Polynomial-degree-robust a posteriori estimates in a unified setting

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Outline

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Previous results, $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

General result

- Prager and Synge (1947):

$$\|\nabla u + \boldsymbol{\sigma}_h\|^2 + \|\nabla(u - u_h)\|^2 = \|\nabla u_h + \boldsymbol{\sigma}_h\|^2$$

for **any** $u_h \in H_0^1(\Omega)$ and **any** $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ s.t. $\nabla\cdot\boldsymbol{\sigma}_h = f$

- a posteriori estimate: **how to practically construct $\boldsymbol{\sigma}_h$?**
- Hlaváček, Haslinger, Nečas, and Lovíšek (1979) & Repin (1997): global construction: unprecise/costly

Local flux reconstructions

- Ladevèze and Leguillon (1983), equilibrated face fluxes
- Destuynder and Métivet (1999), discrete flux $\boldsymbol{\sigma}_h$
- Luce and Wohlmuth (2004), local efficiency
- Vejchodský (2006), mixed approach
- Kim (2007) & Ern, Nicaise, and Vohralík (2007), discontinuous Galerkin method elementwise prescription
- Braess and Schöberl (2008), Vohralík (2008), Ern and Vohralík (2009), **local Neumann MFE problems**

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Previous results

Local potential reconstructions ($u_h \notin H^1_0(\Omega)$)

- Achdou, Bernardi, and Coquel (2003) & Karakashian and Pascal (2003), by prescription
- Carstensen and Merdon (2013), **local Neumann MFE problems**

Unified frameworks

- Ainsworth and Oden (1993)
- Carstensen (2005)
- Ainsworth (2010)
- Ern and Vohralík (heat equation 2010, Stokes equation 2012, nonlinear Laplace equation 2013)

Polynomial-degree-robust estimates

- Braess, Pillwein, and Schöberl (2009), conforming finite elements

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Model problem

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$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega,\\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

Weak formulation

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (constraint)
- $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$ (constraint)
- $\sigma = -\nabla u$ (constitutive law)
- $\nabla\cdot\sigma = f$ (equilibrium)

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Outline

A posteriori error estimate

Theorem (A guaranteed a posteriori error estimate)

- *Let $u \in H_0^1(\Omega)$ be the weak solution,*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega), v|_K \in H^1(K)\ \forall K \in \mathcal{T}_h\}$ *be arbitrary,*
- $s_h \in H_0^1(\Omega)$ *and* $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *with* $(\nabla\cdot\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K$ *for all* $K \in \mathcal{T}_h$ *be arbitrary.*

Then

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K\in\mathcal{T}_h} \left(\|\nabla u_h + \boldsymbol{\sigma}_h\|_K + \frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K \right)^2 + \sum_{K\in\mathcal{T}_h} \|\nabla(u_h - s_h)\|_K^2.$$

Proof (Spirit of Prager–Synge (1947)).

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

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A posteriori error estimate

Proof (continuation).

- projection definition of s:

$$\|\nabla(u-u_h)\|^2 = \underbrace{\sup_{\varphi\in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla(u-u_h),\nabla\varphi)^2}_{\text{dual norm of the residual}} + \underbrace{\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- nonconformity upper bound:

$$\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\| \le \|\nabla(u_h-s_h)\|$$

- weak solution definition, equilibrated flux, Green theorem:

$$(\nabla(u-u_h),\nabla\varphi) = (f,\varphi)-(\nabla u_h,\nabla\varphi) = (f-\nabla\cdot\boldsymbol{\sigma}_h,\varphi)-(\nabla u_h+\boldsymbol{\sigma}_h,\nabla\varphi)$$

- Cauchy–Schwarz and Poincaré inequalities:

$$-(\nabla u_h+\boldsymbol{\sigma}_h,\nabla\varphi) \le \sum_{K\in\mathcal{T}_h} \|\nabla u_h+\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K,$$

$$(f-\nabla\cdot\boldsymbol{\sigma}_h,\varphi) = \sum_{K\in\mathcal{T}_h} (f-\nabla\cdot\boldsymbol{\sigma}_h,\varphi-\varphi_K)_K \le \sum_{K\in\mathcal{T}_h} \frac{h_K}{\pi} \|f-\nabla\cdot\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K$$

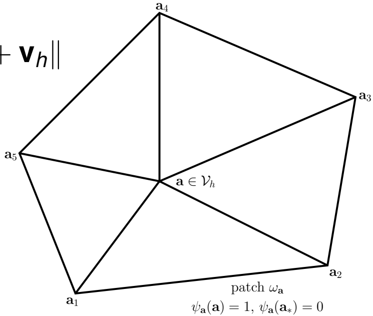
Potential and flux reconstruction

Ideally

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

- . . . too expensive



Partition of unity

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

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- **local minimizations**

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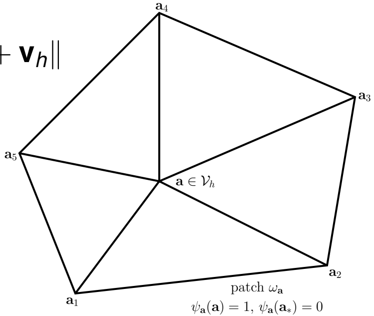

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Local flux reconstruction

Assumption A (Galerkin orthogonality)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}.$$

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving **the local MFE problem**

$$\begin{aligned} (\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla{\cdot}\mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ (\nabla{\cdot}\boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} &= (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}{\cdot}\nabla u_h, q_h)_{\omega_{\mathbf{a}}} && \forall q_h \in Q_h^{\mathbf{a}}, \end{aligned}$$

with mixed finite element spaces $\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$ (homogeneous Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$ and on $\partial\omega_{\mathbf{a}} \setminus \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$). Set

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Comments

$\mathbf{H}(\mathrm{div},\Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility condition gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

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Local potential reconstruction ($d = 2$)

Definition (Construction of s_h)

For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving **the local MFE problem**

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with mixed finite element spaces $\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$ (hom. Neumann BC on $\partial\omega_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{V}_h$). Set

$$\begin{aligned} -\mathrm{R}_{\frac{\pi}{2}} \nabla s_h^{\mathbf{a}} &= \boldsymbol{\sigma}_h^{\mathbf{a}}, \\ s_h^{\mathbf{a}} &= 0 \text{ on } \partial\omega_{\mathbf{a}}, \\ s_h &:= \sum_{\mathbf{a} \in \mathcal{V}_h} s_h^{\mathbf{a}}. \end{aligned}$$

Remark

- The same problems, only RHS/BC different.

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$$\begin{aligned} (\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\mathrm{R}_{\frac{\pi}{2}}\nabla(\psi_{\mathbf{a}} u_h), \mathbf{v}_h)_{\omega_{\mathbf{a}}} && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ (\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} &= (0, q_h)_{\omega_{\mathbf{a}}} && \forall q_h \in Q_h^{\mathbf{a}}, \end{aligned}$$

with mixed finite element spaces $\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$ (hom. Neumann BC on $\partial\omega_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{V}_h$). Set

$$\begin{aligned} -\mathrm{R}_{\frac{\pi}{2}}\nabla s_h^{\mathbf{a}} &= \boldsymbol{\sigma}_h^{\mathbf{a}}, \\ s_h^{\mathbf{a}} &= 0 \text{ on } \partial\omega_{\mathbf{a}}, \\ s_h &:= \sum_{\mathbf{a}\in\mathcal{V}_h} s_h^{\mathbf{a}}. \end{aligned}$$

Remark

- The same problems, only RHS/BC different.

Local potential reconstruction ($d = 2$)

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For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving **the local MFE problem**

$$
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- The same problems, only RHS/BC different.

Outline

Continuous efficiency, flux reconstruction

Theorem (Cont. efficiency Carstensen & Funken (1999), Braess, Pillwein, & Schöberl (2009)**)**

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $\boldsymbol{\sigma}^{\mathbf{a}} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}})$ and $\bar{r}^{\mathbf{a}} \in L^2_*(\omega_{\mathbf{a}})$ be given by

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with

- $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$: $L^2_*(\omega_{\mathbf{a}}) := L^2(\omega_{\mathbf{a}})$ *with zero mean value;* $\mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}) := \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ *with zero normal trace on* $\partial\omega_{\mathbf{a}}$*;*
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Then there exists a constant $C_{\mathrm{cont,PF}} > 0$ only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

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Continuous efficiency, potential reconstruction ($d = 2$)

Assumption B (Weak continuity)

There holds

$$\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$$

Theorem (Continuous efficiency)

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ satisfying Assumption B be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $\boldsymbol{\sigma}^{\mathbf{a}} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}})$ and $\bar{r}^{\mathbf{a}} \in L^2_*(\omega_{\mathbf{a}})$ be given by

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Continuous efficiency, potential reconstruction ($d = 2$)

Proof (sketch).

- equivalent primal formulation: $\|\boldsymbol{\sigma}^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$, where $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}}); (v,1)_{\omega_{\mathbf{a}}} = 0\}$ solves
$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h), \nabla v)_{\omega_{\mathbf{a}}} \qquad \forall v \in H^1_*(\omega_{\mathbf{a}})$$
- dual norm characterization
$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \sup_{v \in H^1_*(\omega_{\mathbf{a}});\, \|\nabla v\|_{\omega_{\mathbf{a}}}=1} (\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}}$$
- arbitrary $\tilde{u} \in H^1(\omega_{\mathbf{a}})$ with $(\tilde{u},1)_{\omega_{\mathbf{a}}} = (u_h,1)_{\omega_{\mathbf{a}}}$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and $\tilde{u} = 0$ on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$:
$$(\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} \tilde{u}), \nabla v)_{\omega_{\mathbf{a}}} = 0$$
- Cauchy–Schwarz:
$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = (\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h)), \nabla v)_{\omega_{\mathbf{a}}} \leq \|\nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h))\|_{\omega_{\mathbf{a}}}$$
- broken Poincaré–Friedrichs inequality:
$$\|\nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h))\|_{\omega_{\mathbf{a}}} \leq (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}) \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}$$

Mixed finite elements stability ($d = 2$)

Assumption C (Piecewise polynomial approximation and data)

The approximation u_h and the datum f are piecewise polynomial and the MFE reconstructions are chosen correspondingly.

Theorem (MFE stability / continuous right inverse of the divergence operator Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010)**)**

Let u be the weak solution and let u_h, f, and the reconstructions satisfy Assumption C. Then there exists a constant $C_{\rm st} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\sigma_h^{\mathbf{a}} + \tau_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\sigma^{\mathbf{a}} + \tau_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}},$$

with $\tau_h^{\mathbf{a}} = \psi_{\mathbf{a}} \nabla u_h$ for the flux reconstruction and $\tau_h^{\mathbf{a}} = \mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h)$ for the potential reconstruction.

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Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency)

Let u be the weak solution and let Assumptions A, B, and C hold. Then

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \le C_{\rm st} C_{\rm cont,PF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}},$$

$$\|\nabla(u_h - s_h)\|_K \le C_{\rm st} C_{\rm cont,bPF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}.$$

Remarks

- $C_{\rm st}$ can be bounded by solving the local Neumann problems by conforming FEs: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}} \subset H^1_*(\omega_{\mathbf{a}})$ s.t.

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}};$$

  then $C_{\rm st} \le \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} / \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$
- $\Rightarrow$ maximal overestimation factor guaranteed

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Outline

Conforming finite elements

Conforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$
- $V_h \subset H_0^1(\Omega)$: $s_h := u_h$, no need for Assumption B

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$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p-1}(e), \forall e \in \mathcal{E}_h$$

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Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

Find $u_h \in V_h$ such that

$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

$$+ \sum_{e\in\mathcal{E}_h}\langle\alpha h_e^{-1}[\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h$$

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  - estimates for the discrete gradient

    $$\mathfrak{G}(u_h) := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_h)]^2$

    $$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_h)]^2$$

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Discontinuous Galerkin finite elements: Assumption B

Nonsymmetric and incomplete versions

- broken Poincaré–Friedrichs inequality with jumps:

$$\|\nabla(\psi_{\mathbf{a}}(\tilde{u}-u_h))\|_{\omega_{\mathbf{a}}} \leq (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}})\|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}} + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \left\{ \sum_{e\in\mathcal{E}_h^{\mathrm{int}},\, \mathbf{a}\in e} h_e^{-1} \|\Pi_e^0 [\![u_h]\!]\|_e^2 \right\}^{1/2}$$

- include the jump terms in the error and estimators

Symmetric version

- discrete gradient $\mathfrak{G}$ satisfies

$$(\mathfrak{G}(u_h), \mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h$$

- modified potential reconstruction: local MFE problems with $\tau_h^{\mathbf{a}} := \psi_{\mathbf{a}} \mathrm{R}_{\frac{\pi}{2}} \mathfrak{G}(u_h)$ and $g^{\mathbf{a}} := (\mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}}) \cdot \mathfrak{G}(u_h)$
- local efficiency

$$\|\mathfrak{G}(u_h - s_h)\|_K \leq C_{\mathrm{st}} C_{\mathrm{cont,P}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\mathfrak{G}(u-u_h)\|_{\omega_{\mathbf{a}}}$$

Discontinuous Galerkin finite elements: Assumption B

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Mixed finite elements

Mixed finite elements
Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) &= 0 \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h,\\ (\nabla\cdot\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \qquad && \forall q_h \in Q_h.\end{aligned}$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, $v_h \in V_h$ satisfy

$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p'}(e), \forall e \in \mathcal{E}_h$$

- Assumption A: no need for flux reconstruction, $\boldsymbol{\sigma}_h$ comes from the discretization
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Outline

Numerics: discontinuous Galerkin

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega :=]0,1[\times]0,1[, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

incomplete interior penalty discontinuous Galerkin method

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Estimates, errors, effectivity indices (calc. V. Dolejší)

h	p	$\|\nabla(u-u_h)\|$	$\|u-u_h\|_{DG}$	$\|\nabla u_h+\sigma_h\|$	$\|\nabla(u_h-s_h)\|$	η_{osc}	η	η_{DG}	I^{eff}	I^{eff}_{DG}
$h_0/1$	1	1.21E+00	1.22E+00	1.24E+00	1.07E-01	5.56E-02	1.30E+00	1.31E+00	1.07	1.07
$h_0/2$		6.18E-01	6.22E-01	6.38E-01	5.09E-02	7.02E-03	6.47E-01	6.50E-01	1.05	1.05
		(0.97)	(0.97)	(0.96)	(1.07)	(2.99)	(1.01)	(1.01)		
$h_0/4$		3.12E-01	3.13E-01	3.22E-01	2.43E-02	8.80E-04	3.24E-01	3.25E-01	1.04	1.04
		(0.99)	(0.99)	(0.99)	(1.07)	(3.00)	(1.00)	(1.00)		
$h_0/8$		1.56E-01	1.57E-01	1.61E-01	1.18E-02	1.10E-04	1.62E-01	1.63E-01	1.04	1.04
		(1.00)	(1.00)	(1.00)	(1.05)	(3.00)	(1.00)	(1.00)		
$h_0/1$	2	1.50E-01	1.53E-01	1.49E-01	2.76E-02	5.10E-03	1.56E-01	1.59E-01	1.04	1.04
$h_0/2$		3.85E-02	3.92E-02	3.83E-02	7.99E-03	3.22E-04	3.94E-02	4.01E-02	1.03	1.02
		(1.96)	(1.96)	(1.96)	(1.79)	(3.98)	(1.98)	(1.98)		
$h_0/4$		9.70E-03	9.88E-03	9.68E-03	2.12E-03	2.02E-05	9.93E-03	1.01E-02	1.02	1.02
		(1.99)	(1.99)	(1.98)	(1.92)	(4.00)	(1.99)	(1.99)		
$h_0/8$		2.43E-03	2.48E-03	2.43E-03	5.42E-04	1.26E-06	2.49E-03	2.54E-03	1.02	1.02
		(1.99)	(1.99)	(1.99)	(1.96)	(4.00)	(1.99)	(1.99)		
$h_0/1$	3	1.32E-02	1.34E-02	1.29E-02	2.52E-03	3.58E-04	1.35E-02	1.37E-02	1.03	1.03
$h_0/2$		1.67E-03	1.69E-03	1.65E-03	3.13E-04	1.13E-05	1.70E-03	1.71E-03	1.01	1.01
		(2.98)	(2.98)	(2.97)	(3.01)	(4.99)	(3.00)	(3.00)		
$h_0/4$		2.11E-04	2.13E-04	2.09E-04	3.83E-05	3.53E-07	2.12E-04	2.15E-04	1.01	1.01
		(2.99)	(2.99)	(2.99)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/8$		2.64E-05	2.67E-05	2.61E-05	4.69E-06	1.10E-08	2.66E-05	2.69E-05	1.01	1.01
		(3.00)	(3.00)	(3.00)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/1$	4	9.36E-04	9.54E-04	9.05E-04	2.41E-04	2.12E-05	9.57E-04	9.74E-04	1.02	1.02
$h_0/2$		5.93E-05	6.05E-05	5.77E-05	1.68E-05	3.36E-07	6.04E-05	6.16E-05	1.02	1.02
		(3.98)	(3.98)	(3.97)	(3.84)	(5.98)	(3.99)	(3.98)		
$h_0/4$		3.72E-06	3.80E-06	3.63E-06	1.10E-06	5.31E-09	3.80E-06	3.87E-06	1.02	1.02
		(3.99)	(3.99)	(3.99)	(3.94)	(5.98)	(3.99)	(3.99)		
$h_0/8$		2.33E-07	2.38E-07	2.27E-07	7.02E-08	8.30E-11	2.38E-07	2.43E-07	1.02	1.02
		(4.00)	(4.00)	(4.00)	(3.97)	(6.00)	(4.00)	(3.99)		
$h_0/1$	5	5.41E-05	5.50E-05	5.22E-05	1.38E-05	1.06E-06	5.50E-05	5.58E-05	1.02	1.02
$h_0/2$		1.70E-06	1.72E-06	1.65E-06	4.39E-07	9.35E-09	1.72E-06	1.74E-06	1.01	1.01
		(4.99)	(5.00)	(4.98)	(4.98)	(6.82)	(5.00)	(5.00)		
$h_0/4$		5.32E-08	5.39E-08	5.19E-08	1.40E-08	7.67E-11	5.38E-08	5.45E-08	1.01	1.01
		(5.00)	(5.00)	(4.99)	(4.97)	(6.93)	(5.00)	(5.00)		
$h_0/8$		1.66E-09	1.69E-09	1.62E-09	4.41E-10	5.99E-13	1.68E-09	1.70E-09	1.01	1.01
		(5.00)	(5.00)	(5.00)	(4.99)	(7.00)	(5.00)	(5.00)		

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Conclusions and future directions

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- polynomial-degree robust estimates, unified framework for most standard numerical methods

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- convergence and optimality
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Thank you for your attention!