Error control and adaptivity in numerical simulations

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Outline

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Numerical approximations of PDEs:

Setting

- u: unknown exact PDE solution
- u_h : known numerical approximation on mesh $\mathcal{T}_h$

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Suggested answers

1. Computable **a posteriori** error **estimates**.

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Suggested answers

1. Computable **a posteriori** error **estimates**.
2. Identification of **error components**.
3. **Balancing** error components, **adaptivity** (working where needed).

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A posteriori error estimates: error control

Laplace equation in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $f \in L^2(\Omega)$

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Guaranteed error upper bound (reliability) ($u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, FEs)

$$\underbrace{\|\nabla(u - u_h)\|}_{\text{unknown error}} \qquad \underbrace{\eta(u_h)}_{\text{computable estimator}}$$

error lower bound (efficiency, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

$$\eta(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|$$

- C_{eff} a generic constant only dependent on shape regularity of $\mathcal{T}_h$ and thus independent of Ω, u, u_h, h, p

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$$\eta_K(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\omega_K} \qquad \forall K \in \mathcal{T}_h$$

- C_{eff} a generic constant only dependent on shape regularity of $\mathcal{T}_h$ and thus independent of Ω, u, u_h, h, p
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How **large** is the overall error? (model pb, known smooth solution)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\lVert\nabla u_h\rVert}$	$\lVert\nabla(u-u_h)\rVert$	rel. error $\frac{\lVert\nabla(u-u_h)\rVert}{\lVert\nabla u_h\rVert}$	$I^{\rm eff} = \frac{\eta(u_h)}{\lVert\nabla(u-u_h)\rVert}$
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$\approx h_0/8$	4	2.60×10^{-7}	5.9×10^{-6}%	2.58×10^{-7}	5.8×10^{-6}%	**1.01**

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How **large** is the overall error? (model pb, known smooth solution)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	$\|\nabla(u-u_h)\|$	rel. error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	$I^{\mathrm{eff}} = \frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	1.25	28%	1.07	24%	**1.17**
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	**1.09**
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	**1.06**
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	**1.04**
$\approx h_0/2$	2	4.23×10^{-2}	9.5×10^{-1}%	4.07×10^{-2}	9.2×10^{-1}%	**1.04**
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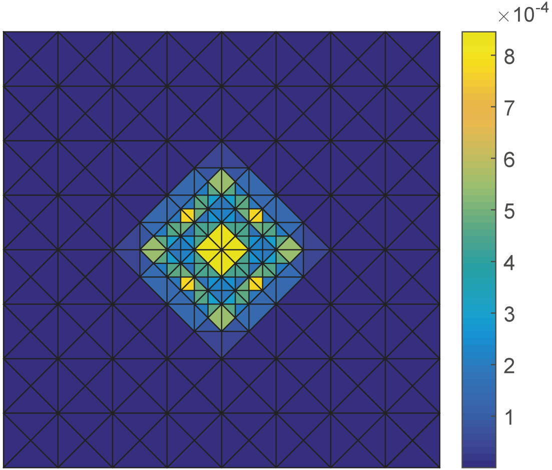
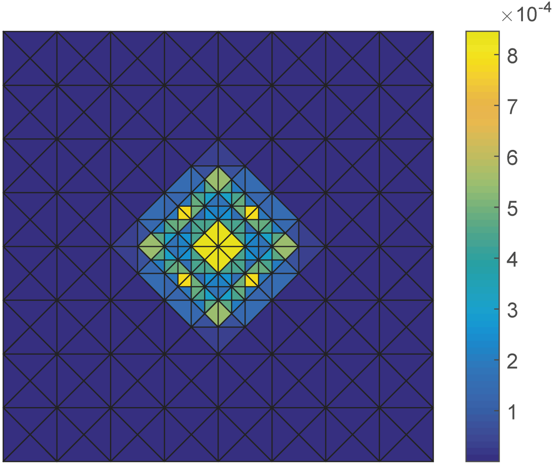
A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

Where (in space) is the error **localized**? (known smooth solution)



Estimated error distribution $\eta_K(u_h)$



Exact error distribution $\|\nabla(u - u_h)\|_K$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Error characterization

Theorem (Error characterization)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\|\nabla(u-u_h)\|^2 = \underbrace{\min_{\substack{\boldsymbol{\sigma}\in \boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{\sigma}=f}} \|\nabla u_h + \boldsymbol{\sigma}\|^2}_{=\max_{\substack{\varphi\in H_0^1(\Omega)\\ \|\nabla\varphi\|=1}}[(f,\varphi)-(\nabla u_h,\nabla\varphi)]^2} + \min_{s\in H_0^1(\Omega)} \|\nabla(u_h - s)\|^2.$$

Comments

- It is enough to choose suitable (discrete, piecewise polynomial) $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{\sigma}_h = f$ and $s_h \in H_0^1(\Omega)$ to get a guaranteed upper bound.
- Local construction of $\boldsymbol{\sigma}_h$ and s_h?

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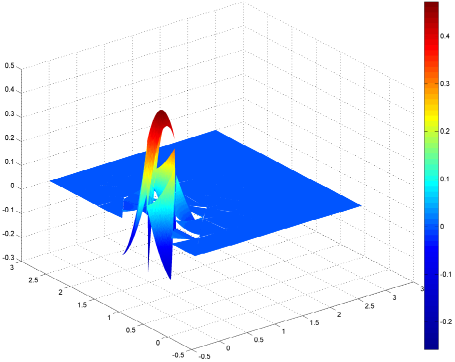
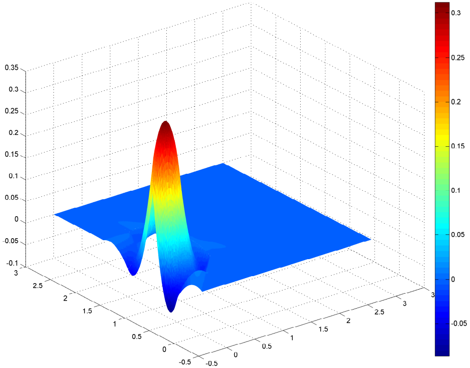
Outline

Potential reconstruction



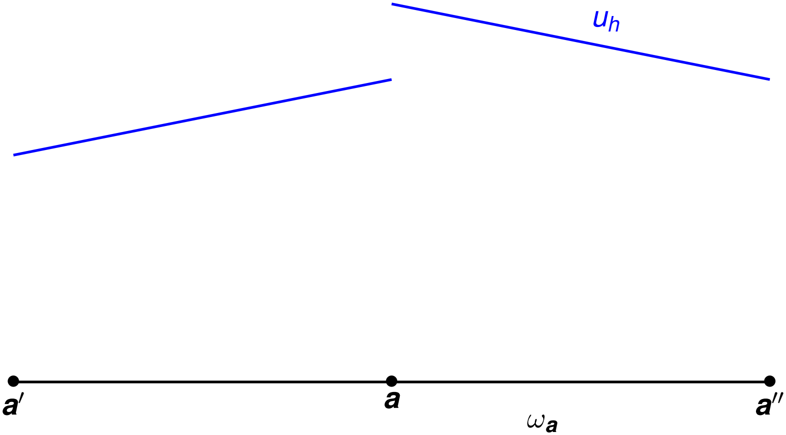
Potential u_h



Potential reconstruction s_h

$$u_h \in \mathcal{P}_p(\mathcal{T}_h) \rightarrow s_h \in \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_0^1(\Omega)$$

Potential reconstruction in 1D, $p = 1$



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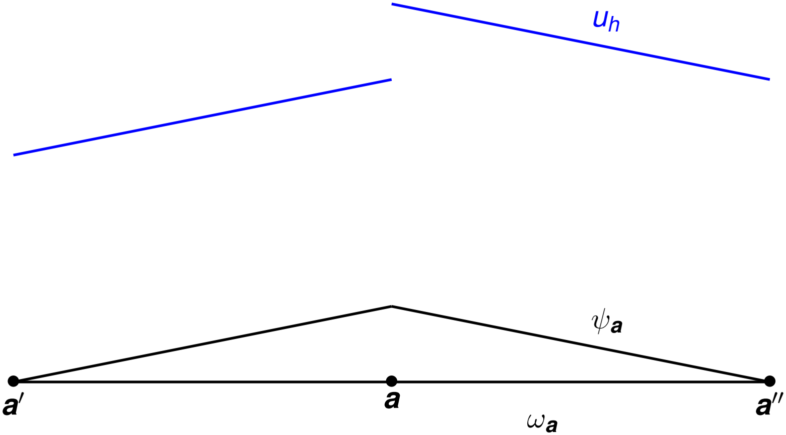

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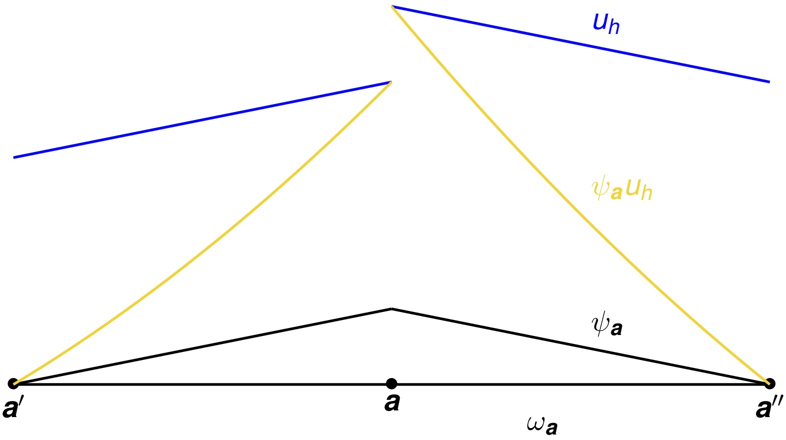

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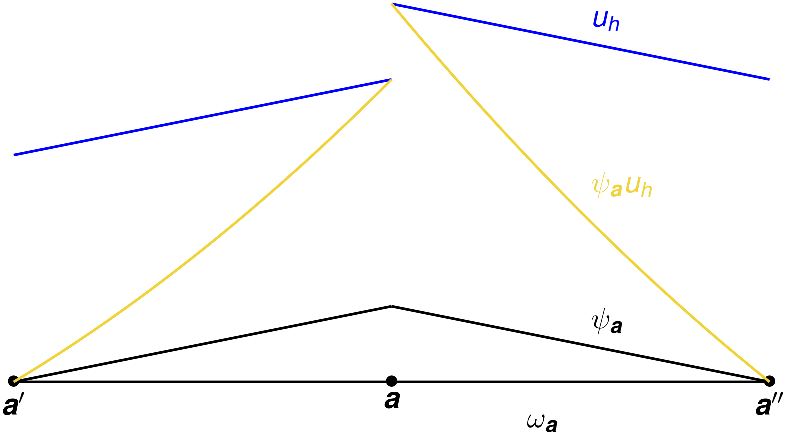

Potential reconstruction: datum $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathcal{P}_{p+1}(\mathcal{T}^{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla(\psi_{\boldsymbol{a}} u_h - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

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Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla(\psi_{\boldsymbol{a}} u_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

Key points

- localization to patches $\mathcal{T}^{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}} u_h$ to a conforming space
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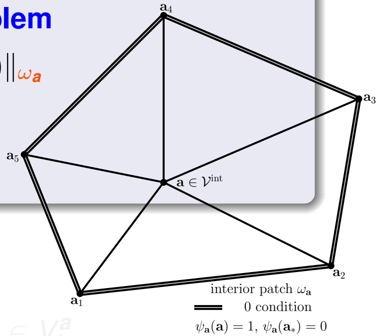

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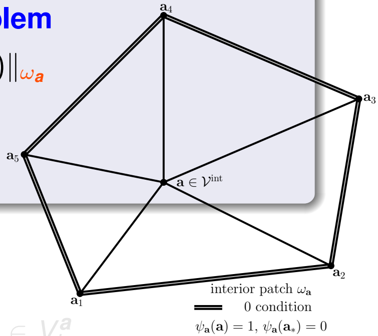

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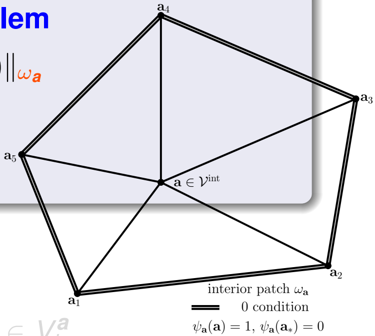

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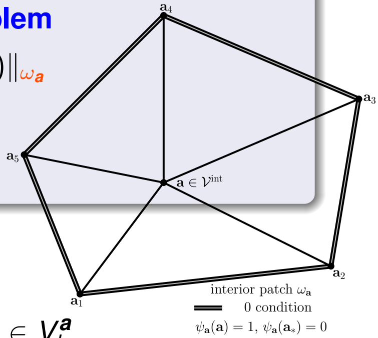

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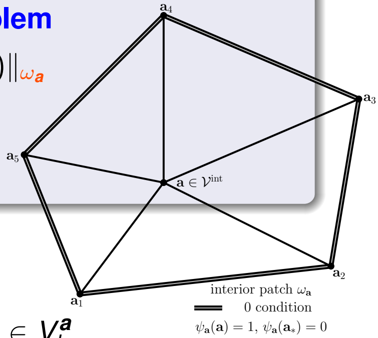

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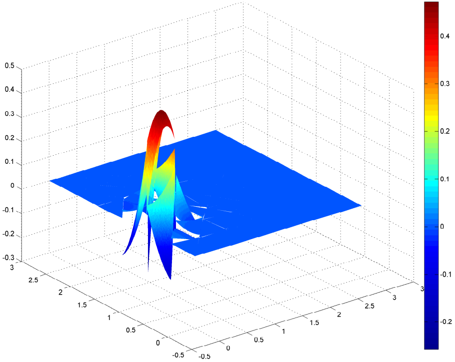
Potential reconstruction



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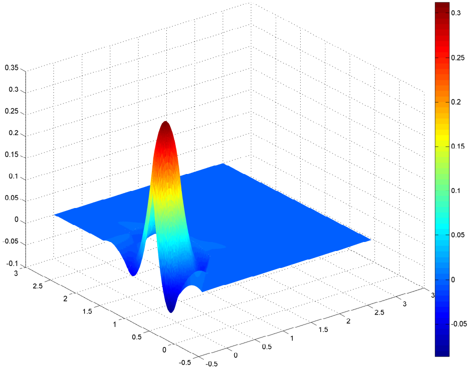
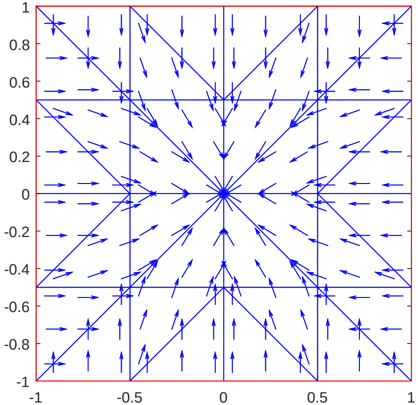


Potential reconstruction s_h

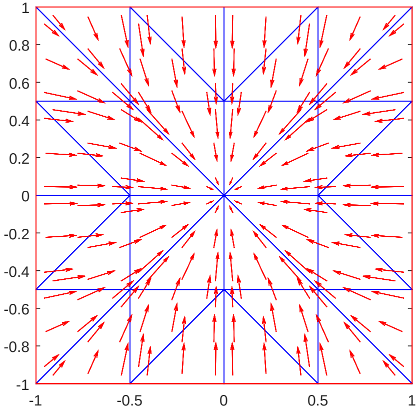
$$u_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_0^1(\Omega)$$

Outline

Equilibrated flux reconstruction



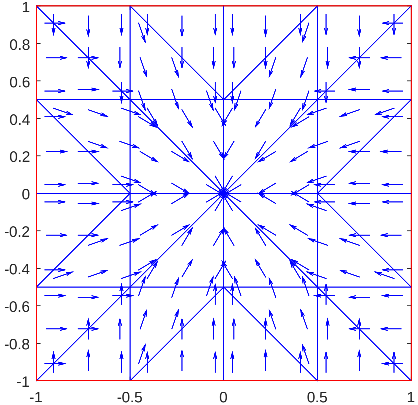
Flux $-\nabla u_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{-\nabla u_h \in \mathcal{RT}_{p-1}(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega), \nabla \cdot \boldsymbol{\sigma}_h = \Pi_p f$$

Equilibrated flux reconstruction



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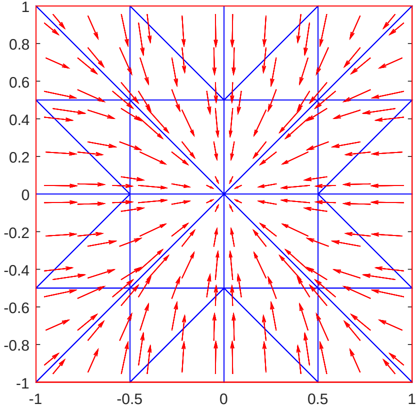

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Flux reconstruction: $-\nabla u_h \in \mathcal{RT}_{p-1}(\mathcal{T}_h)$, $p \geq 1$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h^{\text{int}}.$$

Definition (Construction of $\boldsymbol{\sigma}_h$, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}} = \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\text{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_p(f\psi_{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi_{\boldsymbol{a}})}} \|\psi_{\boldsymbol{a}} \nabla u_h + \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}.$$

Key points

- homogeneous Neumann BC on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\text{div}, \Omega)$
- **equilibrium** $\nabla \cdot \boldsymbol{\sigma}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \Pi_p(f\psi_{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi_{\boldsymbol{a}}) = \Pi_p f$

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Key points

- homogeneous Neumann BC on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\text{div}, \Omega)$
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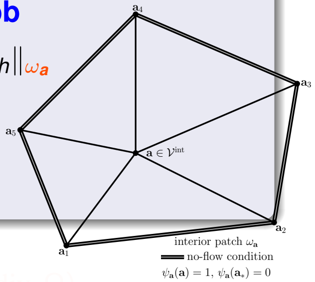

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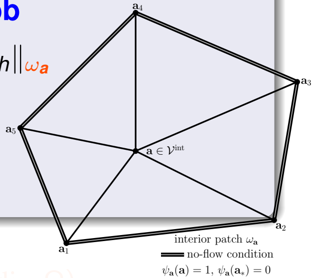

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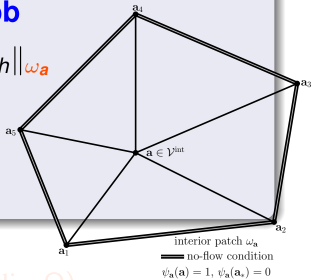

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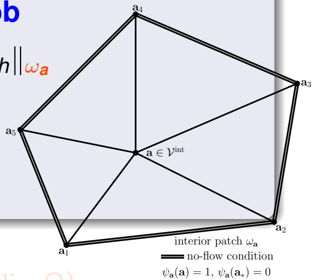

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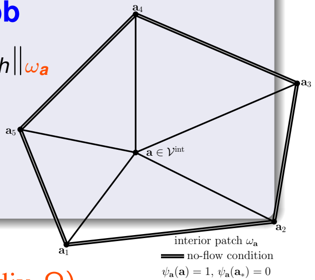

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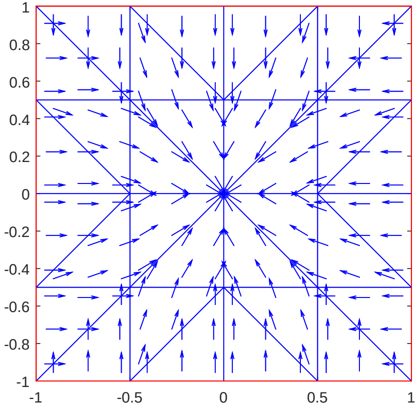
Equilibrated flux reconstruction



Flux $-\nabla u_h$

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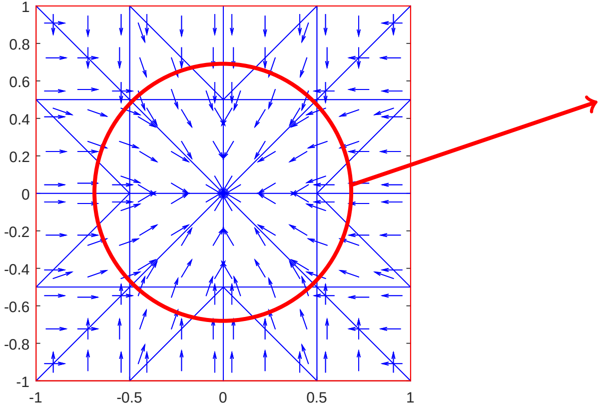

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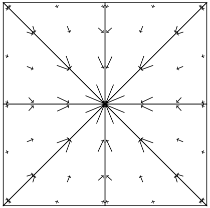

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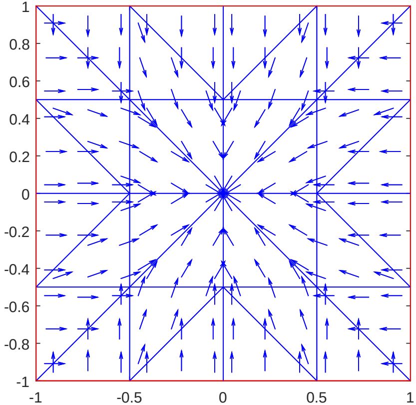

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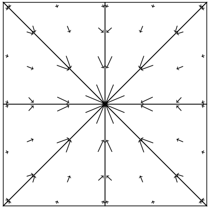

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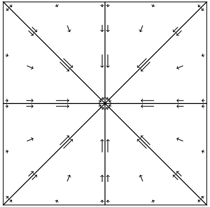

$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

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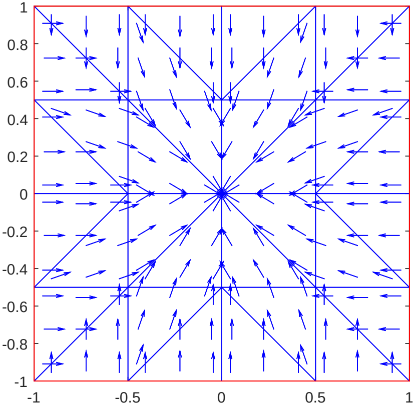

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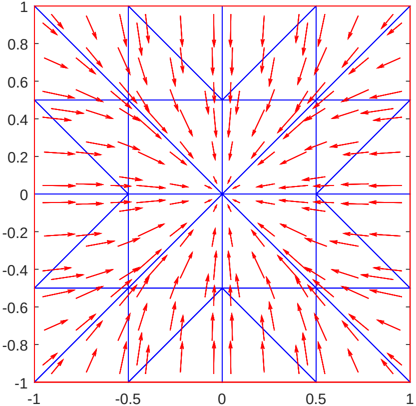

Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

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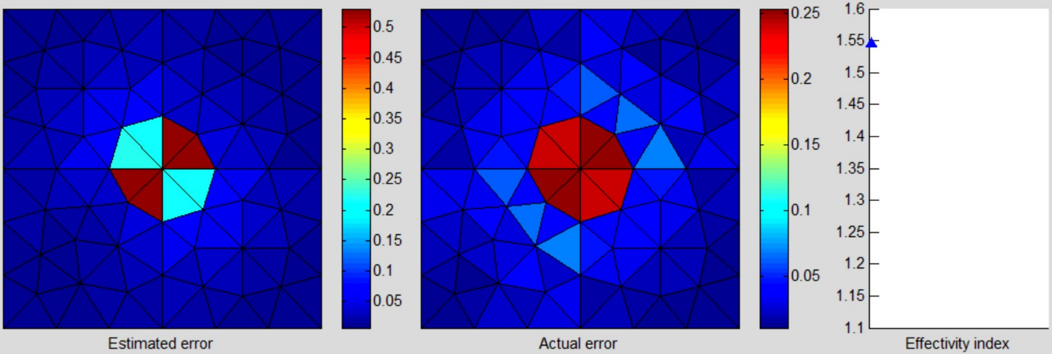
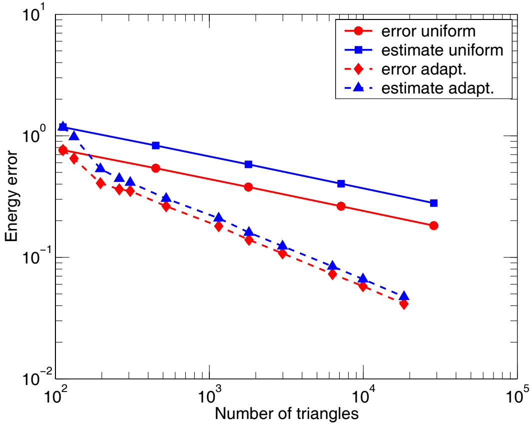
Outline

Can we **decrease the error efficiently**? (adaptive mesh refinement)

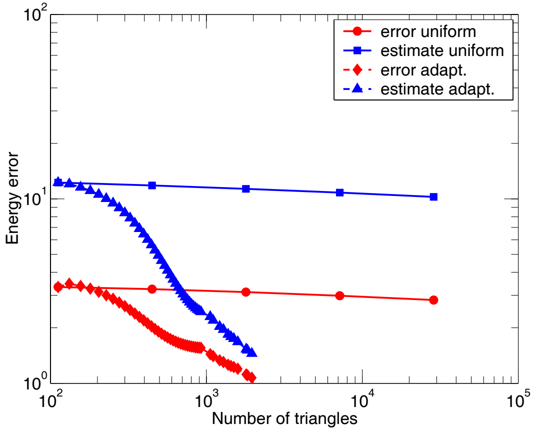


M. Vohralík, SIAM Journal on Numerical Analysis (2007)

Estimated and actual error against the number of elements in uniformly/adaptively refined meshes (singular solutions)



$H^{1.54}$ singularity



$H^{1.13}$ singularity

M. Vohralík, SIAM Journal on Numerical Analysis (2007)

Adaptive mesh refinement

Adaptive mesh refinement $\hookrightarrow$ Talk by Dirk Praetorius

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$$\sum_{K\in\mathcal{T}_\ell} \eta_K(u_\ell)^2 = \quad \eta(u_\ell)^2$$

Adaptive mesh refinement

Adaptive mesh refinement $\hookrightarrow$ Talk by Dirk Praetorius

- Dörfler marking: subset $\mathcal{M}_\ell$ containing θ-fraction of the estimates

$$\sum_{K \in \mathcal{M}_\ell} \eta_K(u_\ell)^2 \geq \theta^2 \sum_{K \in \mathcal{T}_\ell} \eta_K(u_\ell)^2 = \theta^2 \eta(u_\ell)^2$$

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Convergence on a sequence of adaptively refined meshes

- $$\|\nabla(u-u_\ell)\| \to 0$$
- some mesh elements may not be refined at all: $\not{h} \searrow \not{0}$
- Babuška & Miller (1987), Dörfler (1996)

Adaptive mesh refinement

Adaptive mesh refinement ↪ Talk by Dirk Praetorius

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Optimal error decay rate wrt degrees of freedom

- $$\|\nabla(u-u_\ell)\| \lesssim |\mathrm{DoF}_\ell|^{-p/d} \qquad \text{(replaces } h^p\text{)}$$
- same for smooth & singular solutions: ~~higher-order only pay-off for sm. sol.~~
- decays to zero as fast as on a best-possible sequence of meshes
- Morin, Nochetto, Siebert (2000), Stevenson (2005, 2007), Cascón, Kreuzer, Nochetto, Siebert (2008), Canuto, Nochetto, Stevenson, Verani (2017)

Outline

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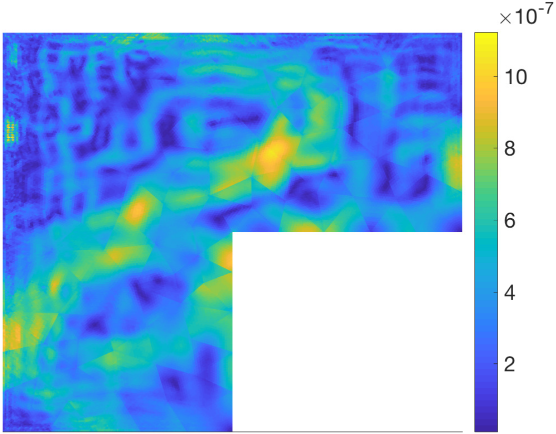
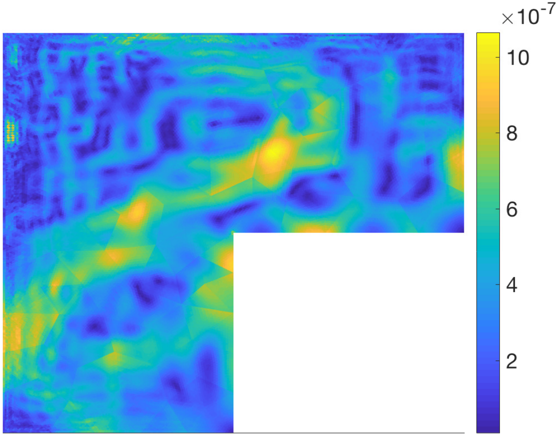
Including **algebraic error**: $\mathbb{A}_\ell \mathsf{U}_\ell^i \neq \mathsf{F}_\ell$



Estimated algebraic errors $\eta_{\mathrm{alg},K}(u_\ell^i)$



Exact algebraic errors $\|\nabla(u_\ell - u_\ell^i)\|_K$

J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

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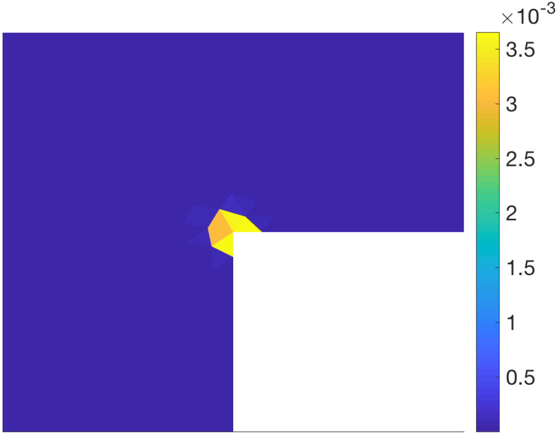
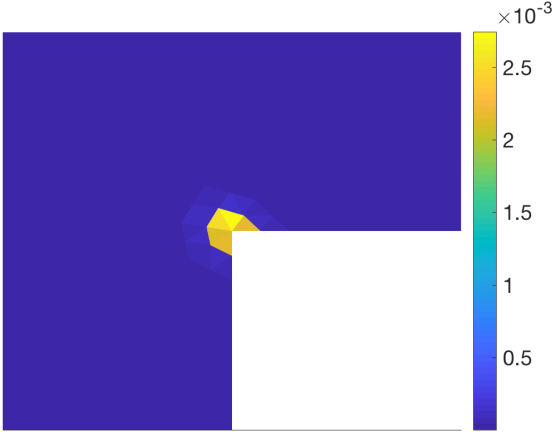


Estimated total errors $\eta_K(u_\ell^i)$



Exact total errors $\|\nabla(u - u_\ell^i)\|_K$

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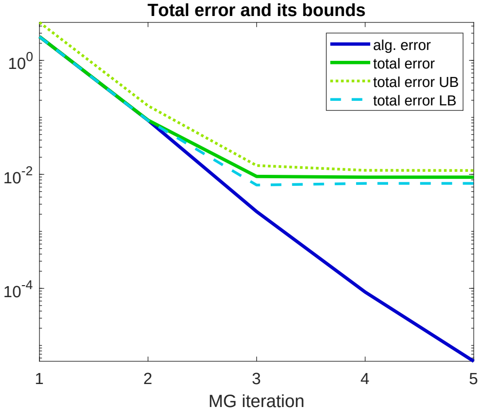
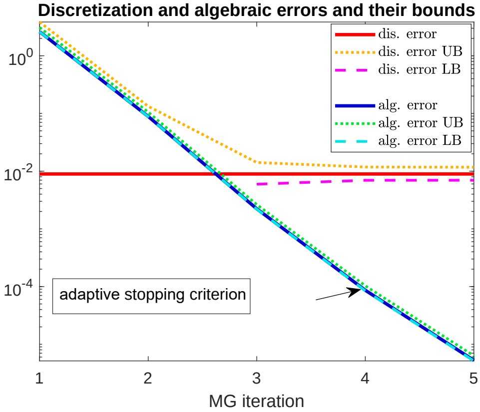


Total error



Error components and adaptive st. crit.

J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

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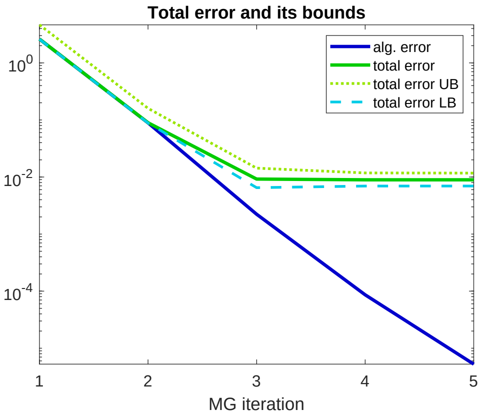

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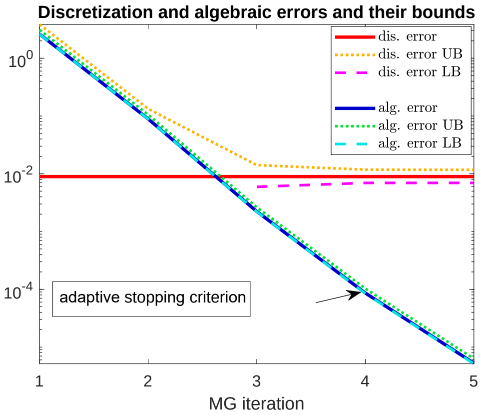

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J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

$\hookrightarrow$ Talk by Ani Miraçi

Nonlinear pb $-\nabla\cdot\sigma(\nabla u) = f$: including **linearization** and **algebraic error**: $\mathcal{A}_\ell(\mathrm{U}_\ell^{k,i}) \neq \mathrm{F}_\ell$, $\mathbb{A}_\ell^{k-1}\mathrm{U}_\ell^{k,i} \neq \mathrm{F}_\ell^{k-1}$

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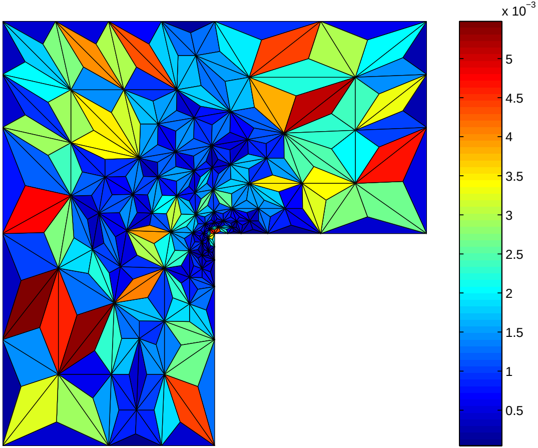
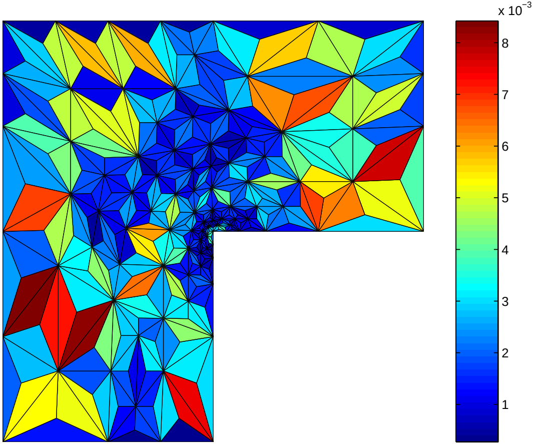


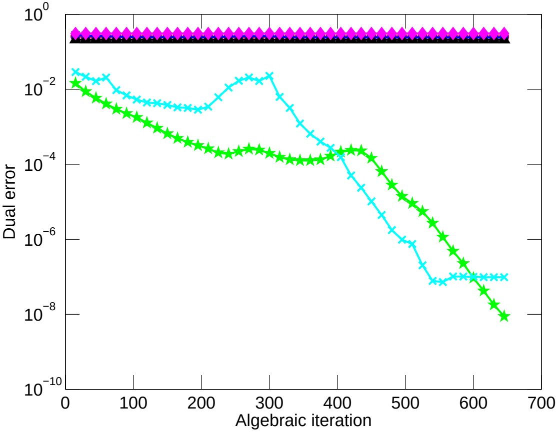
Estimated errors $\eta_K(u_\ell^{k,i})$

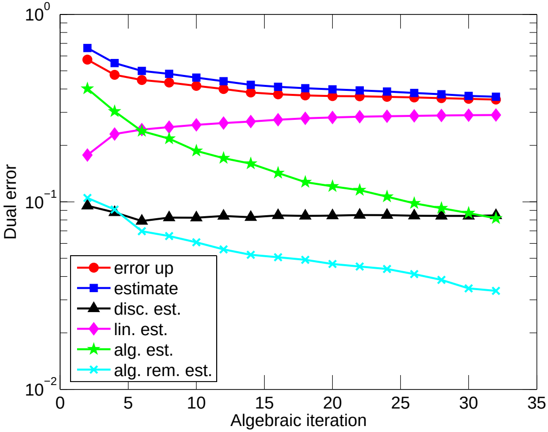


Exact errors $\|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_\ell^{k,i})\|_{q,K}$

A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2013)

Nonlinear pb $-\nabla\cdot\sigma(\nabla u) = f$: including **linearization** and **algebraic error**: $\mathcal{A}_\ell(\mathrm{U}_\ell^{k,i}) \neq \mathrm{F}_\ell$, $\mathbb{A}_\ell^{k-1}\mathrm{U}_\ell^{k,i} \neq \mathrm{F}_\ell^{k-1}$





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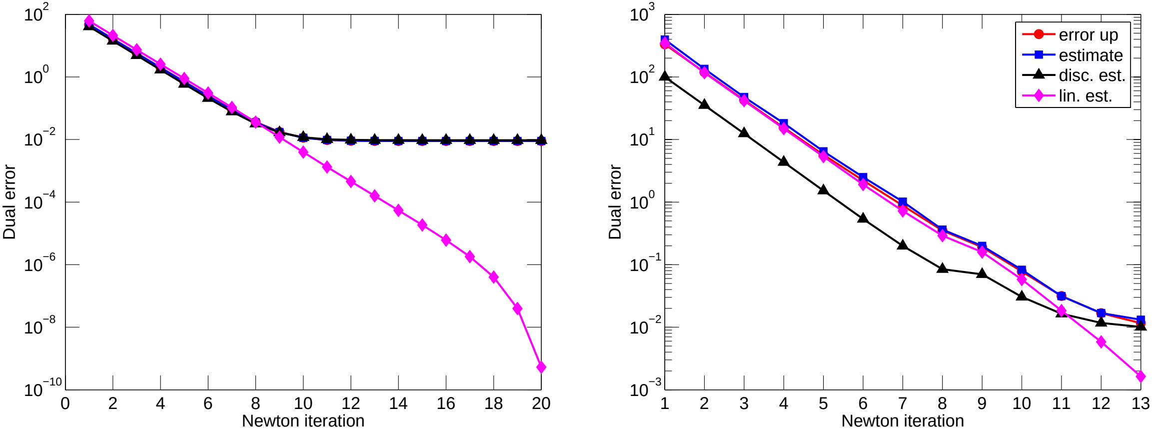

Outline

Solver adaptivity (nonlinear problem, inexact solvers)

Fully adaptive algorithm (adaptive inexact Newton method)

- **total error estimate** on mesh $\mathcal{T}_\ell$, linearization step k, algebraic solver step i

$$\underbrace{\|u - u_\ell^{k,i}\|_*}_{\text{total error}} \le \underbrace{\eta_{\ell,\mathrm{disc}}^{k,i}}_{\text{discretization estimate}} + \underbrace{\eta_{\ell,\mathrm{lin}}^{k,i}}_{\text{linearization estimate}} + \underbrace{\eta_{\ell,\mathrm{alg}}^{k,i}}_{\text{algebraic estimate}}$$

- **balancing error components**: work where needed

$\eta_{\ell,\mathrm{alg}}^{k,i} \le \gamma_{\mathrm{alg}} \eta_{\ell,\mathrm{lin}}^{k,i}$	stopping criterion linear solver
$\eta_{\ell,\mathrm{lin}}^{k,i} \le \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$	stopping criterion nonlinear solver
$\theta \eta_{\ell,\mathrm{disc}}^{k,i} \le \eta_{\mathrm{disc},\mathcal{M}_\ell}^{k,i}$	adaptive mesh refinement

- link – inexact Newton method: Bank & Rose (1982), Hackbusch & Reusken (1989), Deuflhard (1991), Eisenstat & Walker (1994)

Convergence, optimal **error decay rate** wrt **DoFs**

- Gantner, Haberl, Praetorius, & Stiftner (2018), Heid & Wihler (2019)

Optimal error decay rate wrt overall computational cost

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Outline

The reaction–diffusion equation: $f \in L^2(\Omega)$, $\varepsilon > 0$, $\kappa \geq 0$ parameters

Find $u : \Omega \to \mathbb{R}$ such that ($\varepsilon \ll \kappa$ **singular perturbation**)

$$
\begin{aligned}
-\varepsilon^2 \Delta u + \kappa^2 u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Guaranteed error upper bound (reliability) ($u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, FEs)

$$\underbrace{|||u - u_h|||}_{\text{unknown error}} \qquad \underbrace{\eta(u_h)}_{\text{computable estimator}}$$

error lower bound (efficiency, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

$$\eta\ (u_h) \leq C_{\text{eff}} |||u - u_h|||$$

- C_{eff} a generic constant independent of Ω, u, u_h, h,

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Equilibrated flux and potential reconstructions

Definition (Flux σ_h and potential ϕ_h)

For each vertex $\boldsymbol{a} \in \mathcal{V}$, let

$$(\sigma_h^{\boldsymbol{a}}, \phi_h^{\boldsymbol{a}}) := \arg \min_{(\boldsymbol{v}_h, q_h) \in \mathcal{RT}_p(\mathcal{T}^{\boldsymbol{a}}) \times \mathcal{P}_p(\mathcal{T}^{\boldsymbol{a}}) \subset \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \times L^2(\omega_{\boldsymbol{a}})} J^{\boldsymbol{a}}_{u_h}(\boldsymbol{v}_h, q_h)$$

$$J^{\boldsymbol{a}}_{u_h}(\boldsymbol{v}_h, q_h) := w_{\boldsymbol{a}}^2 \|\varepsilon \psi_{\boldsymbol{a}} \nabla u_h + \varepsilon^{-1} \boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}} + \|\kappa [\Pi_h(\psi_{\boldsymbol{a}} u_h) - q_h]\|^2_{\omega_{\boldsymbol{a}}}$$

Comments

- **local discrete** constrained minimization problems
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with the weight $w_{\boldsymbol{a}} := \min\left\{1, C_* \sqrt{\frac{\varepsilon}{\kappa h_{\omega_{\boldsymbol{a}}}}}\right\}$. Combine

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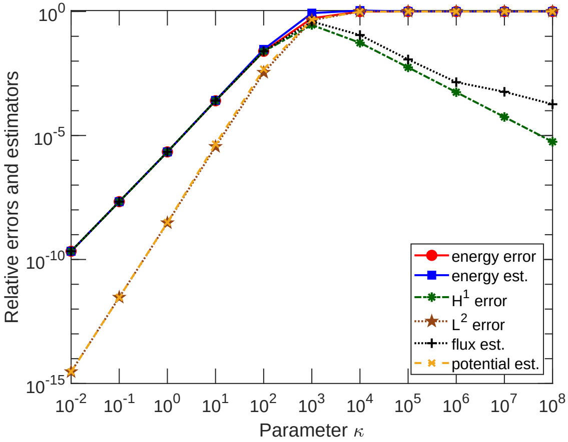
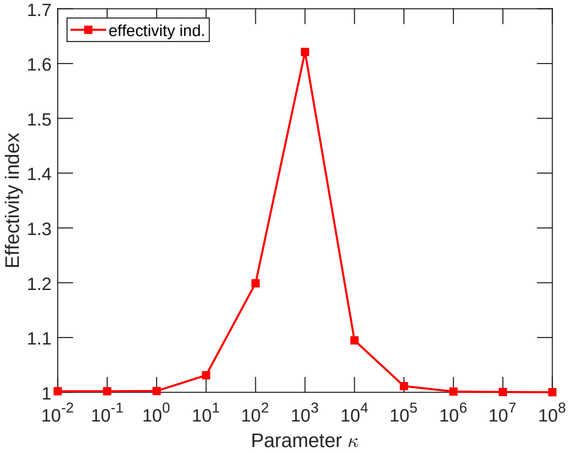
Boundary layer, solution $u(x,y) = e^{-\frac{\kappa}{\varepsilon}x} + e^{-\frac{\kappa}{\varepsilon}y}$, $p = 2$



Relative energy errors and estimates



Effectivity indices $\eta(u_h)/|||u - u_h|||$

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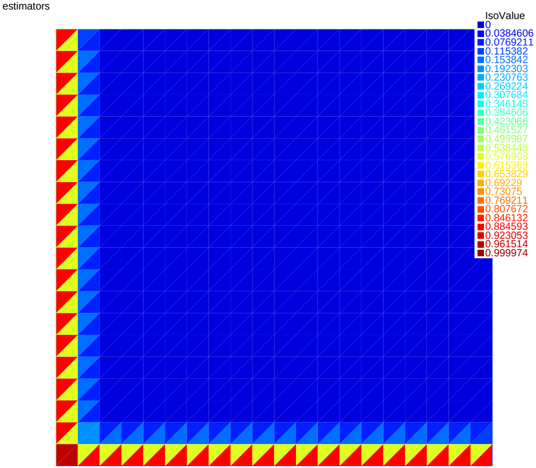
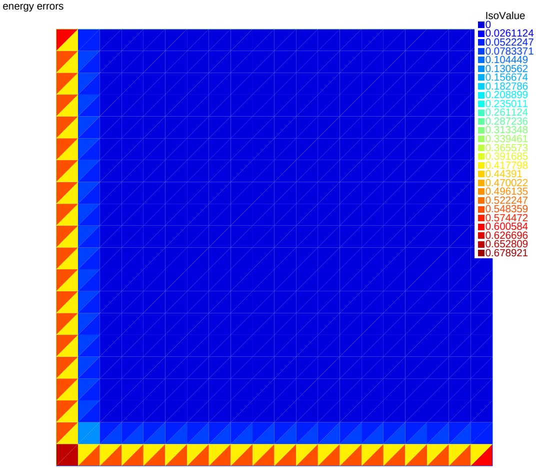


Estimated error distribution $\eta_K(u_h)$



Exact error distribution $|||u - u_h|||_K$

Outline

The heat equation ($f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$)

The heat equation

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Y norm error is the dual X norm of the residual + initial condition error

$$\|u - u_{h\tau}\|_Y^2 = \sup_{v \in X, \|v\|_X = 1} \left[\int_0^T (f,v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v) \,\mathrm{d}t \right]^2 + \|u_0 - u_{h\tau}(0)\|^2$$

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Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_{h\tau}^{\mathbf{a},n} \\ \nabla\cdot\mathbf{v}_h = \psi_{\mathbf{a}}(f - \partial_t \mathcal{I} u_{h\tau}) - \nabla\psi_{\mathbf{a}}\cdot\nabla u_{h\tau}}} \int_{I_n} \|\mathbf{v}_h + \psi_{\mathbf{a}} \nabla u_{h\tau}\|_{\omega_{\mathbf{a}}}^2 \,\mathrm{d}t.$$

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Outline

Eigenvalue problems

$\hookrightarrow$ Talks by Tomáš Vejchodský and Philip Lederer

Outline

Conclusions

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- a posteriori **error control**
- **full adaptivity**: space mesh, time step, linear solver, nonlinear solver, polynomial degree
- recovering **mass balance** in any situation

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Thank you for your attention!

Outline

- Motivation
- Polynomial-degree (p) adaptivity

CDG Terminal 2E collapse in 2004 (opened in 2003)

- no earthquake, flooding, tsunami, heavy rain, extreme temperature
- deterministic, steady problem, PDE known, data known, implementation OK

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CDG Terminal 2E collapse in 2004 (opened in 2003)

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probably **numerical simulations done with insufficient precision**,
I believe **without error control**

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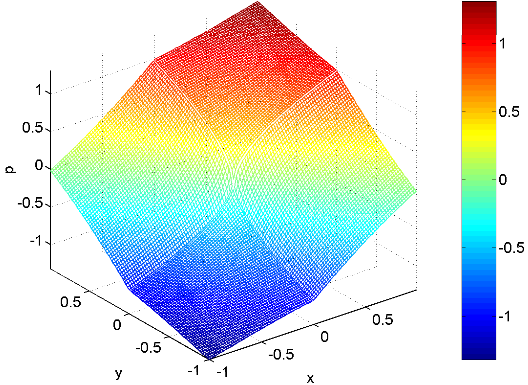
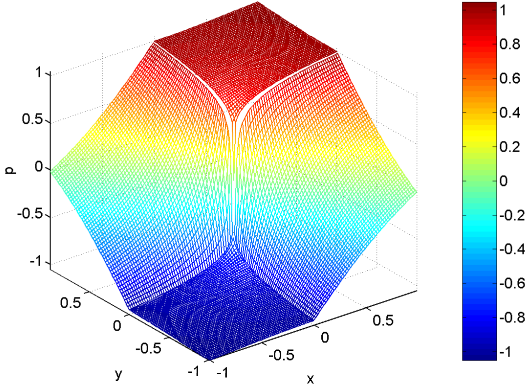
Singular solutions



$H^{1.54}$ singularity

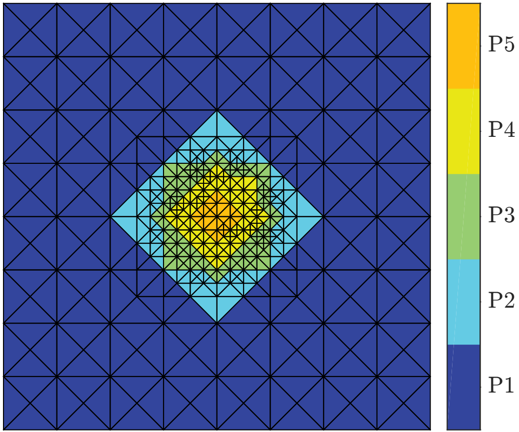


$H^{1.13}$ singularity

Outline

- Motivation
- Polynomial-degree (p) adaptivity

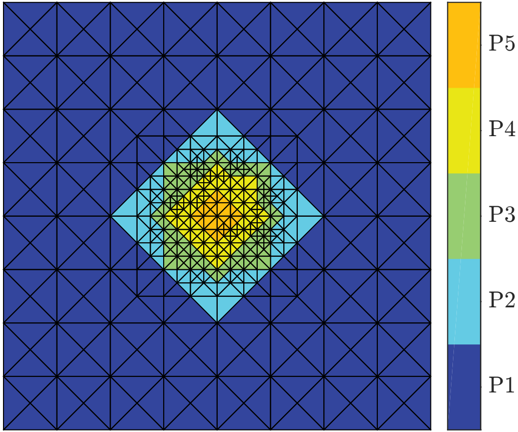
Best-possible error decrease: *hp* **adaptivity**, (**smooth** solution)



Mesh $\mathcal{T}_\ell$ and pol. degrees p_K

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (20[illegible]

Best-possible error decrease: *hp* **adaptivity**, (**smooth** solution)



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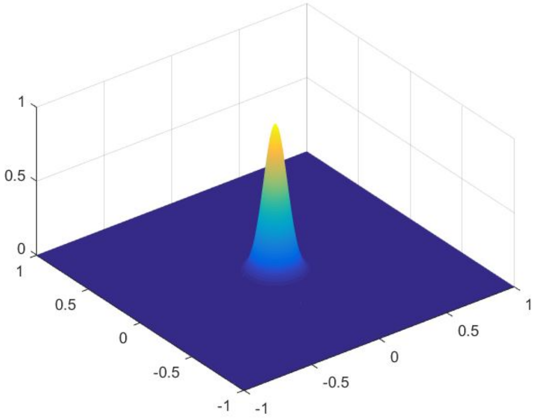
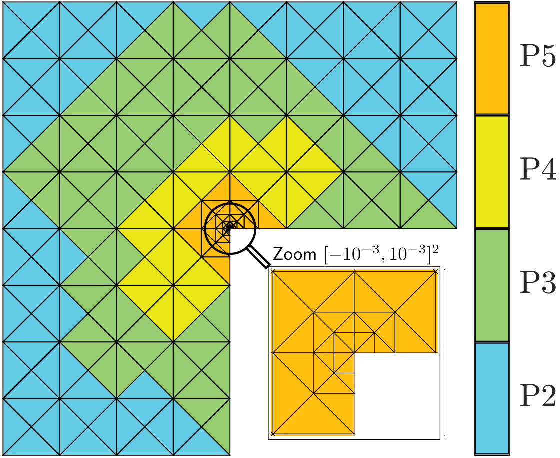


Exact solution

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

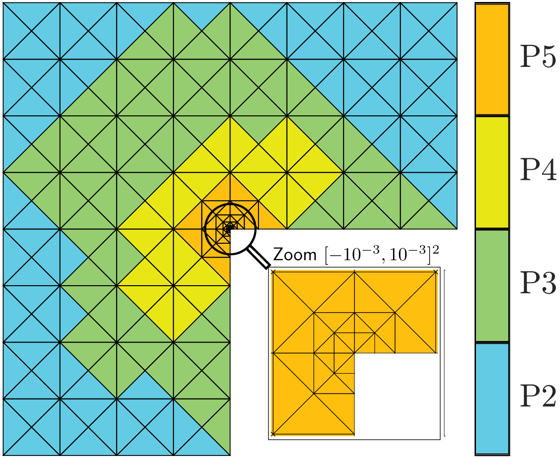
Best-possible error decrease: *hp* **adaptivity**, (**singular** solution)



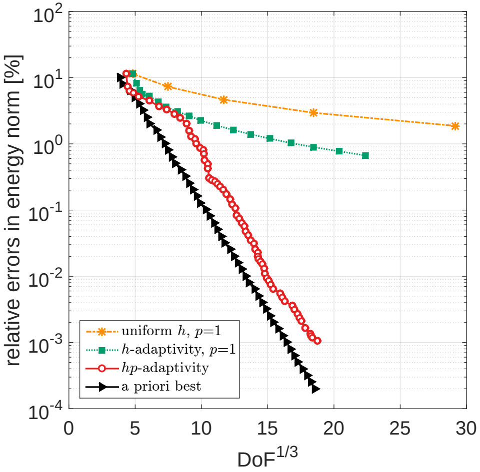
Mesh $\mathcal{T}_\ell$ and polynomial degrees p_K

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (20

Best-possible error decrease: *hp* **adaptivity**, (**singular** solution)



Mesh $\mathcal{T}_\ell$ and polynomial degrees p_K



Relative error as a function of DoF

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)