p-uniform discrete Poincaré inequalities

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Outline

1. Introduction
2. Equivalent results
   - Stability of discrete constrained minimizations
   - Discrete inf-sup conditions
   - Operator norms of piecewise polynomial vector potential operators
   - Existence of graph-stable commuting projectors
3. Proofs and behavior of constants
   - Using broken polynomial extension operators
   - Using stable commuting projectors
   - Using piecewise Piola transformations
4. Conclusions and outlook

Setting

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- ω: three-dimensional, open, bounded, connected, Lipschitz, polyhedral (diameter h_ω)

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- $\boldsymbol{H}(\mathbf{curl},\omega)$: vector-valued functions from $\boldsymbol{L}^2(\omega)$ with weak curl in $\boldsymbol{L}^2(\omega)$
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Versions with homogeneous boundary conditions available

Poincaré inequalities

Easy case

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- **curl** and div have **infinite-dimensional kernels**

Discrete Poincaré inequalities

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- $p \geq 0$: polynomial degree

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$$\|\boldsymbol{u}_{\mathcal{T}}\| \leq C_{\mathrm{P}}^{\mathrm{d},1} h_\omega \|\mathbf{curl}\, \boldsymbol{u}_{\mathcal{T}}\| \qquad \forall \boldsymbol{u}_{\mathcal{T}} \in \mathcal{N}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \text{ s.t. } \langle \boldsymbol{u}_{\mathcal{T}}, \boldsymbol{v}_{\mathcal{T}} \rangle = 0$$
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- the discrete spaces are **not subspaces of the continuous ones**: $C_{\mathrm{P}}^{\mathrm{d},l} \not\leq C_{\mathrm{P}}^{l}$, $l \in \{1{:}2\}$

Unified writing – Poincaré inequalities – PI

$$\|u\| \leq C_{\mathrm{P}}^{l} h_\omega \|d^l u\| \qquad \forall u \in V^l(\omega) \text{ such that } \langle u, v \rangle = 0$$
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- **values** of $C_{\mathrm{P}}^{\mathrm{d},l}$, $l \in \{1{:}2\}$
- their **dependence** on
  - C_{P}^{l}
  - $|\mathcal{T}_\omega|$ – the number of tetrahedra in $\mathcal{T}_\omega$
  - p – the polynomial degree
  - $\rho_{\mathcal{T}_\omega}$ – the shape regularity of $\mathcal{T}_\omega$

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  - $\rho_{\mathcal{T}_\omega}$ – the shape regularity of $\mathcal{T}_\omega$
- **optimal answer**: $C_{\mathrm{P}}^{\mathrm{d},l} = C_{\mathrm{P}}^{\mathrm{d},l}(C_{\mathrm{P}}^{l}, \rho_{\mathcal{T}_\omega}) = C_{\mathrm{P}}^{\mathrm{d},l}(\rho_{\mathcal{T}_\omega})$

Available results

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- Girault and Raviart (1986), Monk and Demkowicz (2001), Arnold, Falk, and Winther (2006), Ern and Guermond (2021), ...

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Operator norms of a piecewise polynomial vector potential operators

- *p*-version: Demkowicz and Babuška (2003), Gopalakrishnan and Demkowicz (2004), Demkowicz and Buffa (2005), Costabel and McIntosh (2010), Boffi, Costabel, Dauge, Demkowicz, and Hiptmair (2011) ...
- *hp*-version: Gopalakrishnan, Pasciak, and Demkowicz (2004), Braess, Pillwein, and Schöberl (2009), Ern, and Vohralík (2020), Chaumont-Frelet and Vohralík (2026), Demkowicz and Vohralík (2026)

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Stable commuting projectors

- *h*-stable: Schöberl, Christiansen and Winther, Falk and Winther, Ern and Guermond, Licht, Arnold and Guzmán, Ern, Gudi, Smears, and Vohralík, Chaumont-Frelet and Vohralík, ...
- *hp*-stable: Demkowicz and Vohralík (2026)

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For $r_{\mathcal{T}} \in d^l(V_p^l(\mathcal{T}_\omega))$: Find

$$u_{\mathcal{T}}^* := \underset{\substack{v_{\mathcal{T}} \in V_p^l(\mathcal{T}_\omega) \\ d^l v_{\mathcal{T}} = r_{\mathcal{T}}}}{\arg\min} \|v_{\mathcal{T}}\|^2. \tag{1}$$

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Theorem (Equivalence: DPI – stability of discrete constrained minimization)

The DPI are equivalent to the stability of (1) *in the sense*

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Proof.

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- **Euler optimality conditions** of (1):

$$\begin{cases} \text{Find } u_{\mathcal{T}}^* \in V_p^l(\mathcal{T}_\omega) \text{ with } d^l u_{\mathcal{T}}^* = r_{\mathcal{T}} \text{ such that} \\ \langle u_{\mathcal{T}}^*, v_{\mathcal{T}} \rangle = 0 \quad \forall v_{\mathcal{T}} \in V_p^l(\mathcal{T}_\omega) \text{ with } d^l v_{\mathcal{T}} = 0 \end{cases} \tag{2}$$

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Discrete inf-sup conditions

For $r_{\mathcal{T}} \in d^l(V_p^l(\mathcal{T}_\omega))$, consider

$$\begin{cases} \text{Find } u_{\mathcal{T}}^* \in V_p^l(\mathcal{T}_\omega) \text{ and } s_{\mathcal{T}}^* \in d^l(V_p^l(\mathcal{T}_\omega)) \text{ such that} \\ \langle u_{\mathcal{T}}^*, v_{\mathcal{T}} \rangle - \langle s_{\mathcal{T}}^*, d^l v_{\mathcal{T}} \rangle \qquad = 0 \qquad\qquad \forall v_{\mathcal{T}} \in V_p^l(\mathcal{T}_\omega), \\ \langle d^l u_{\mathcal{T}}^*, t_{\mathcal{T}} \rangle \qquad\qquad\qquad = \langle r_{\mathcal{T}}, t_{\mathcal{T}} \rangle \qquad \forall t_{\mathcal{T}} \in d^l(V_p^l(\mathcal{T}_\omega)). \end{cases} \tag{3}$$

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Theorem (Equivalence: DPI – discrete inf-sup conditions)

The DPI are equivalent to the discrete inf-sup conditions of (3)*:*

$$\inf_{t_{\mathcal{T}} \in d^l(V_p^l(\mathcal{T}_\omega))} \sup_{v_{\mathcal{T}} \in V_p^l(\mathcal{T}_\omega)} \frac{\langle t_{\mathcal{T}}, d^l v_{\mathcal{T}} \rangle}{\|t_{\mathcal{T}}\| \|v_{\mathcal{T}}\|} \geq \frac{1}{C_{\mathrm{P}}^{\mathrm{d},l} h_\omega}.$$

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Proof.

- (3) is equivalent to (2)
- discrete inf-sup conditions $\Leftrightarrow$ stability of discrete constrained minimization

Outline

Piecewise polynomial vector potential operators (Poincaré maps)

Piecewise polynomial vector potential operators:

$$\Phi^l_{\mathcal{T}} : d^l(V^l_p(\mathcal{T}_\omega)) \to V^l_p(\mathcal{T}_\omega), \quad r_{\mathcal{T}} \to \Phi^l_{\mathcal{T}}(r_{\mathcal{T}}) \text{ s.t. } d^l(\Phi^l_{\mathcal{T}}(r_{\mathcal{T}})) = r_{\mathcal{T}}.$$

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$L^2(\omega)$-norm minimizing operators:

$$\Phi^{l,*}_{\mathcal{T}}(r_{\mathcal{T}}) := \arg\min_{\substack{v_{\mathcal{T}} \in V^l_p(\mathcal{T}_\omega) \\ d^l v_{\mathcal{T}} = r_{\mathcal{T}}}} \|v_{\mathcal{T}}\|^2. \tag{4}$$

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Operator norm:

$$\left\|\left|\Phi^{l,*}_{\mathcal{T}}\right|\right\| := \max_{v_{\mathcal{T}} \in V^l_p(\mathcal{T}_\omega)} \frac{\|\Phi^{l,*}_{\mathcal{T}}(d^l v_{\mathcal{T}})\|}{\|d^l v_{\mathcal{T}}\|}.$$

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Theorem (Equivalence: DPI – bounds of operator norms)

Best discrete Poincaré inequality constant $C^{\mathrm{d},l}_{\mathrm{P}} h_\omega = $ *the operator norm* $\left|\!\left|\!\left| \Phi^{l,*}_{\mathcal{T}} \right|\!\right|\!\right|$.

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Best discrete Poincaré inequality constant $C^{\mathrm{d},l}_{\mathrm{P}} h_\omega = $ *the operator norm* $\left\|\left|\Phi^{l,*}_{\mathcal{T}}\right|\right\|$.

Proof.

- the $L^2(\omega)$-norm minimizing operator is the minimizer from (1)

Outline

Graph-stable commuting projectors

Theorem (Equivalence: DPI – graph-stable commuting projectors)

The DPI are equivalent to the existence of projectors $\Pi_p^m : V^m(\omega) \to V_p^m(\mathcal{T}_\omega)$, $m \in \{1{:}3\}$, *satisfying for all* $l \in \{1{:}2\}$ *the* **commuting**

$$d^l(\Pi_p^l(u)) = \Pi_p^{l+1}(d^l u) \qquad \forall u \in V^l(\omega)$$

*and the graph-***stability**

$$\|\Pi_p^l(u)\|_{V^l(\omega)} \leq C_{\mathrm{st}} \|u\|_{V^l(\omega)} \qquad \forall u \in V^l(\omega).$$

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Proof.

- stable projectors $\Longrightarrow$ DPI: shown later
- DPI $\Longrightarrow$ stable projectors:
  - define the (global) projector

$$\Pi_p^l(u) := \operatorname*{arg\,min}_{\substack{v_\mathcal{T} \in V_p^l(\mathcal{T}_\omega) \\ d^l v_\mathcal{T} = \Pi_p^{l+1}(d^l u)}} \|u - v_\mathcal{T}\|^2$$

  - show it is stable in the graph norm

Outline

Setting and notation

PI

$$\|u\| \leq C_{\mathrm{P}}^{l} h_{\omega} \|d^{l} u\| \qquad \forall u \in V^{l}(\omega) \text{ such that } \langle u, v \rangle = 0$$
$$\forall v \in V^{l}(\omega) \text{ with } d^{l} v = 0, \qquad l \in \{1{:}2\}$$

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$$\|u_{\mathcal{T}}\| \le C_{\mathrm{P}}^{\mathrm{d},l} h_\omega \|d^l u_{\mathcal{T}}\| \qquad \forall u_{\mathcal{T}} \in V_p^l(\mathcal{T}_\omega) \text{ such that } \langle u_{\mathcal{T}}, v_{\mathcal{T}}\rangle = 0$$
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$\mathfrak{Z} V_p^l(\mathcal{T}_\omega) := \{v_{\mathcal{T}} \in V_p^l(\mathcal{T}_\omega) : d^l v_{\mathcal{T}} = 0\}$

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Discrete kernel orthogonal complement

$\mathfrak{Z}^{\perp} V_p^l(\mathcal{T}_\omega) := \{ u_{\mathcal{T}} \in V_p^l(\mathcal{T}_\omega) : \langle u_{\mathcal{T}}, v_{\mathcal{T}} \rangle = 0 \quad \forall v_{\mathcal{T}} \in \mathfrak{Z} V_p^l(\mathcal{T}_\omega) \}$

Outline

Broken polynomial extension operators $\Longrightarrow$ DPI

Theorem (Broken polynomial extension operators $\Longrightarrow$ DPI)

Assume that there is $C^l_{\min}$ such that, for all $r_\mathcal{T} \in d^l(V^l_p(\mathcal{T}_\omega))$, there holds

$$\underbrace{\min_{\substack{v_\mathcal{T} \in V^l_p(\mathcal{T}_\omega)\\ d^l v_\mathcal{T} = r_\mathcal{T}}}}_{u^*_\mathcal{T} := \arg} \|v_\mathcal{T}\| \leq C^l_{\min} \underbrace{\min_{\substack{v \in V^l(\omega)\\ d^l v = r_\mathcal{T}}}}_{u^* := \arg} \|v\| .$$

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Then the DPI hold with constants $C^{d,l}_{\mathrm{P}} \leq C^l_{\min} C^l_{\mathrm{P}}$.

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Then the DPI hold with constants $C^{\mathrm{d},l}_{\mathrm{P}} \leq C^l_{\min} C^l_{\mathrm{P}}$.

Proof.

Let $u_\mathcal{T} \in \mathfrak{Z}^\perp V^l_p(\mathcal{T}_\omega)$. Set $r_\mathcal{T} := d^l u_\mathcal{T} \in d^l(V^l_p(\mathcal{T}_\omega))$. Then

$$\|u_\mathcal{T}\| = \|u^*_\mathcal{T}\| \leq C^l_{\min}\|u^*\| \leq C^l_{\min} C^l_{\mathrm{P}} h_\omega \|d^l u^*\| = C^l_{\min} C^l_{\mathrm{P}} h_\omega \|d^l u_\mathcal{T}\|.$$

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- $r_\mathcal{T}$ are piecewise polynomials: **broken polynomial extension operators** by Braess, Pillwein, and Schöberl (2009), Ern and Vohralík (2020), Chaumont-Frelet, Ern, and Vohralík (2022), Chaumont-Frelet and Vohralík (2026), Demkowicz and Vohralík (2026) if $\overline{\omega}$ is contractible
- $C^l_{\min}$ only **depend** on the number of tetrahedra $|\mathcal{T}_\omega|$ and the shape-regularity $\rho_{\mathcal{T}_\omega}$ (only on $\rho_{\mathcal{T}_\omega}$ for patches tetrahedra)
- $C^l_{\min}$, and consequently $C^{\mathrm{d},l}_{\mathrm{P}}$, is **independent** of the polynomial degree p

Outline

Stable commuting projectors $\Longrightarrow$ DPI

Theorem (Stable commuting projectors $\Longrightarrow$ DPI)

Assume that there are projectors $\Pi_p^m : V^m(\omega) \to V_p^m(\mathcal{T}_\omega)$, $m \in \{1{:}3\}$, *satisfying, for all* $l \in \{1{:}2\}$, **commuting**

$$d^l(\Pi_p^l(u)) = \Pi_p^{l+1}(d^l u) \qquad \forall u \in V^l(\omega),$$

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and L^2-**stability** *on functions whose* **image by d^l is piecewise polynomial**

$$\|\Pi_p^l(u)\| \le C_{\rm st}\|u\| \qquad \forall u \in V^l(\omega) \text{ such that } d^l u \in V_p^{l+1}(\mathcal{T}_\omega).$$

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Proof.

Let $u_{\mathcal{T}} \in \mathfrak{Z}^\perp V_p^l(\mathcal{T}_\omega)$. Set $r_{\mathcal{T}} := d^l u_{\mathcal{T}} \in d^l(V_p^l(\mathcal{T}_\omega))$.

- $\Pi_p^l(u^*) \in V_p^l(\mathcal{T}_\omega)$ by definition, $d^l u^* = r_{\mathcal{T}} \in V_p^{l+1}(\mathcal{T}_\omega)$
- $d^l(\Pi_p^l(u^*)) = \Pi_p^{l+1}(d^l u^*) = \Pi_p^{l+1}(r_{\mathcal{T}}) = r_{\mathcal{T}}$, using commuting and projection
- L^2-stability and PI imply

$$\|u_{\mathcal{T}}\| = \|u_{\mathcal{T}}^*\| \leq \|\Pi_p^l(u^*)\| \leq C_{\mathrm{st}}\|u^*\| \leq C_{\mathrm{st}} C_{\mathrm{P}}^l h_\omega \|r_{\mathcal{T}}\|$$

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- L^2- (piecewise polynomial data) **stable commuting projectors** by Arnold, and Guzmán (2021), Ern, Gudi, Smears, and Vohralík (2022), Chaumont-Frelet and Vohralík (2024), Ern, Guzmán, Potu, and Vohralík (2026): $C_{\rm st}^l$ only **depend** on the polynomial degree p and the shape-regularity $\rho_{\mathcal{T}_\omega}$ but are **independent** of the number of tetrahedra $|\mathcal{T}_\omega|$

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- Demkowicz and Vohralík (2026): C_{st}^2, and consequently $C_{\mathrm{P}}^{\mathrm{d},2}$, ($\boldsymbol{H}(\mathrm{div}, \omega)$ case), **only depend** on $\rho_{\mathcal{T}_\omega}$ and are thus **independent** of p and $|\mathcal{T}_\omega|$

Outline

Piecewise Piola transformations $\Longrightarrow$ DPI

Theorem (Piecewise Piola transformations $\Longrightarrow$ DPI)

DPI hold with constants that only **depend** *on the shape-regularity parameter* $\rho_{\mathcal{T}_\omega}$, *the number of tetrahedra* $|\mathcal{T}_\omega|$, *and the polynomial degree* p, *but* **do not need to invoke** *the constants* C_{P}^{l}.

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DPI hold with constants that only **depend** *on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, the number of tetrahedra $|\mathcal{T}_\omega|$, and the polynomial degree p, but* **do not need to invoke** *the constants C_{P}^{l}.*

Proof.

- collection of reference patches
- piecewise Piola transformations
- equivalence of norms in finite dimension

Outline

Conclusions and outlook

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- equivalence of five seemingly disconnected properties
- optimal result: $C_{\mathrm{P}}^{\mathrm{d},2}$ **only depends** on $\rho_{\mathcal{T}_\omega}$

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- $C_{\mathrm{P}}^{\mathrm{d},1}$ **only depends** on $\rho_{\mathcal{T}_\omega}$
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References

ERN A., GUZMÁN J., POTU, P., AND VOHRALÍK M., Discrete Poincaré inequalities: a review on proofs, equivalent formulations, and behavior of constants, *IMA J. Numer. Anal.* (2025), DOI 10.1093/imanum/draf089.

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References

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Thank you for your attention!