A posteriori error estimates with inexact solvers and recovering mass balance in any situation

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Inria Paris & Ecole des Ponts

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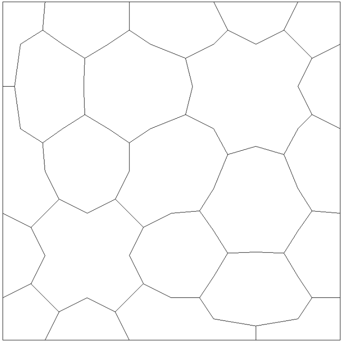
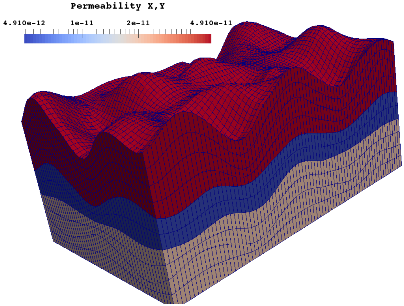
Outline

Context

General polygonal/polyhedral meshes, arbitrary locally conservative scheme





Multi-phase, multi-compositional porous media flows

- **unsteady nonlinear** degenerate **systems** of PDEs
- algebraic constraints

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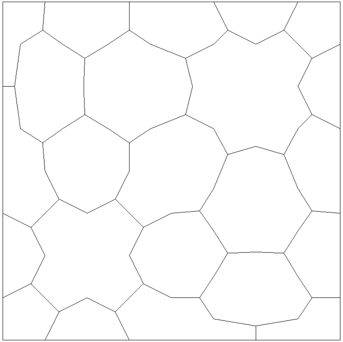

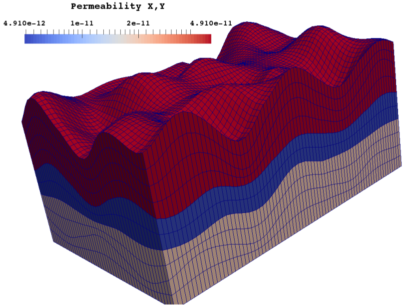

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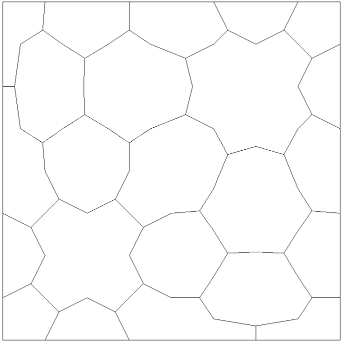

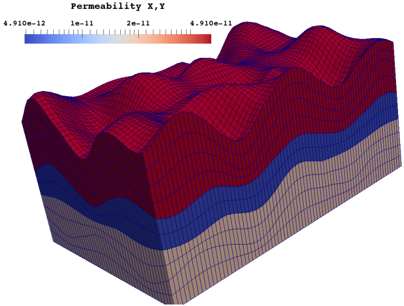

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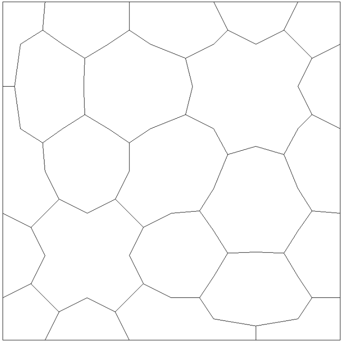

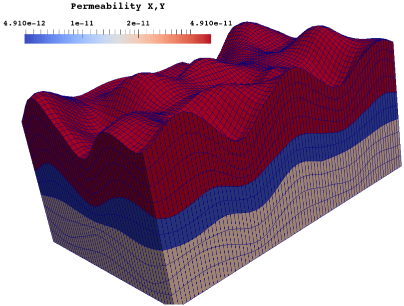

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- **unsteady nonlinear** degenerate **systems** of PDEs
- algebraic constraints, phase appearance/disappearance

Example: steady nonlinear Darcy flow $\nabla\cdot(-\underline{\mathbf{K}}(\nabla p)\nabla p) = f$

Discretization: system of nonlinear algebraic eqs

Find $\mathsf{P} \in \mathbb{R}^N$ such that

$$\underbrace{\mathcal{U}}_{\text{nonlin. op.}}(\mathsf{P}) = \mathsf{F}$$

Linearization: system of linear algebraic eqs

Find $\mathsf{P}^k \in \mathbb{R}^N$ such that

$$\underbrace{\mathbb{U}^{k-1}}_{\text{matrix}}\mathsf{P}^k = \mathsf{F}^{k-1}$$

Algebraic solver:

On step i, one has $\mathsf{P}^{k,i} \in \mathbb{R}^N$ such that

$$\mathbb{U}^{k-1}\mathsf{P}^{k,i} = \mathsf{F}^{k-1} - \underbrace{\mathsf{R}^{k,i}}_{\text{algebraic residual vector}}$$

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- *linearization stopping crit.:* $\|\mathsf{P}^k - \mathsf{P}^{k-1}\|_\infty$ *small*
- *algebraic stopping crit.:* $\|\mathsf{R}^{k,i}\|_2/\|\mathsf{R}^{k,0}\|_2$ *small*
- **comparing apples and oranges**, *not comparing in the right norm*

- *locally conservative scheme*

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- guaranteed **a posteriori** error **estimate**

$$\left\| \mathbf{u}|_{I_n} - \mathbf{u}_h^{n,k,i} \right\| \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}$$

- valid at each step: time n, linearization k, linear solver i
- distinguishing different error components, all estimators with the same (flux) physical units
- easy to code, fast to evaluate, cosy to use in practice
- full adaptivity (stopping criteria for linear and nonlinear solvers, mesh refinement, time step adjustment)

Construction of the estimates interconnected with **recovering mass balance** at each step n, k, i

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Linear Darcy flow

Steady linear Darcy flow

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla p) = f \qquad \text{in } \Omega,$$
$$p = 0 \qquad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d \geq 1$, polytope
- $f \in L^2(\Omega)$ source term, pw constant for simplicity
- $\underline{\mathbf{K}} \in [L^\infty(\Omega)]^{d\times d}$ symmetric elliptic diffusion-dispersion tensor (pw constant)

Unknowns

- p pressure head
- $\mathbf{u} := -\underline{\mathbf{K}}\nabla p$ Darcy velocity (flux)

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Locally conservative discretization

Assumption A (Locally conservative discretization)

1. *There is one pressure* $(\mathrm{P})_K \in \mathbb{R}$ *per element* $K \in \mathcal{T}_H$ *and one face normal flux* $(\mathrm{U})_\sigma \in \mathbb{R}$ *per face* $\sigma \in \mathcal{E}_H$.
2. *The flux balance is satisfied, with* $(\mathrm{F})_K := (f, 1)_K$*:*

$$\sum_{\sigma \in \mathcal{E}_K} (\mathrm{U})_\sigma \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_\sigma = (\mathrm{F})_K \quad \forall K \in \mathcal{T}_H.$$

- $(\mathrm{U})_\sigma \approx \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_\sigma = \int_\sigma \mathbf{u} \cdot \mathbf{n}$
- **any** (lowest-order) **locally conservative method**
- the link between $(\mathrm{U})_\sigma$ and $(\mathrm{P})_K$ is not important for the a posteriori error estimate upper bound

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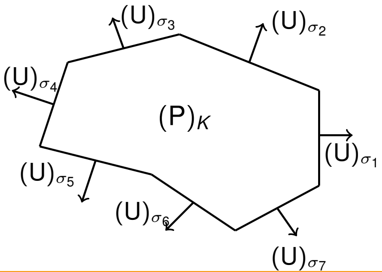

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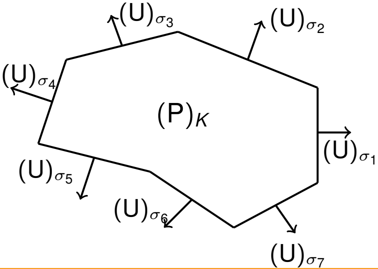

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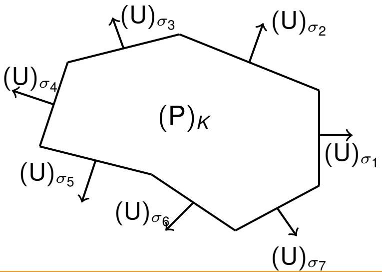

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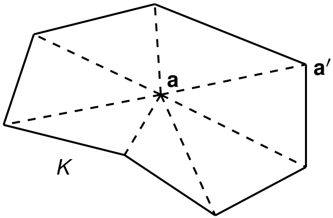
Three element **matrices** easily computable from the geometry of K



1. finite element stiffness matrix

$$(\mathbb{S}_K)_{\mathbf{a},\mathbf{a}'} := (\underline{\mathbf{K}}\nabla\psi_{\mathbf{a}'}, \nabla\psi_{\mathbf{a}})_K \qquad \mathbf{a}, \mathbf{a}' \in \mathcal{V}_{K,h}$$

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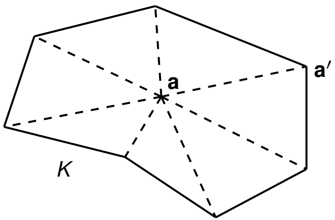

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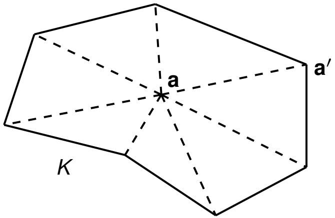

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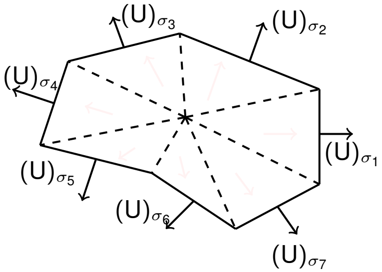

3. mixed finite element local static condensation matrix

$$\Leftrightarrow \quad \mathbf{u}_h|_K := \arg \min_{\mathbf{v}_h;\, \langle \mathbf{v}_h \cdot \mathbf{n}, 1\rangle_\sigma = (\mathrm{U})_\sigma\ \nabla\cdot\mathbf{v}_h = \text{constant}} \left\| \underline{\mathbf{K}}^{-\frac{1}{2}} \mathbf{v}_h \right\|_K$$

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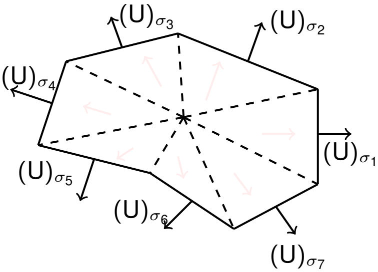
Three **element matrices** easily computable from the geometry of K



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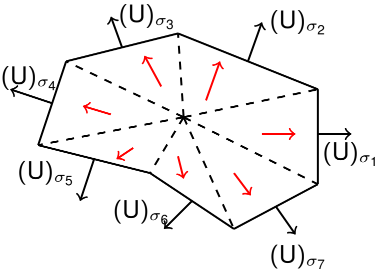

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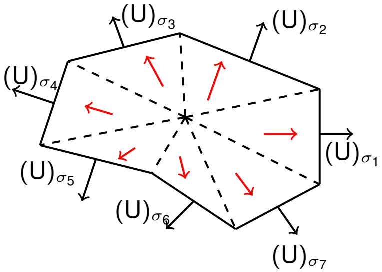

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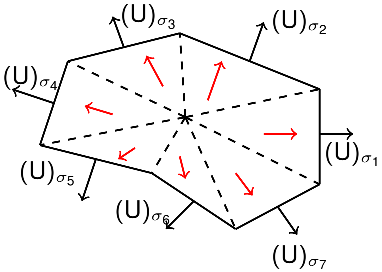

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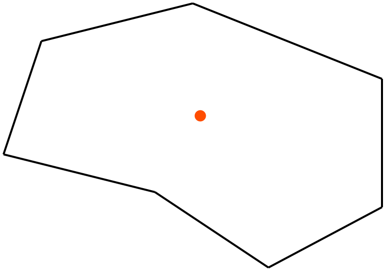
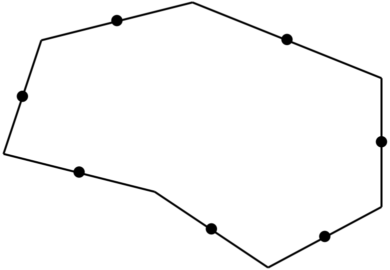
Element pressure and flux **vectors**



- cell pressure $(P)_K$

Element pressure and flux vectors



$\mathsf{S}_K^{\mathrm{ext}}$

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- $\mathsf{S}_K^{\mathrm{ext}}$: face pressures, local averages of neighbor $(\mathsf{P})_{K'}$

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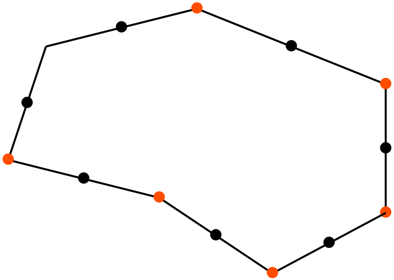

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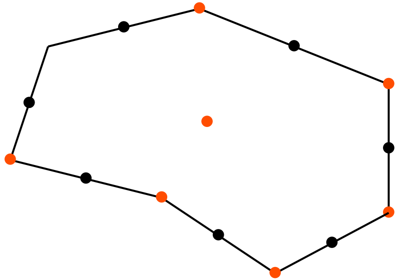

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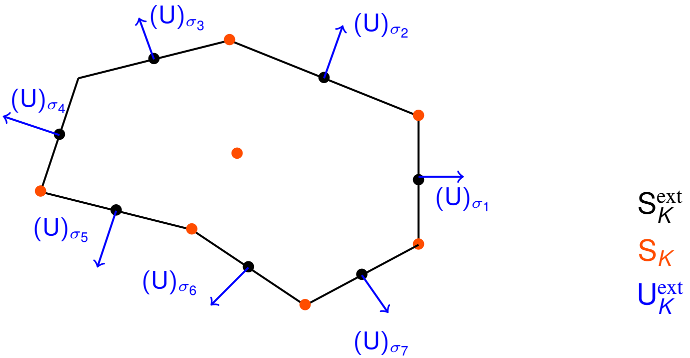

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A posteriori error estimate

Theorem (Linear Darcy flow)

Under Assumption A, there holds

$$\|\mathbf{K}^{-\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)\| \leq \left\{\sum_{K\in\mathcal{T}_H} \eta_K^2\right\}^{\frac{1}{2}},$$

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Comments

- **guaranteed upper bound** on the Darcy velocity error
- price: **matrix-vector multiplication** on each element (no (local) linear system, no (potential or flux) reconstruction)

A posteriori error estimate

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Proof (1)

- Prager–Synge equality:

$$\left\|\underline{\mathbf{K}}^{-\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)\right\| = \inf_{v\in H_0^1(\Omega)} \left\|\underline{\mathbf{K}}^{-\frac{1}{2}}\mathbf{u}_h + \underline{\mathbf{K}}^{\frac{1}{2}}\nabla v\right\|$$

- consequently, for an arbitrary $s_h \in H_0^1(\Omega)$:

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- Green theorem:

$$\begin{aligned}(\mathbf{u}_h, \nabla s_h)_K &= \langle \mathbf{u}_h\cdot\mathbf{n}, s_h\rangle_{\partial K} - (\nabla\cdot\mathbf{u}_h, s_h)_K \\ &= (\mathsf{U}_K^{\mathrm{ext}})^{\mathrm{t}}\mathsf{S}_K^{\mathrm{ext}} - (\mathsf{F})_K|K|^{-1}1^{\mathrm{t}}\mathbb{M}_K\mathsf{S}_K\end{aligned}$$

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   - A posteriori estimate
   - Numerical experiments
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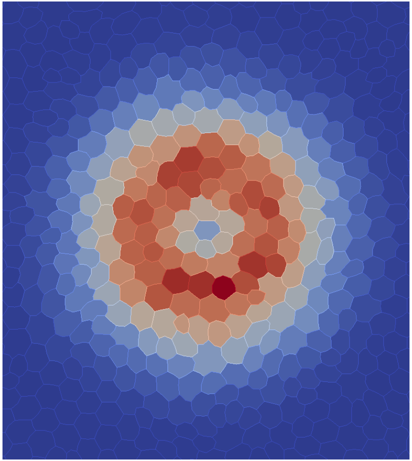
5. Conclusions

Numerical experiment

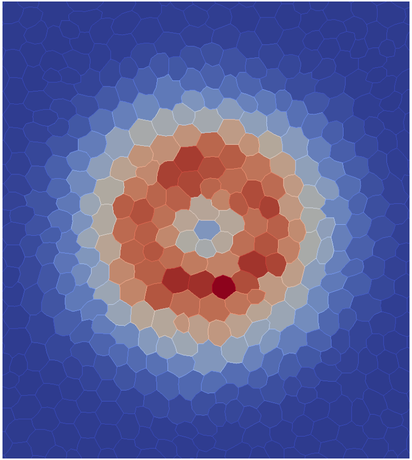
Setting

- $-\Delta p = f$
- $\Omega = (0,1)^2$
- analytic solution $2^{4\alpha} x^\alpha (1-x)^\alpha y^\alpha (1-y)^\alpha$, $\alpha = 200$
- hybrid finite volume (HFV) discretization (Droniou, Eymard, Gallouët, Herbin (2010))

Energy error & reference estimate (triangular submesh)



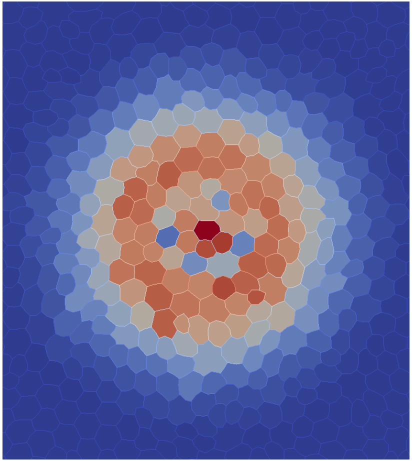
Energy error



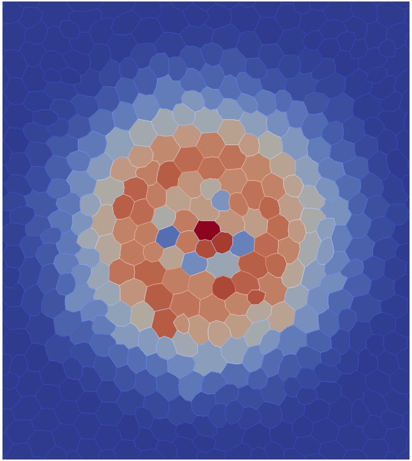
Reference estimate (Vohralík (2008))

M. Vohralík, S. Yousef, Computer Methods in Applied Mechanics and Engineering (2020)

Simple polygonal estimates



Using the MFE element matrix $\mathbb{A}_K$



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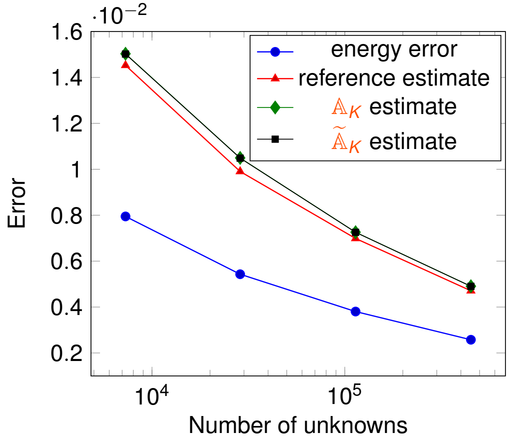
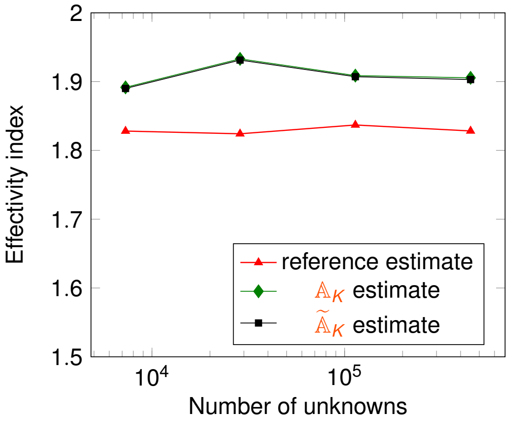
M. Vohralík, S. Yousef, Computer Methods in Applied Mechanics and Engineering (2020)

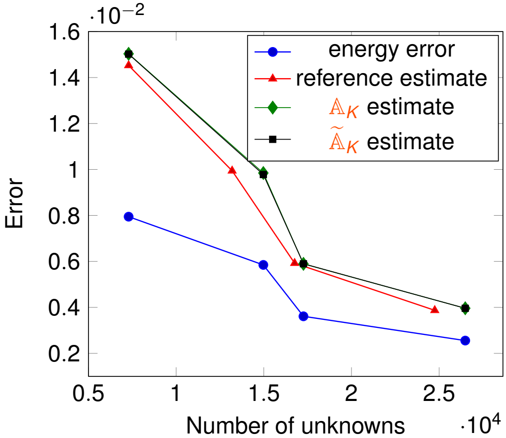
Uniform mesh refinement

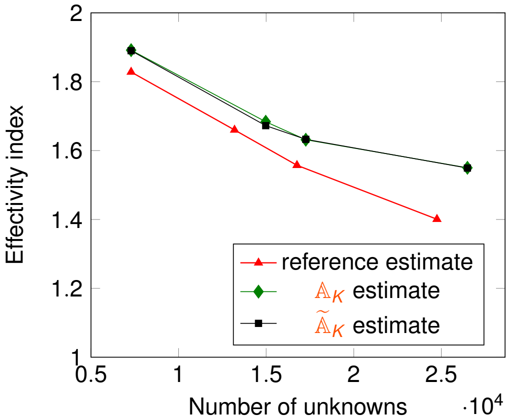




M. Vohralík, S. Yousef, Computer Methods in Applied Mechanics and Engineering (2020)

Adaptive mesh refinement





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Outline

Nonlinear Darcy flow

Steady nonlinear Darcy flow

$$
\begin{aligned}
-\nabla\cdot(\underline{\mathbf{K}}(\nabla p)\nabla p) &= f \qquad \text{in } \Omega,\\
p &= 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$

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$$p = 0 \qquad \text{on } \partial\Omega.$$

Assumptions

- invertible nonlinearity

$$\mathbf{v} = -\underline{\mathbf{K}}(\mathbf{w})\mathbf{w} \iff \mathbf{w} = -\underline{\tilde{\mathbf{K}}}(\mathbf{v})\mathbf{v}, \qquad \forall \mathbf{v},\mathbf{w} \in \mathbb{R}^d$$

- strong monotonicity

$$c_{\underline{\tilde{\mathbf{K}}}}|\mathbf{v}-\mathbf{w}|^2 \le (\mathbf{v}-\mathbf{w})\cdot(\underline{\tilde{\mathbf{K}}}(\mathbf{v})\mathbf{v} - \underline{\tilde{\mathbf{K}}}(\mathbf{w})\mathbf{w}), \qquad \forall \mathbf{v},\mathbf{w} \in \mathbb{R}^d$$

- Lipschitz-continuity

$$|\underline{\tilde{\mathbf{K}}}(\mathbf{v})\mathbf{v} - \underline{\tilde{\mathbf{K}}}(\mathbf{w})\mathbf{w}| \le C_{\underline{\tilde{\mathbf{K}}}}|\mathbf{v}-\mathbf{w}|, \qquad \forall \mathbf{v},\mathbf{w} \in \mathbb{R}^d$$

- for simple matrix-vector multiplication:

$$c_{\underline{\tilde{\mathbf{K}}}}|\mathbf{v}|^2 \le \mathbf{v}\cdot\underline{\tilde{\mathbf{K}}}(\mathbf{w})\mathbf{v}, \qquad |\underline{\tilde{\mathbf{K}}}(\mathbf{w})\mathbf{v}| \le C_{\underline{\tilde{\mathbf{K}}}}|\mathbf{v}|, \qquad \forall \mathbf{v},\mathbf{w} \in \mathbb{R}^d$$

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Steady nonlinear Darcy flow

$$-\nabla\cdot(\underline{\mathbf{K}}(\nabla p)\nabla p) = f \quad \text{in } \Omega,$$
$$p = 0 \quad \text{on } \partial\Omega.$$

Weak solution

$p \in H_0^1(\Omega)$ such that

$$(\underline{\mathbf{K}}(\nabla p)\nabla p, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Darcy velocity

$$\mathbf{u} := -\underline{\mathbf{K}}(\nabla p)\nabla p \in \mathbf{H}(\mathrm{div}, \Omega)$$

Inverse relation

$$\nabla p = -\underline{\tilde{\mathbf{K}}}(\mathbf{u})\mathbf{u}$$

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Discretization, linearization, and algebraic resolution

Discretization

$$\sum_{\sigma\in\mathcal{E}_K} (\mathrm{U}(\mathrm{P}))_\sigma \mathbf{n}_{K,\sigma}\cdot\mathbf{n}_\sigma = (\mathrm{F})_K \quad \forall K\in\mathcal{T}_H \qquad \mathcal{U}(\mathrm{P}) = \mathrm{F}$$

- system of $|\mathcal{T}_H|$ **nonlinear** algebraic equations

Linearization (step $k \geq 1$)

$$\sum_{\sigma\in\mathcal{E}_K} (\mathrm{U}^{k-1}(\mathrm{P}^k))_\sigma \mathbf{n}_{K,\sigma}\cdot\mathbf{n}_\sigma = (\mathrm{F})_K \quad \forall K\in\mathcal{T}_H \qquad \mathbb{U}^{k-1}\mathrm{P}^k = \mathrm{F}^{k-1}$$

- linearized face normal fluxes $\mathrm{U}^{k-1}(\mathrm{P}^k)$: affine functions of P^k
- system of $|\mathcal{T}_H|$ **linear** algebraic equations

Algebraic resolution (step $i \geq 1$)

$$\sum_{\sigma\in\mathcal{E}_K} (\mathrm{U}^{k-1}(\mathrm{P}^{k,i}))_\sigma \mathbf{n}_{K,\sigma}\cdot\mathbf{n}_\sigma = (\mathrm{F})_K - (\mathrm{R}^{k,i})_K \quad \forall K\in\mathcal{T}_H \qquad \mathbb{U}^{k-1}\mathrm{P}^{k,i} = \mathrm{F}^{k-1} - \mathrm{R}^{k,i}$$

- $(\mathrm{R})^{k,i}$: algebraic residual vector

Discretization, linearization, and algebraic resolution

Discretization

$$\sum_{\sigma\in\mathcal{E}_K} (\mathrm{U}(\mathrm{P}))_\sigma \mathbf{n}_{K,\sigma}\cdot\mathbf{n}_\sigma = (\mathrm{F})_K \quad \forall K \in \mathcal{T}_H$$

$$\boxed{\mathcal{U}(\mathrm{P}) = \mathrm{F}}$$

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Algebraic resolution (step $i \geq 1$)

$$\boxed{\mathbb{U}^{k-1}\mathrm{P}^{k,i} = \mathrm{F}^{k-1} - \mathrm{R}^{k,i}}$$

$$\sum_{\sigma\in\mathcal{E}_K} (\mathrm{U}^{k-1}(\mathrm{P}^{k,i}))_\sigma \mathbf{n}_{K,\sigma}\cdot\mathbf{n}_\sigma = (\mathrm{F})_K - (\mathrm{R}^{k,i})_K \quad \forall K \in \mathcal{T}_H$$

- $(\mathrm{R})^{k,i}$: algebraic residual vector

Discretization, linearization, and algebraic resolution

Discretization

$$\sum_{\sigma\in\mathcal{E}_K} (\mathsf{U}(\mathsf{P}))_\sigma \mathbf{n}_{K,\sigma}\cdot\mathbf{n}_\sigma = (\mathsf{F})_K \quad \forall K \in \mathcal{T}_H$$

$$\mathcal{U}(\mathsf{P}) = \mathsf{F}$$

- system of $|\mathcal{T}_H|$ **nonlinear** algebraic equations

Linearization (step $k \geq 1$)

$$\mathbb{U}^{k-1}\mathsf{P}^k = \mathsf{F}^{k-1}$$

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Estimating the algebraic error via additional solver steps

Perform $j \geq 1$ additional algebraic solver steps

$$\sum_{\sigma \in \mathcal{E}_K} (\mathsf{U}^{k-1}(\mathsf{P}^{k,i+j}))_\sigma \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_\sigma = (\mathsf{F})_K - (\mathsf{R}^{k,i+j})_K \quad \forall K \in \mathcal{T}_H$$

Outline

Face fluxes

Discretization face normal flux

$$(\mathrm{U}^{k,i})_\sigma := (\mathrm{U}(\mathrm{P}^{k,i}))_\sigma$$

Linearization error face normal flux

$$(\mathrm{U}^{k,i}_{\mathrm{lin}})_\sigma := (\mathrm{U}^{k-1}(\mathrm{P}^{k,i}))_\sigma - (\mathrm{U}(\mathrm{P}^{k,i}))_\sigma$$

Algebraic error face normal flux

$$(\mathrm{U}^{k,i}_{\mathrm{alg}})_\sigma := (\mathrm{U}^{k-1}(\mathrm{P}^{k,i+j}))_\sigma - (\mathrm{U}^{k-1}(\mathrm{P}^{k,i}))_\sigma$$

One number per face **immediately available** from the scheme on each step $k \geq 1$, $i \geq 1$.

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One number per face **immediately available** from the scheme on each step $k \geq 1$, $i \geq 1$.

A posteriori error estimate

Theorem (Nonlinear Darcy flow)

Under Assumption A, there holds

$$c_{\tilde{\underline{\mathbf{K}}}}^{\frac{1}{2}} \|\mathbf{u} - \mathbf{u}_h^{k,i}\| \le \eta_{\mathrm{sp}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i}$$

with $\eta_\bullet^{k,i} = \left\{\sum_{K\in\mathcal{T}_H} \left(\eta_{\bullet,K}^{k,i}\right)^2\right\}^{\frac{1}{2}}$, $\bullet = \{\mathrm{sp}, \mathrm{lin}, \mathrm{alg}, \mathrm{rem}\}$, *and*

$$\left(\eta_{\mathrm{sp},K}^{k,i}\right)^2 := (\mathsf{U}_K^{k,i})^{\mathrm{t}} \mathbb{A}_K \mathsf{U}_K^{k,i} + (\mathsf{S}_K^{k,i})^{\mathrm{t}} \mathbb{S}_K \mathsf{S}_K^{k,i} + 2 c_{\tilde{\underline{\mathbf{K}}}}^{-1} C_{\tilde{\underline{\mathbf{K}}}} \left[(\mathsf{U}_K^{k,i,\mathrm{ext}})^{\mathrm{t}} \mathsf{S}_K^{k,i,\mathrm{ext}} - (\mathsf{F})_K |K|^{-1} 1^{\mathrm{t}} \mathbb{M}_K \mathsf{S}_K^{k,i}\right],$$

$$\left(\eta_{\mathrm{lin},K}^{k,i}\right)^2 := (\mathsf{U}_{\mathrm{lin},K}^{k,i})^{\mathrm{t}} \mathbb{A}_K \mathsf{U}_{\mathrm{lin},K}^{k,i},$$

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Nonlinear Darcy flow estimate

Comments

- **guaranteed upper bound** on the Darcy velocity error
- same element matrices $\mathbb{S}_K$, $\mathbb{M}_K$, and $\mathbb{A}_K$ or $\widetilde{\mathbb{A}}_K$
- price: **matrix-vector multiplication** on each element
- **error components distinction**

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Outline

Recovering mass balance: up to working precision

Algebraic error (face normal) flux reconstruction

Backward problem: for a given residual vector $\mathrm{R}^{k,i}$, find a (face normal) flux s.t.

$$\sum_{\sigma\in\mathcal{E}_K} (\mathrm{U}^{k,i}_{\mathrm{alg}})_\sigma \mathbf{n}_{K,\sigma}\cdot\mathbf{n}_\sigma = (\mathrm{R}^{k,i})_K \quad \forall K \in \mathcal{T}_H$$

or

$$\nabla\cdot(\mathbf{u}^{k,i}_{h,\mathrm{alg}}) = |K|^{-1}(\mathrm{R})^{k,i}_K \quad \forall K \in \mathcal{T}_H$$

$\Longrightarrow$ Mass balance

On each mesh $\mathcal{T}_H$, linearization step k, and algebraic step i, there holds

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Domain decomposition solver

- exact **coarse solve**
- backward residual problem on each **subdomain**
- S. Hassan, C. Japhet, M. Kern, M. Vohralík, Comput. Methods Appl. Math. (2018)

Multilevel solver

- exact **coarse solve**
- backward residual problem on each **parent element** (patch) on each **level**
- J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, HAL Preprint 01662944 (2020)

Removes the term $\eta_{\mathrm{rem}}^{k,i}$ from the a posteriori error estimate.

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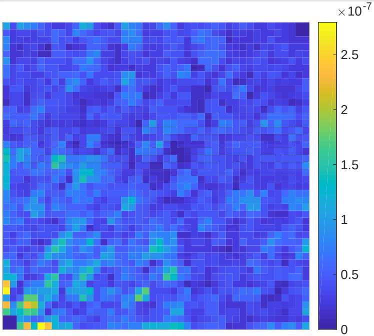
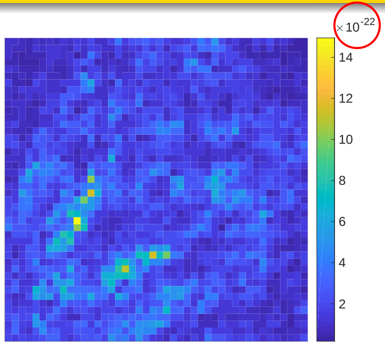
Recovering mass balance in a 2-phase Darcy porous media flow



original mass balance misfit $(\mathrm{m}^2\mathrm{s}^{-1})$



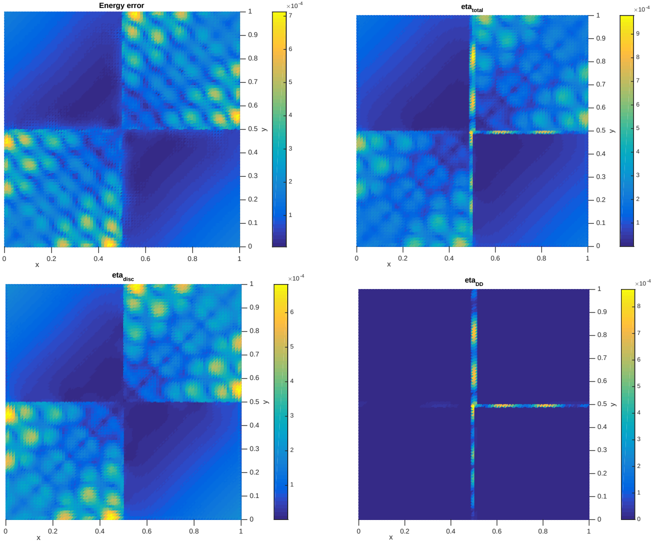
corrected mass balance misfit $(\mathrm{m}^2\mathrm{s}^{-1})$

Setting

- iterative coupling/IMPES discretization of a two-phase flow
- vertex-centered finite volumes on a square mesh
- time step 260, 1st iterative coupling linearization, CG iteration 171

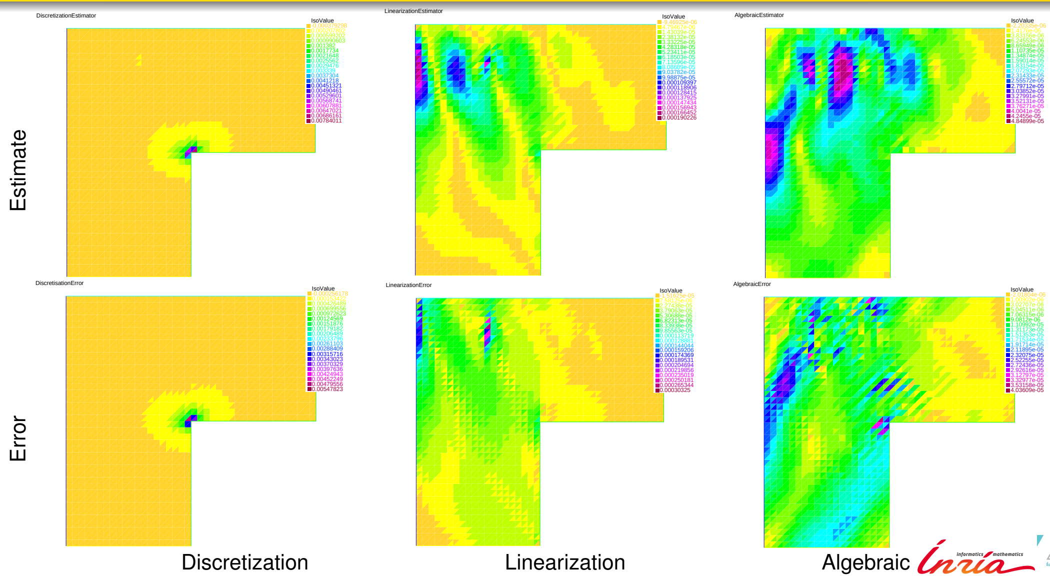
J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, HAL Preprint 01662944 (2020)

Domain decomposition for a 1-phase Darcy porous media flow



S. Ali Hassan, C. Japhet, M. Kern, M. Vohralík, Computational Methods in Applied Mathematics (2018)

Finite elements & 4-Laplacian (FreeFem++ implementation Z. Tang)



Outline

Multi-phase multi-compositional flows

Unknowns

- reference pressure P
- phase saturations $\boldsymbol{S} := (S_p)_{p\in\mathcal{P}}$
- component molar fractions $\boldsymbol{C}_p := (C_{p,c})_{c\in\mathcal{C}_p}$ of phase $p \in \mathcal{P}$

Constitutive laws

- phase pressure = reference pressure + capillary pressure

$$P_p := P + P_{c_p}(\boldsymbol{S})$$

- Darcy's law

$$\mathbf{u}_p(P_p) := -\underline{\mathbf{K}}(\nabla P_p + \rho_p g \nabla z)$$

- component fluxes

$$\theta_c := \sum_{p\in\mathcal{P}_c} \theta_{p,c}, \qquad \theta_{p,c} := \nu_p C_{p,c} \mathbf{u}_p(P_p)$$

- amount of moles of component c per unit volume

$$l_c = \phi \sum_{p\in\mathcal{P}_c} \zeta_p S_p C_{p,c}$$

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Multi-phase multi-compositional flows

Governing PDEs

- conservation of mass for components

$$\partial_t l_c + \nabla\cdot\boldsymbol{\theta}_c = q_c \qquad \forall c \in \mathcal{C}$$

- + boundary & initial conditions

Closure algebraic equations

- conservation of pore volume: $\sum_{p\in\mathcal{P}} S_p = 1$
- conservation of the quantity of the matter: $\sum_{c\in\mathcal{C}_p} C_{p,c} = 1$ for all $p \in \mathcal{P}$
- thermodynamic equilibrium

Mathematical issues

- coupled system PDE – algebraic constraints
- unsteady, nonlinear
- elliptic–degenerate parabolic type
- dominant advection

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Theorem (Multi-phase multi-compositional Darcy flow)

Under Assumption A, there holds

$$\text{dual residual norm} \leq \left\{ \sum_{c\in\mathcal{C}} \left(\eta_{\mathrm{sp},c}^{n,k,i} + \eta_{\mathrm{tm},c}^{n,k,i} + \eta_{\mathrm{lin},c}^{n,k,i} + \eta_{\mathrm{alg},c}^{n,k,i} + \eta_{\mathrm{rem},c}^{n,k,i}\right)^2 \right\}^{\frac{1}{2}}$$

$$\text{with } \eta_{\bullet,c}^{n,k,i} := \left\{ \int_{I_n} \sum_{K\in\mathcal{T}_H^n} \left(\eta_{\bullet,K,c}^{n,k,i}\right)^2 \mathrm{d}t \right\}^{\frac{1}{2}}, \quad \bullet = \mathrm{sp}, \mathrm{tm}, \mathrm{lin}, \mathrm{alg}, \mathrm{rem}.$$

Comments

- immediate extension of the results of the steady case
- still matrix-vector multiplication on each element
- same element matrices $\mathbb{S}_K$, $\mathbb{M}_K$, and $\mathbb{A}_K$ or $\widetilde{\mathbb{A}}_K$
- input: available normal face fluxes, reference pressure, phase saturations, and component molar fractions
- same physical units of estimators of all error components

A posteriori error estimate

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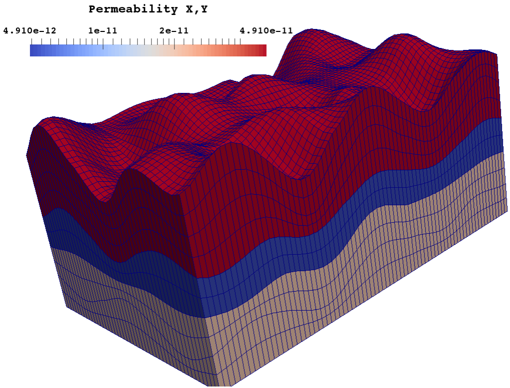
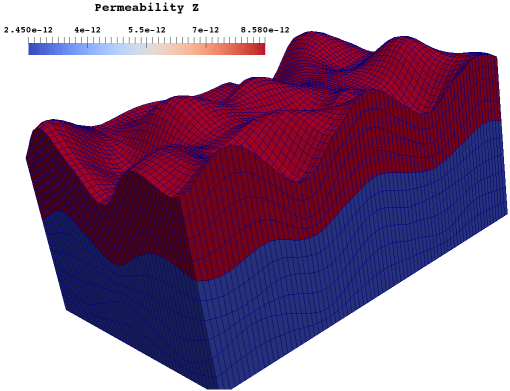
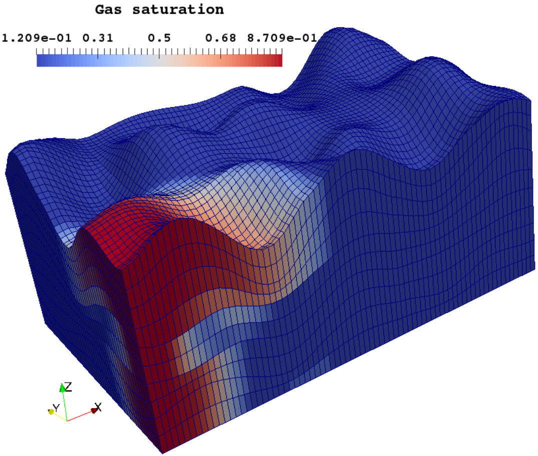
Outline

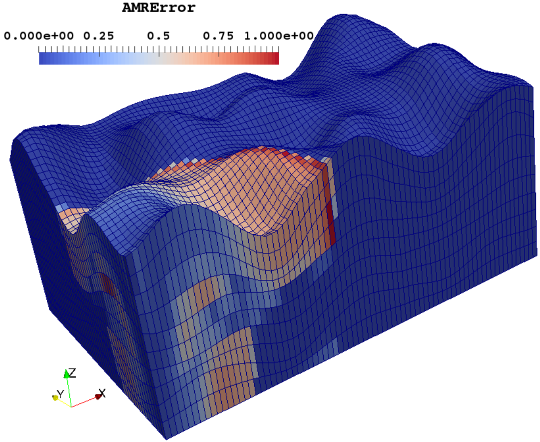
3 phases, 3 components (black-oil) problem: permeability



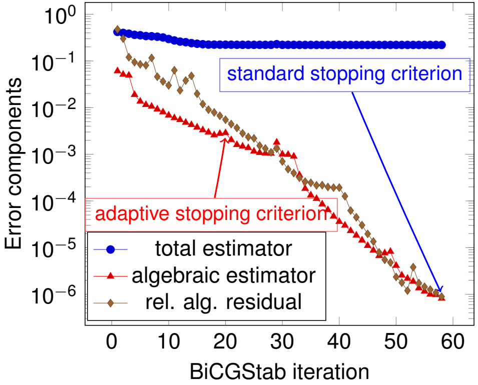


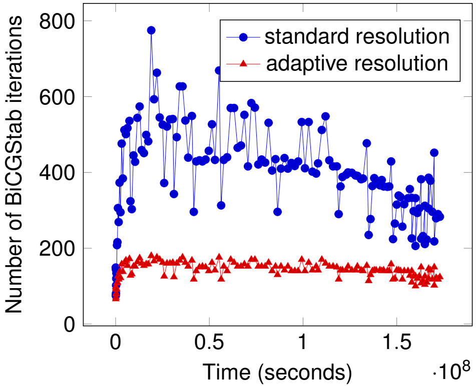
3 phases, 3 components (black-oil) problem: gas saturation and a posteriori estimate





3 phases, 3 components (black-oil): alg. solver & mesh adaptivity





	Linear solver steps	Resolution time	AMR time	Estimators evaluation	Gain factor
Standard resolution	66386	1023s	-	-	-
Adaptive resolution	20184	201s	42s	26s	3.8

M. Vohralík, S. Yousef, Computer Methods in Applied Mechanics and Engineering (2020)

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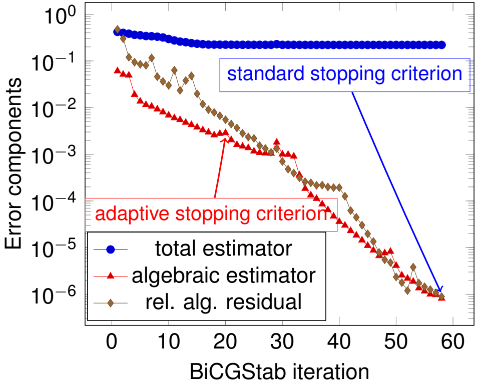

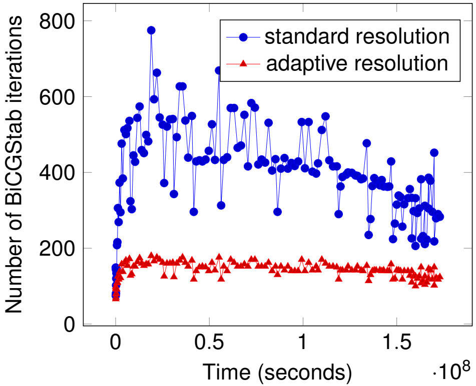

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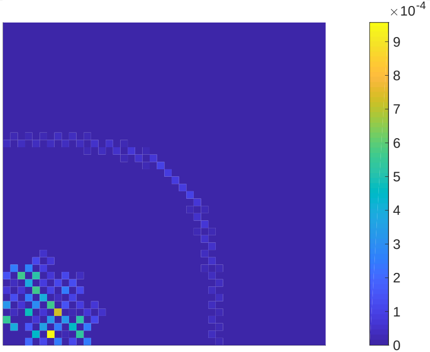
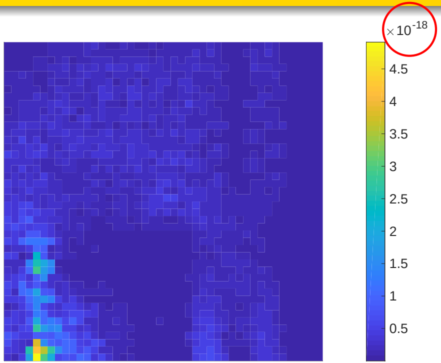
2 phases: recovering water mass balance

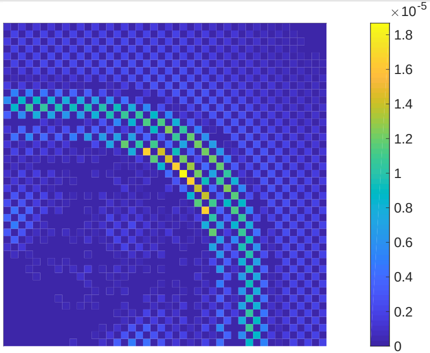


original mass balance misfit $(\mathrm{m}^2\mathrm{s}^{-1})$



corrected mass balance misfit $(\mathrm{m}^2\mathrm{s}^{-1})$

2 phases: recovering oil mass balance



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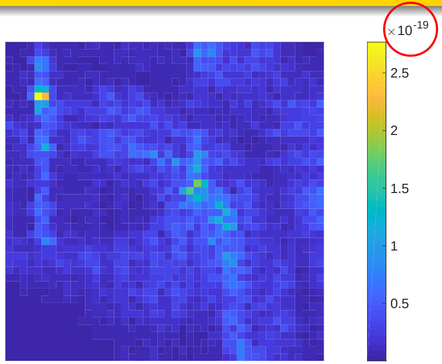

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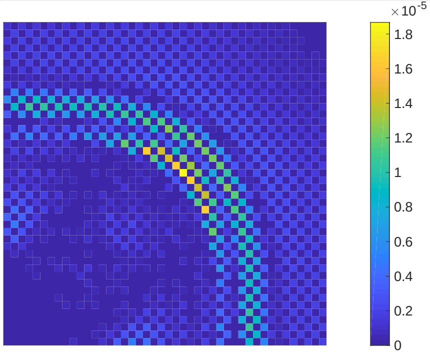
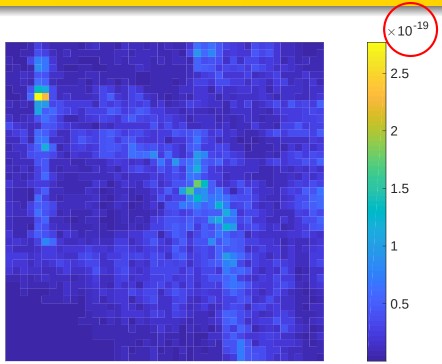
2 phases: recovering oil mass balance



original mass balance misfit ($\mathrm{m}^2\mathrm{s}^{-1}$)



corrected mass balance misfit ($\mathrm{m}^2\mathrm{s}^{-1}$)

Setting

- fully implicit discretization
- cell-centered finite volumes on a square mesh
- time step 260 (60 days), 1st Newton linearization, GMRes iteration 195

J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, HAL Preprint 01662944 (2020)

Outline

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- **simple** estimates on polygonal/polyhedral meshes
- **full adaptivity**: linear solver, nonlinear solver, time step, space mesh
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VOHRALÍK M., YOUSEF S., A simple a posteriori estimate on general polytopal meshes with applications to complex porous media flows, *Comput. Methods Appl. Mech. Engrg.* **331** (2018), 728–760.

HASSAN S., JAPHET C., KERN M., VOHRALÍK M., A posteriori stopping criteria for optimized Schwarz domain decomposition algorithms in mixed formulations, *Comput. Methods Appl. Math.* **18** (2018), 495–519.

PAPEŽ J., RÜDE U., VOHRALÍK M., WOHLMUTH B., Sharp algebraic and total a posteriori error bounds for h and p finite elements via a multilevel approach. Recovering mass balance in any situation, HAL Preprint 01662944 (2020).

Thank you for your attention!

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