\}}}{p^{s-1}} \|v\|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

Approximation error estimates in the $H^1(\Omega)$ context

hp approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, \not{p}) \ln(p) \frac{h^{\min\{p,s-1\}}}{p^{s-1}} \|v\|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

Approximation error estimates in the $H^1(\Omega)$ context

hp approximation estimate

Let $v \in H^s(\Omega)$, $s \geq 1$. *Then*

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) \ln(p) \frac{h^{\min\{p, s-1\}}}{p^{s-1}} \|v\|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ *with*

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$.

- varying polynomial degree: $\underline{p}_K := \min_{L\in\widetilde{\mathcal{T}}_K}\{p_L\}$ is the smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ *with*

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$. *Then*

$$\|\nabla(v - P_h^{p,\mathrm{grad}} v)\|_K^2 \leq C(\kappa_{\mathcal{T}_h}, \kappa_p, d, s)^2 \sum_{L \in \widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(\underline{p}_K, s_L - 1)}}{\underline{p}_K^{s_L - 1}} \|v\|_{H^{s_L}(L)} \right)^2 \qquad \forall K \in \mathcal{T}_h.$$

- varying polynomial degree: $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$ is the smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ *with*

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$. *Then*

$$\|\nabla(v - P_h^{p,\mathrm{grad}} v)\|_K^2 \leq C(\kappa_{\mathcal{T}_h}, \kappa_p, d, s)^2 \sum_{L \in \widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(\underline{p}_K, s_L - 1)}}{\underline{p}_K^{s_L - 1}} \|v\|_{H^{s_L}(L)} \right)^2 \qquad \forall K \in \mathcal{T}_h.$$

- varying polynomial degree: $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$ is the smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

Outline

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div}, \Omega)$ context

h approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) \quad h^{\min\{p+1,s\}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994), rectangular meshes
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div},\Omega)$ context

h approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) \quad h^{\min\{p+1,s\}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994), rectangular meshes
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div},\Omega)$ context

hp approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, \not{p}) \quad \frac{h^{\min\{p+1,s\}}}{(p+1)^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994), rectangular meshes
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div},\Omega)$ context

hp approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, \not{p}) \ln(p) \frac{h^{\min\{p+1,s\}}}{(p+1)^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994), ~~rectangular meshes~~
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div}, \Omega)$ context

hp approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, \cancel{p})\cancel{\ln(p)} \frac{h^{\min\{p+1,s\}}}{(p+1)^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994),
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div},\Omega)$ context

h approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{H}^s(\Omega)$, ~~$s > 1/2$~~. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_h^p(\nabla \cdot \boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p+1,s\}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994),
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div},\Omega)$ context

hp approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \le C(\kappa_{\mathcal{T}_h}, d, s, p) \ln(p) \frac{h^{\min\{p+1,s\}}}{(p+1)^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994),
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla{\cdot}\boldsymbol{v})|_K \in [\mathcal{P}_p(K)]^3 \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$.

- varying polynomial degree can be added

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla\cdot\boldsymbol{v})|_K \in [\mathcal{P}_p(K)]^3 \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h = \nabla\cdot\boldsymbol{v}}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s)^2 \sum_{K\in\mathcal{T}_h} \left(\frac{h_K^{\min\{p+1, s_K\}}}{(p+1)^{s_K}} \|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)} \right)^2.$$

- varying polynomial degree can be added

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla \cdot \boldsymbol{v})|_K \in \boldsymbol{H}^{t_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ *and* $t_K \geq 0$.

- varying polynomial degree can be added

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla\cdot\boldsymbol{v})|_K \in \boldsymbol{H}^{t_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ *and* $t_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h=\Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p(\nabla\cdot\boldsymbol{v})\|_K\right)^2\Bigg]$$

$$\leq C(\kappa_{\mathcal{T}_h},d,s,t)^2\sum_{K\in\mathcal{T}_h}\left[\left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2+\left(\frac{h_K}{p+1}\frac{h_K^{\min\{p+1,t_K\}}}{(p+1)^{t_K}}\|\nabla\cdot\boldsymbol{v}\|_{\boldsymbol{H}^{t_K}(K)}\right)^2\right].$$

- varying polynomial degree can be added

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla\cdot\boldsymbol{v})|_K \in \boldsymbol{H}^{t_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ *and* $t_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p(\nabla\cdot\boldsymbol{v})\|_K\right)^2\Bigg]$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s, t)^2 \sum_{K\in\mathcal{T}_h}\left[\left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2 + \left(\frac{h_K}{p+1}\frac{h_K^{\min\{p+1,t_K\}}}{(p+1)^{t_K}}\|\nabla\cdot\boldsymbol{v}\|_{\boldsymbol{H}^{t_K}(K)}\right)^2\right].$$

- varying polynomial degree can be added

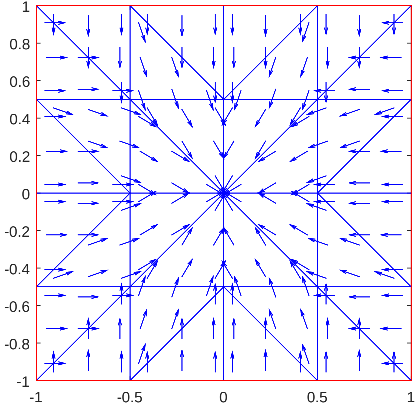
Outline

1. Domain, differential operators, Sobolev spaces, and de Rham sequences
2. Meshes and piecewise polynomial spaces
3. *p*-stable local commuting projectors
   - *p*-stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
4. *p*-robust global-best–local-best equivalence
   - *p*-robust global-best–local-best equivalence in $H^1(\Omega)$
   - *p*-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
5. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div}, \Omega)$
6. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - *p*-stable (broken) polynomial extensions
   - *p*-stable decompositions
7. Conclusions

Outline

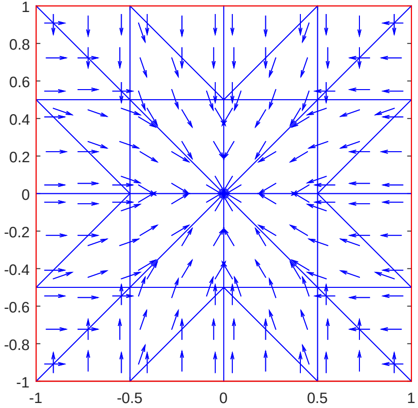
1. Domain, differential operators, Sobolev spaces, and de Rham sequences
2. Meshes and piecewise polynomial spaces
3. p-stable local commuting projectors
   - p-stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
4. p-robust global-best–local-best equivalence
   - p-robust global-best–local-best equivalence in $H^1(\Omega)$
   - p-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
5. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div}, \Omega)$
6. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - p-stable (broken) polynomial extensions
   - p-stable decompositions
7. Conclusions

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla \cdot \boldsymbol{\tau}_h \neq f$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)

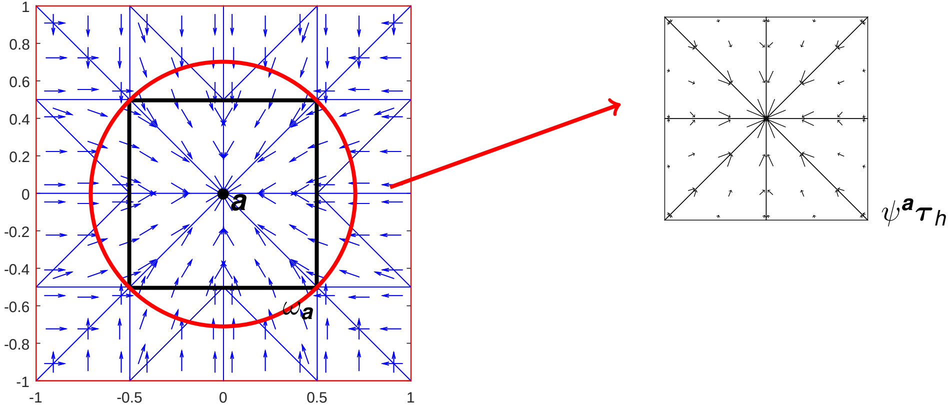


Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$

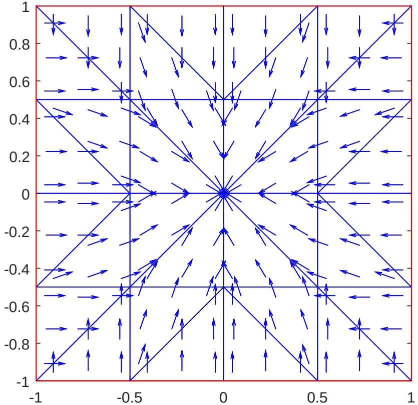
Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



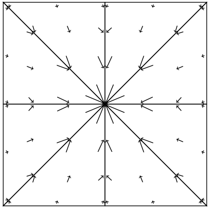
Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$

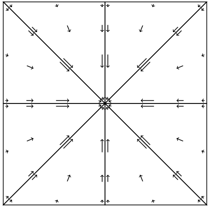
$$\underbrace{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



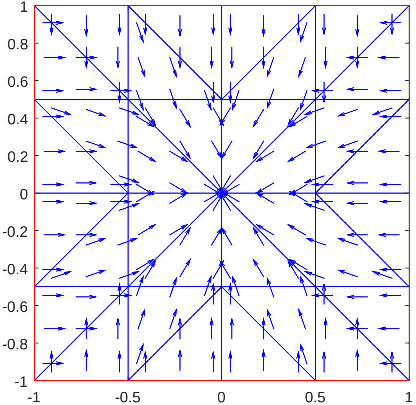
Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$


$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$

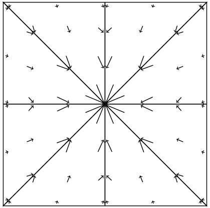

$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

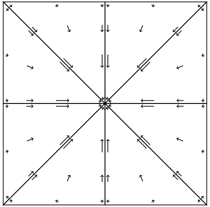
Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



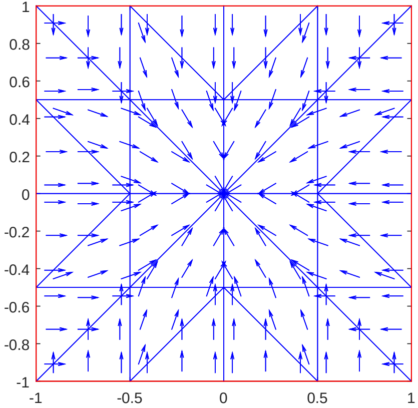
$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$



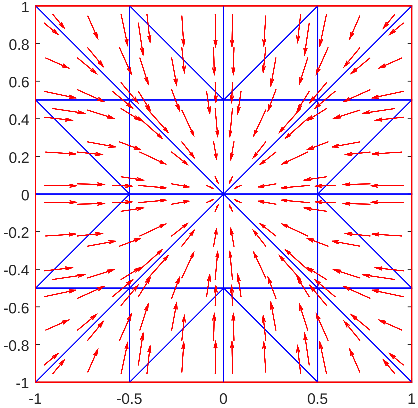
$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \qquad \boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=f\psi^{\boldsymbol{a}}+\boldsymbol{\tau}_h\cdot\nabla\psi^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h-\boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



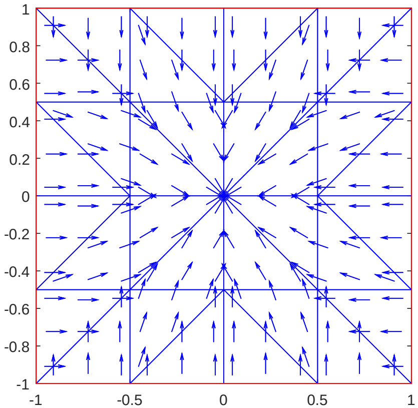
Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



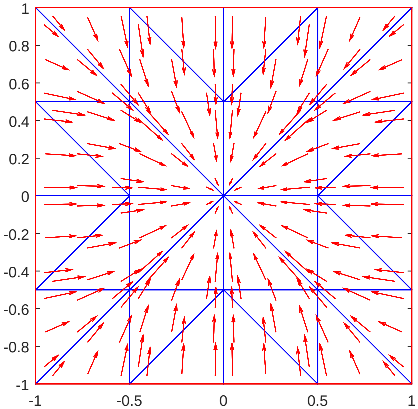
Equilibrated flux rec. $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}),\, \nabla\cdot\boldsymbol{\sigma}_h = f$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



Equilibrated flux rec. $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_h = f$

Outline

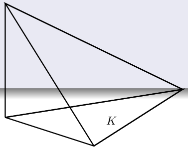
1. Domain, differential operators, Sobolev spaces, and de Rham sequences
2. Meshes and piecewise polynomial spaces
3. *p*-stable local commuting projectors
   - *p*-stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
4. *p*-robust global-best–local-best equivalence
   - *p*-robust global-best–local-best equivalence in $H^1(\Omega)$
   - *p*-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
5. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div}, \Omega)$
6. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - *p*-stable (broken) polynomial extensions
   - *p*-stable decompositions
7. Conclusions

$\boldsymbol{H}(\mathrm{div})$ polynomial extensions on a tetrahedron

Theorem ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020))

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_\mathcal{F} \in \mathcal{P}_p(\Gamma_\mathcal{F})$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{RT}_p(K)\\ \nabla\cdot\boldsymbol{v}_p = r_K\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K)\\ \nabla\cdot\boldsymbol{v} = r_K\\ \boldsymbol{v}\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}\|_K.$$



Comments

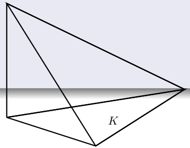
- C_{st} only depends on the shape-regularity of K
- p-robustness: for (pw) p-polynomial data r_K, $r_\mathcal{F}$, minimization over $\mathcal{RT}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{div}, K)$
- extension to a vertex patch: Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)

$\boldsymbol{H}$(div) polynomial extensions on a tetrahedron

Theorem ($\boldsymbol{H}$(div) polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_\mathcal{F} \in \mathcal{P}_p(\Gamma_\mathcal{F})$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{RT}_p(K)\\ \nabla\cdot\boldsymbol{v}_p = r_K\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K)\\ \nabla\cdot\boldsymbol{v} = r_K\\ \boldsymbol{v}\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}\|_K.$$



Comments

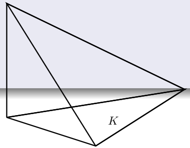
- C_{st} only depends on the shape-regularity of K
- p-robustness: for (pw) p-polynomial data r_K, $r_\mathcal{F}$, minimization over $\mathcal{RT}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{div}, K)$
- extension to a vertex patch: Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)

$\boldsymbol{H}$(div) polynomial extensions on a tetrahedron

Theorem ($\boldsymbol{H}$(div) polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_\mathcal{F} \in \mathcal{P}_p(\Gamma_\mathcal{F})$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{RT}_p(K) \\ \nabla\cdot\boldsymbol{v}_p = r_K \\ \boldsymbol{v}_p\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}_p\|_K \leq C_{\rm st} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K) \\ \nabla\cdot\boldsymbol{v} = r_K \\ \boldsymbol{v}\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}\|_K.$$



Comments

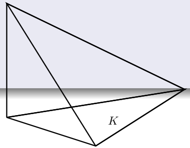
- $C_{\rm st}$ only depends on the shape-regularity of K
- p-robustness: for (pw) p-polynomial data r_K, $r_\mathcal{F}$, minimization over $\mathcal{RT}_p(K)$ is up to $C_{\rm st}$ as good as minimization over the entire $\boldsymbol{H}(\mathrm{div}, K)$
- extension to a vertex patch: Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)

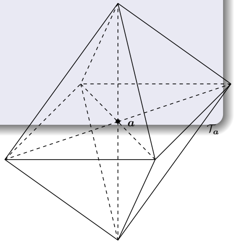
$\boldsymbol{H}(\mathrm{div})$ polynomial extensions on a tetrahedron and on patches

Theorem ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_{\mathcal{F}} \in \mathcal{P}_p(\Gamma_{\mathcal{F}})$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{RT}_p(K) \\ \nabla\cdot\boldsymbol{v}_p = r_K \\ \boldsymbol{v}_p\cdot\boldsymbol{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K) \\ \nabla\cdot\boldsymbol{v} = r_K \\ \boldsymbol{v}\cdot\boldsymbol{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\boldsymbol{v}\|_K.$$





Comments

- C_{st} only depends on the shape-regularity of K
- p-robustness: for (pw) p-polynomial data r_K, $r_{\mathcal{F}}$, minimization over $\mathcal{RT}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{div}, K)$
- extension to a vertex patch: Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)

Outline

$\boldsymbol{H}(\mathrm{div})$ stable decomposition

Theorem ($\boldsymbol{H}(\mathrm{div})$ stable decomposition in 2D; in extension of Schöberl, Melenk, Pechstein, & Zaglmayr (2008)**)**

Let $d = 2$ and let $\overline{\Omega}$ be contractible. Let

$$\boldsymbol{\delta}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_p = 0, \qquad \text{div-free}$$

$$(\boldsymbol{\delta}_p, \boldsymbol{r}_h)_K = 0 \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_h. \qquad \text{vanishing means}$$

Then there exists a decomposition of $\boldsymbol{\delta}_p$ as

$$\boldsymbol{\delta}_p = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\delta}_p^{\boldsymbol{a}}, \qquad \text{decomposition}$$

where

$\boldsymbol{\delta}_p^{\boldsymbol{a}}$ are supported on the vertex patch subdomains $\omega_{\boldsymbol{a}}$, linearly depend on $\boldsymbol{\delta}_p$ on the extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$,

and satisfy

$$\boldsymbol{\delta}_p^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_p^{\boldsymbol{a}} = 0, \qquad \text{local div-free}$$

$$\|\boldsymbol{\delta}_p^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\delta}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \qquad p\text{-stable}$$

Outline

Conclusions

Piecewise polynomial approximation in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$:

- a p-stable **local (commuting) projector**
- p-robust **global**-best–**local**-best **equivalence**
- elementwise *hp* **approximation** error **estimates** under **minimal elementwise Sobolev regularity**

Conclusions

Piecewise polynomial approximation in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$:

- a p-stable **local (commuting) projector**
- p-robust **global**-best–**local**-best **equivalence**
- elementwise *hp* **approximation** error **estimates** under **minimal elementwise Sobolev regularity**
- a posteriori tools in a priori analysis (equilibration)

Conclusions

Piecewise polynomial approximation in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$:

- a p-stable **local (commuting) projector**
- p-robust **global**-best–**local**-best **equivalence**
- elementwise *hp* **approximation** error **estimates** under **minimal elementwise Sobolev regularity**
- a posteriori tools in a priori analysis (equilibration)

Chaumont-Frelet T., Vohralík M. p-robust equilibrated flux reconstruction in $\boldsymbol{H}$(curl) based on local minimizations. Application to a posteriori analysis of the curl–curl problem. *SIAM Journal on Numerical Analysis* **61** (2023), 1783–1818.

Demkowicz L., Vohralík M. p-robust equivalence of global continuous constrained and local discontinuous unconstrained approximation, a p-stable local commuting projector, and optimal elementwise *hp* approximation estimates in $\boldsymbol{H}$(div). *Math. Comp.* (2026), DOI 10.1090/mcom/4199.

Vohralík M. p-robust equivalence of global continuous and local discontinuous approximation, a p-stable local projector, and optimal elementwise *hp* approximation estimates in H^1. HAL Preprint 04436063, 2024.

Conclusions

Piecewise polynomial approximation in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div},\Omega)$:

- a p-stable **local (commuting) projector**
- p-robust **global**-best–**local**-best **equivalence**
- elementwise *hp* **approximation** error **estimates** under **minimal elementwise Sobolev regularity**
- a posteriori tools in a priori analysis (equilibration)

Chaumont-Frelet T., Vohralík M. p-robust equilibrated flux reconstruction in $\boldsymbol{H}$(curl) based on local minimizations. Application to a posteriori analysis of the curl–curl problem. *SIAM Journal on Numerical Analysis* **61** (2023), 1783–1818.

Demkowicz L., Vohralík M. p-robust equivalence of global continuous constrained and local discontinuous unconstrained approximation, a p-stable local commuting projector, and optimal elementwise *hp* approximation estimates in $\boldsymbol{H}$(div). *Math. Comp.* (2026), DOI 10.1090/mcom/4199.

Vohralík M. p-robust equivalence of global continuous and local discontinuous approximation, a p-stable local projector, and optimal elementwise *hp* approximation estimates in H^1. HAL Preprint 04436063, 2024.

Thank you for your attention!