A posteriori algebraic error estimates and nonoverlapping domain decomposition in mixed formulations: energy coarse grid balancing, local mass conservation on each step, and line search

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Outline

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1. Introduction
   - The Darcy model problem and its mixed finite element approximation
   - (DD) solvers for mixed finite elements
2. Flux equilibration: coarse mesh constrained energy minimization & subdomain Neumann solves
3. Nonoverlapping domain decomposition: a posteriori error estimates, local mass conservation on each step, and Pythagorean error decrease via line search
4. Properties
5. Numerical experiments
6. Conclusions

The model problem

The Darcy porous media flow problem

Find the pressure head $p : \Omega \to \mathbb{R}$ *and the Darcy velocity* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *such that*

$$\begin{aligned} \boldsymbol{u} &= -\boldsymbol{S}\nabla p && \text{in } \Omega, \\ \nabla\cdot \boldsymbol{u} &= f && \text{in } \Omega, \\ p &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

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Setting

- $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$: interval/polygon/polyhedron
- $\boldsymbol{S}$: symmetric and positive definite diffusion tensor, piecewise constant for simplicity
- f: source term, piecewise constant for simplicity
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- $|||\boldsymbol{v}|||_{\mathcal{D}} := \left\|\boldsymbol{S}^{-\frac{1}{2}}\boldsymbol{v}\right\|_{\mathcal{D}}$: energy norm

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Dual finite element approximation

$$\boxed{\mathbf{u}_h := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_{h,g_{\mathrm{N}}} \\ \nabla\cdot \mathbf{v}_h = f}} |||\mathbf{v}_h|||^2}$$

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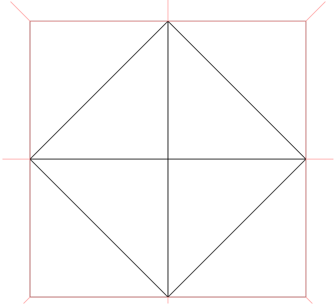
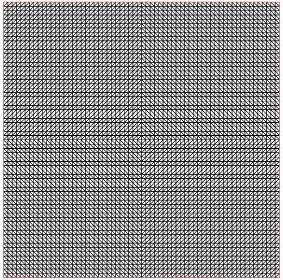


Coarse mesh $\mathcal{T}_H$: subdomains Ω_i



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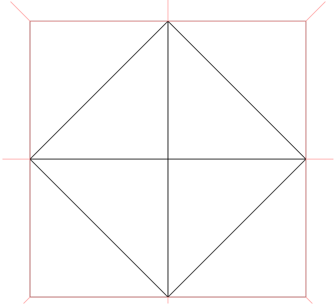

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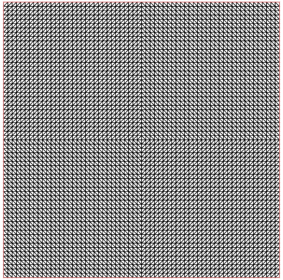

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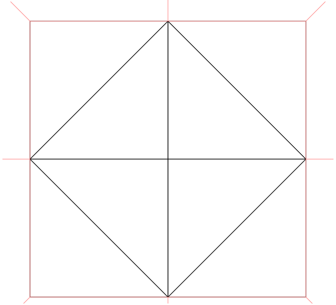

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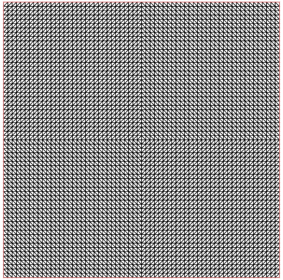

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Domain decomposition solvers for mixed finite elements

Saddle-point solvers

- after a choice of basis: find algebraic vectors U and P such that

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- symmetric and positive definite system matrix
- preconditioned conjugate gradients possible, DD possible but Λ (face pressure heads) are nonconforming and in non-nested spaces (in the multigrid setting cf. Brenner (1992), Chen (1996), Wheeler, Yotov (2000), . . .)

A few central reflections

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  - system of linear algebraic equations, **properties depend** on the **basis**
  - analysis restricted to **linear algebraic information** and **tools**
- saddle-point **indefinite** matrix / SPD system on **nonconforming non-nested spaces**:
  - problems of **balancing**
  - problems of **interior node compatibility**

Our approach

- basis-independent approach:
  - functional writing, **basis-independent**
  - analysis can exploit **function spaces information** and **tools**
- natural **balancing (equilibration)**:
  - **coarse mesh**
  - **constrained energy minimization**
  - **subdomain Neumann** solvers
- additive Schwarz: **subdomain Dirichlet**
- **line search**:
  - **optimal step size**
  - **Pythagoras formula** for **error decrease** on **each iteration**

A few central reflections

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- first choose a basis:
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  - **Pythagoras formula** for **error decrease** on **each iteration**
- **built-in** a posteriori **estimate** on the **algebraic error**: solver **adaptivity**

Outline

Equilibration: the principle

Equilibration

Let $(\boldsymbol{u}_h^j, p_h^j) \in \boldsymbol{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot \boldsymbol{u}_h^j \neq f$ *be* **arbitrary**.

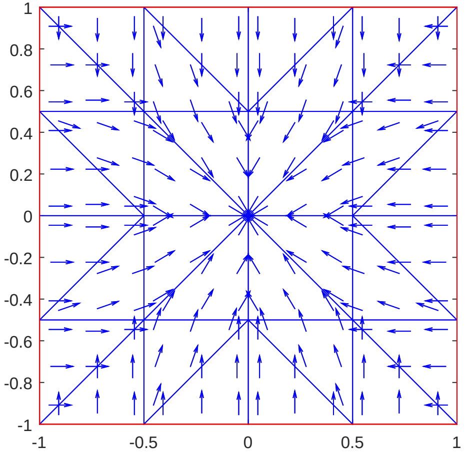
Equilibration: the principle

Equilibration

Let $(\boldsymbol{u}_h^j, p_h^j) \in \boldsymbol{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot \boldsymbol{u}_h^j \neq f$ *be* **arbitrary**. *Construct*

$$\boxed{\mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) \in \boldsymbol{V}_{h,g_{\mathrm{N}}}, \nabla\cdot \mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) = f.}$$

Equilibration: example



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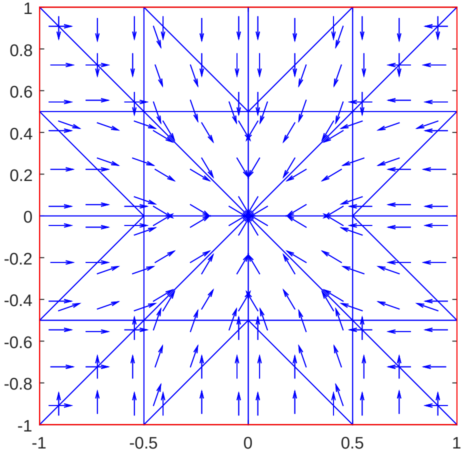

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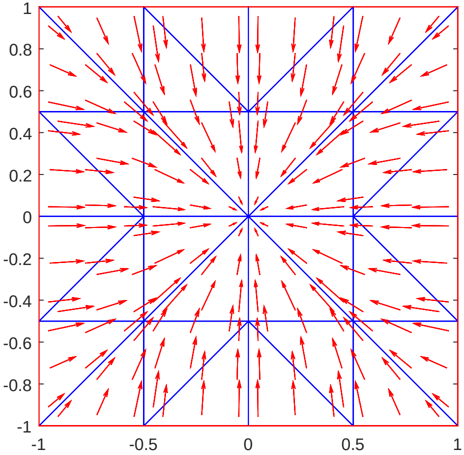

$\mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) \in \boldsymbol{V}_{h,g_{\mathrm{N}}}, \; \nabla\cdot \mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) = f$

Equilibration: details 1/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

Let $(\mathbf{u}_h^j, p_h^j) \in \mathbf{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot \mathbf{u}_h^j \neq f$, *be* **arbitrary**.

Equilibration: details 1/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

Let $(\mathbf{u}_h^j, p_h^j) \in \mathbf{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot \mathbf{u}_h^j \neq f$, *be* **arbitrary**. *Proceed in four steps.*

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1. *Averaging on mesh faces*

Equilibration: details 1/4

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1. *Averaging on mesh faces*

Create $\boldsymbol{u}_h^{j,1} \in \boldsymbol{V}_{h,g_{\mathrm{N}}}$ *such that*

$$\boldsymbol{u}_h^{j,1}\cdot\boldsymbol{n}_F := \begin{cases} \{\!\{\boldsymbol{u}_h^j\cdot\boldsymbol{n}_F\}\!\} & F \in \mathcal{F}_h^{\mathrm{int}}, \\ \boldsymbol{u}_h^j\cdot\boldsymbol{n}_F & F \in \mathcal{F}_h^{\mathrm{D}}, \\ g_{\mathrm{N}} & F \in \mathcal{F}_h^{\mathrm{N}}, \end{cases}$$

Equilibration: details 1/4

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Equilibration: details 1/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

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Equilibration: details 1/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

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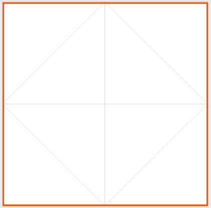
There holds $\boldsymbol{u}_h^{j,1} \in \boldsymbol{V}_{h,g_{\mathrm{N}}}$ *but* $\nabla\cdot\boldsymbol{u}_h^{j,1} \neq f$.

Equilibration: details 2/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

2 *Coarse grid solver*

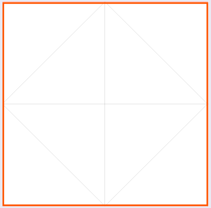


Equilibration: details 2/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

② *Coarse grid solver*

$$\delta_H^{j,2} := \arg \min_{\substack{\mathbf{v}_H \in \mathbf{V}_{H,0} \\ \nabla\cdot \mathbf{v}_H = \Pi_H(f - \nabla\cdot \mathbf{u}_h^{j,1})}} |||\mathbf{v}_H + \mathbf{u}_h^{j,1}|||^2$$



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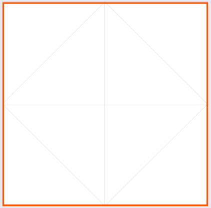
$$(\nabla\cdot \delta_H^{j,2}, w_H) = \overbrace{(f - \nabla\cdot \boldsymbol{u}_h^{j,1}, w_H)}^{\text{residual}} \qquad \forall w_H \in W_H.$$



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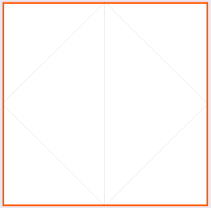
Denote

$$\boldsymbol{u}_h^{j,2} := \boldsymbol{u}_h^{j,1} + \delta_H^{j,2} \in \boldsymbol{V}_{h,g_{\mathrm{N}}}, \quad p_h^{j,2} := p_h^{j,1} + r_H^{j,2} \in W_h;$$



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$\boldsymbol{u}_h^{j,2}$ *satisfies the weak divergence constraint*

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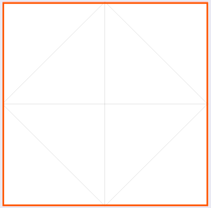

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Equilibration via coarse constrained energy minimization & subdomain Neumann s.

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If $\nabla\cdot \mathbf{u}_h^{j,1} = f$,

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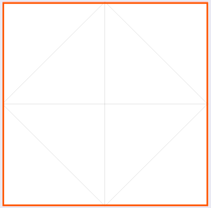


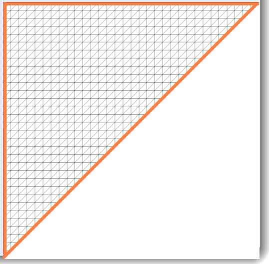
Equilibration: details 3/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

③ *Subdomain Neumann solver*

On all subdomains Ω_i, find $(\delta_h^{j,3}, r_h^{j,3})|_{\Omega_i} \in \mathbf{V}_{i,h,0} \times W_{i,h}$ such that

$$\begin{aligned}(\mathbf{S}^{-1}\delta_h^{j,3}, \mathbf{v}_h)_{\Omega_i} - (r_h^{j,3}, \nabla\cdot \mathbf{v}_h)_{\Omega_i} &= (p_h^{j,2}, \nabla\cdot \mathbf{v}_h)_{\Omega_i} - (\mathbf{S}^{-1}\mathbf{u}_h^{j,2}, \mathbf{v}_h)_{\Omega_i} \qquad &\forall \mathbf{v}_h \in \mathbf{V}_{i,h,0},\\ (\nabla\cdot \delta_h^{j,3}, w_h)_{\Omega_i} &= (f - \nabla\cdot \mathbf{u}_h^{j,2}, w_h)_{\Omega_i} \qquad &\forall w_h \in W_{i,h}.\end{aligned}$$



Equilibration: details 3/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

3 *Subdomain Neumann solver*

On all subdomains Ω_i*, find* $(\delta_h^{j,3}, r_h^{j,3})|_{\Omega_i} \in \mathbf{V}_{i,h,0} \times W_{i,h}$ *such that*

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Update

$$\mathbf{u}_h^{j,3} := \mathbf{u}_h^{j,2} + \delta_h^{j,3} \in \mathbf{V}_{h,g_{\mathrm{N}}}, \nabla\cdot\mathbf{u}_h^{j,3} = f,$$

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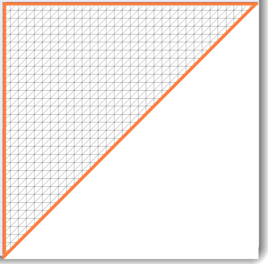

Equilibration: details 3/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

3 *Subdomain Neumann solver*

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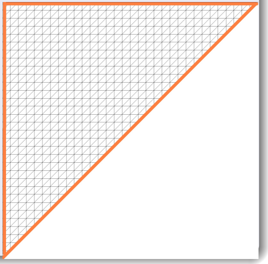


If $\nabla\cdot \boldsymbol{u}_h^{j,2} = f$, *there holds*

$$|||\boldsymbol{u}_h - \boldsymbol{u}_h^{j,3}|||^2 = |||\boldsymbol{u}_h - \boldsymbol{u}_h^{j,2}|||^2 - |||\delta_h^{j,3}|||^2.$$

Equilibration: details 4/4

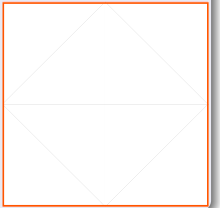
Equilibration via coarse constrained energy minimization & subdomain Neumann s.

4 *Coarse grid correction*

Compute $(\delta_H^{j,4}, r_H^{j,4}) \in \mathbf{V}_{H,0} \times W_H$ *such that*

$$(\mathbf{S}^{-1}\delta_H^{j,4}, \mathbf{v}_H) - (r_H^{j,4}, \nabla\cdot \mathbf{v}_H) = (p_h^{j,3}, \nabla\cdot \mathbf{v}_H) - (\mathbf{S}^{-1}\mathbf{u}_h^{j,3}, \mathbf{v}_H) \qquad \forall \mathbf{v}_H \in \mathbf{V}_{H,0},$$

$$(\nabla\cdot \delta_H^{j,4}, w_H) = 0 \qquad \forall w_H \in \mathbf{V}_{H,0}.$$



Equilibration: details 4/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

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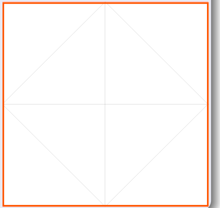
$$(\nabla\cdot \delta_H^{j,4}, w_H) = 0 \qquad \forall w_H \in \boldsymbol{V}_{H,0}.$$

Define

$$\mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) := \boldsymbol{u}_h^{j,3} + \delta_H^{j,4} \in \boldsymbol{V}_{h,g_{\mathrm{N}}}, \nabla\cdot \mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) = f,$$

$$\mathcal{R}_{\mathrm{P}}(\boldsymbol{u}_h^j, p_h^j) := p_h^{j,3} + r_H^{j,4} \in W_h.$$



Equilibration: details 4/4

Equilibration via coarse constrained energy minimization & subdomain Neumann s.

④ *Coarse grid correction*

Compute $(\delta_H^{j,4}, r_H^{j,4}) \in \boldsymbol{V}_{H,0} \times W_H$ *such that*

$$(\boldsymbol{S}^{-1}\delta_H^{j,4}, \boldsymbol{v}_H) - (r_H^{j,4}, \nabla\cdot \boldsymbol{v}_H) = (p_h^{j,3}, \nabla\cdot \boldsymbol{v}_H) - (\boldsymbol{S}^{-1}\boldsymbol{u}_h^{j,3}, \boldsymbol{v}_H) \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_{H,0},$$

$$(\nabla\cdot \delta_H^{j,4}, w_H) = 0 \qquad \forall w_H \in \boldsymbol{V}_{H,0}.$$

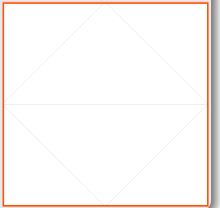
Define

$$\mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) := \boldsymbol{u}_h^{j,3} + \delta_H^{j,4} \in \boldsymbol{V}_{h,g_{\mathrm{N}}}, \nabla\cdot \mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j) = f,$$

$$\mathcal{R}_{\mathrm{P}}(\boldsymbol{u}_h^j, p_h^j) := p_h^{j,3} + r_H^{j,4} \in W_h.$$

There holds

$$\boxed{|||\boldsymbol{u}_h - \mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^j, p_h^j)|||^2 = |||\boldsymbol{u}_h - \boldsymbol{u}_h^{j,3}|||^2 - |||\delta_H^{j,4}|||^2.}$$



Outline

Domain decomposition 0/3

Nonoverlapping domain decomposition

Let $(\mathbf{u}_h^0, p_h^0) \in \mathbf{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot \mathbf{u}_h^0 \neq f$, *be an* **arbitrary initial guess**.

Domain decomposition 0/3

Nonoverlapping domain decomposition

Let $(\boldsymbol{u}_h^0, p_h^0) \in \boldsymbol{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot \boldsymbol{u}_h^0 \neq f$, *be an* **arbitrary initial guess**.

0 *Equilibration*

Set $(\boldsymbol{u}_h^1, p_h^1) := \mathcal{R}_{\mathrm{FP}}(\boldsymbol{u}_h^0, p_h^0)$.

Domain decomposition 0/3

Nonoverlapping domain decomposition

Let $(\mathbf{u}_h^0, p_h^0) \in \mathbf{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot\mathbf{u}_h^0 \neq f$, *be an* **arbitrary initial guess**.

⓪ *Equilibration*

Set $(\mathbf{u}_h^1, p_h^1) := \mathcal{R}_{\mathrm{FP}}(\mathbf{u}_h^0, p_h^0)$. *This gives* $\mathbf{u}_h^1 \in \mathbf{V}_{h,g_{\mathrm{N}}}$ *with* $\nabla\cdot\mathbf{u}_h^1 = f$. *All subsequent iterates will retain* $\mathbf{u}_h^i \in \mathbf{V}_{h,g_{\mathrm{N}}}$ *with* $\nabla\cdot\mathbf{u}_h^i = f$.

Domain decomposition 0/3

Nonoverlapping domain decomposition

Let $(\mathbf{u}_h^0, p_h^0) \in \mathbf{V}_h^{\mathrm{dc}} \times W_h$, $\nabla\cdot \mathbf{u}_h^0 \neq f$, *be an* **arbitrary initial guess**.

0 *Equilibration*

Set $(\mathbf{u}_h^1, p_h^1) := \mathcal{R}_{\mathrm{FP}}(\mathbf{u}_h^0, p_h^0)$. *This gives* $\mathbf{u}_h^1 \in \mathbf{V}_{h,g_{\mathrm{N}}}$ *with* $\nabla\cdot \mathbf{u}_h^1 = f$. *All subsequent iterates will retain* $\mathbf{u}_h^j \in \mathbf{V}_{h,g_{\mathrm{N}}}$ *with* $\nabla\cdot \mathbf{u}_h^j = f$.

Set $j = 1$ *and on each iteration* j*, proceed in three steps.*

Domain decomposition 1/3

Nonoverlapping domain decomposition

1 *Elementwise trace reconstruction*

Compute the associated Lagrange multiplier $\lambda_h^j \in \Psi_h^{\mathrm{dc}}$:

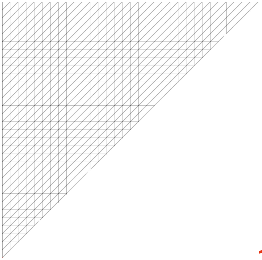
$$\langle \lambda_h^j, \boldsymbol{v}_h \cdot \boldsymbol{n}_K \rangle_F = (p_h^j, \nabla \cdot \boldsymbol{v}_h)_K - (\boldsymbol{S}^{-1} \boldsymbol{u}_h^j, \boldsymbol{v}_h)_K \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(K, F),\ K \in \mathcal{T}_h,\ F \in \mathcal{F}_K.$$

Domain decomposition 2/3

Nonoverlapping domain decomposition

② *Subdomain Dirichlet solver*

On all subdomains Ω_i, *construct* $(\delta_h^j, r_h^j)|_{\Omega_i} \in \mathbf{V}_{i,h} \times W_{i,h}$ *such that*



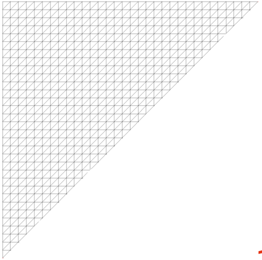
Domain decomposition 2/3

Nonoverlapping domain decomposition

② *Subdomain Dirichlet solver*

On all subdomains Ω_i, construct $(\delta_h^j, r_h^j)|_{\Omega_i} \in \boldsymbol{V}_{i,h} \times W_{i,h}$ such that

$$
\begin{aligned}
(\boldsymbol{S}^{-1}\delta_h^j, \boldsymbol{v}_h)_{\Omega_i} - (r_h^j, \nabla\cdot \boldsymbol{v}_h)_{\Omega_i} = (p_h^j, \nabla\cdot \boldsymbol{v}_h)_{\Omega_i} - (\boldsymbol{S}^{-1}\boldsymbol{u}_h^j, \boldsymbol{v}_h)_{\Omega_i} \\
- \langle \{\!\{\lambda_h^j\}\!\}, \boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega_i} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{i,h}, \\
(\nabla\cdot \delta_h^j, w_h)_{\Omega_i} = 0 \qquad \forall w_h \in W_{i,h}.
\end{aligned}
$$



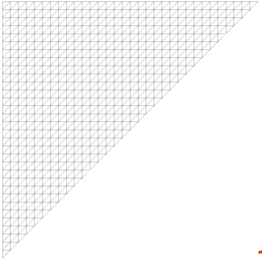
Domain decomposition 2/3

Nonoverlapping domain decomposition

2 *Subdomain Dirichlet solver*

On all subdomains Ω_i, construct $(\delta_h^j, r_h^j)|_{\Omega_i} \in \boldsymbol{V}_{i,h} \times W_{i,h}$ such that ($\delta_h^j \notin \boldsymbol{V}_{h,0}$):

$$\begin{aligned}
(\boldsymbol{S}^{-1}\delta_h^j, \boldsymbol{v}_h)_{\Omega_i} - (r_h^j, \nabla\cdot \boldsymbol{v}_h)_{\Omega_i} &= (p_h^j, \nabla\cdot \boldsymbol{v}_h)_{\Omega_i} - (\boldsymbol{S}^{-1}\boldsymbol{u}_h^j, \boldsymbol{v}_h)_{\Omega_i} \\
&\quad - \langle \{\!\{\lambda_h^j\}\!\}, \boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega_i} \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{i,h}, \\
(\nabla\cdot \delta_h^j, w_h)_{\Omega_i} &= 0 \qquad \forall w_h \in W_{i,h}.
\end{aligned}$$

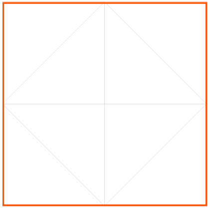


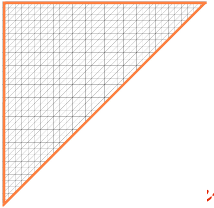
Domain decomposition 3/3

Nonoverlapping domain decomposition

③ *Equilibration and line search*

Equilibration: $(\hat{\boldsymbol{u}}_h^j, \hat{p}_h^j) := \mathcal{R}_{\mathrm{FP}}(\boldsymbol{u}_h^j + \boldsymbol{\delta}_h^j, p_h^j + r_h^j) \in \boldsymbol{V}_{h,g_{\mathrm{N}}} \times W_h,\ \nabla\cdot \hat{\boldsymbol{u}}_h^j = f.$





Domain decomposition 3/3

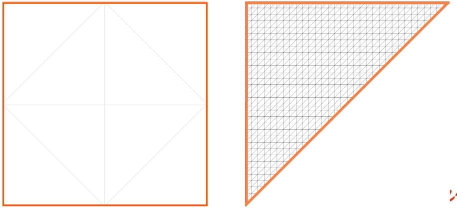
Nonoverlapping domain decomposition

③ *Equilibration and line search*

Equilibration: $(\hat{\boldsymbol{u}}_h^j, \hat{p}_h^j) := \mathcal{R}_{\mathrm{FP}}(\boldsymbol{u}_h^j + \boldsymbol{\delta}_h^j, p_h^j + r_h^j) \in \boldsymbol{V}_{h,g_{\mathrm{N}}} \times W_h,\ \nabla\cdot\hat{\boldsymbol{u}}_h^j = f.$

Line search

$$\alpha^j := \arg\min_{\alpha\in\mathbb{R}} |||\boldsymbol{u}_h - (\boldsymbol{u}_h^j + \alpha(\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j))|||^2$$



Domain decomposition 3/3

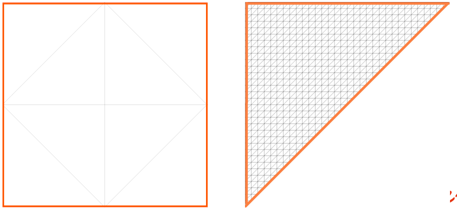
Nonoverlapping domain decomposition

③ *Equilibration and line search*

Equilibration: $(\hat{\boldsymbol{u}}_h^j, \hat{p}_h^j) := \mathcal{R}_{\mathrm{FP}}(\boldsymbol{u}_h^j + \delta_h^j, p_h^j + r_h^j) \in \boldsymbol{V}_{h,g_{\mathrm{N}}} \times W_h$, $\nabla\cdot \hat{\boldsymbol{u}}_h^j = f$.

Line search

$$\alpha^j := \arg\min_{\alpha\in\mathbb{R}} |||\boldsymbol{u}_h - (\boldsymbol{u}_h^j + \alpha(\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j))|||^2 \Longrightarrow \alpha^j := -\frac{(\boldsymbol{S}^{-1}\boldsymbol{u}_h^j, \hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j)}{|||\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j|||^2}.$$



Domain decomposition 3/3

Nonoverlapping domain decomposition

③ *Equilibration and line search*

Equilibration: $(\hat{\boldsymbol{u}}_h^j, \hat{p}_h^j) := \mathcal{R}_{\mathrm{FP}}(\boldsymbol{u}_h^j + \boldsymbol{\delta}_h^j, p_h^j + r_h^j) \in \boldsymbol{V}_{h,g_{\mathrm{N}}} \times W_h,\ \nabla\cdot\hat{\boldsymbol{u}}_h^j = f.$

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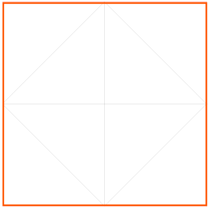
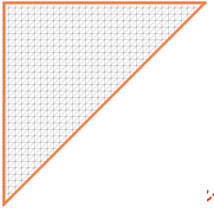
$$\alpha^j := \arg\min_{\alpha\in\mathbb{R}} |||\boldsymbol{u}_h - (\boldsymbol{u}_h^j + \alpha(\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j))|||^2 \Longrightarrow \alpha^j := -\frac{(\boldsymbol{S}^{-1}\boldsymbol{u}_h^j, \hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j)}{|||\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j|||^2}.$$

Update

$$\boldsymbol{u}_h^{j+1} := \boldsymbol{u}_h^j + \alpha^j(\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j) \in \boldsymbol{V}_{h,g_{\mathrm{N}}},\quad \nabla\cdot\boldsymbol{u}_h^{j+1} = f,$$

$$p_h^{j+1} := p_h^j + \alpha^j(\hat{p}_h^j - p_h^j) \in W_h.$$





Outline

Computable error decrease formula

Theorem (Error decrease formula)

There holds

$$|||\mathbf{u}_h - \mathbf{u}_h^{j+1}|||^2 = |||\mathbf{u}_h - \mathbf{u}_h^{j}|||^2 - \underbrace{(\underline{\eta}^j)^2}_{\alpha^j|||\hat{\mathbf{u}}_h^j - \mathbf{u}_h^j|||} .$$

A posteriori estimates on the algebraic error

Theorem (Guaranteed a posteriori algebraic error estimates)

Let

$$\underline{\eta}^j := \alpha^j |||\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j|||$$

and

$$\eta^j := |||\boldsymbol{u}_h^j + \Pi_k^{\mathcal{RTN}}(\boldsymbol{S}\nabla \tilde{p}_h^{j+1})|||, \quad \tilde{p}_h^{j+1} := \widetilde{\mathcal{R}}_{\mathrm{P}}(p_h^{j+1}, \lambda_h^{j+1}).$$

A posteriori estimates on the algebraic error

Theorem (Guaranteed a posteriori algebraic error estimates)

Let

$$\underline{\eta}^j := \alpha^j |||\hat{\boldsymbol{u}}_h^j - \boldsymbol{u}_h^j|||$$

and

$$\eta^j := |||\boldsymbol{u}_h^j + \Pi_k^{\mathcal{RTN}}(\boldsymbol{S}\nabla\tilde{p}_h^{j+1})|||, \quad \tilde{p}_h^{j+1} := \widetilde{\mathcal{R}}_{\mathrm{P}}(p_h^{j+1}, \lambda_h^{j+1}).$$

Then there holds

$$\boxed{\underline{\eta}^j \leq |||\boldsymbol{u}_h - \boldsymbol{u}_h^j||| \leq \eta^j.}$$

Outline

Uniform diffusion

Setting ($k = 0$)

- $\Omega = (0,1) \times (0,1)$
- $\boldsymbol{S} = \mathrm{Id}$
- $f(x,y) = -2(x^2 + y^2) + 2(x+y)$
- $\Gamma_{\mathrm{D}} = \partial\Omega$
- zero initial guess $(\boldsymbol{u}_h^0, p_h^0) = (0,0)$

Uniform diffusion

Setting ($k = 0$)

- $\Omega = (0,1) \times (0,1)$
- $\boldsymbol{S} = \mathrm{Id}$
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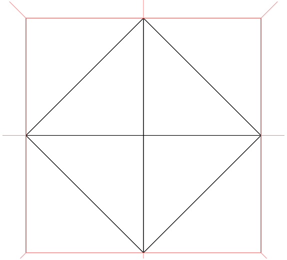
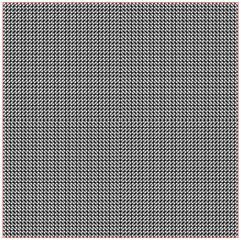
- $\Gamma_{\mathrm{D}} = \partial\Omega$
- zero initial guess $(\boldsymbol{u}_h^0, p_h^0) = (0, 0)$

Meshes

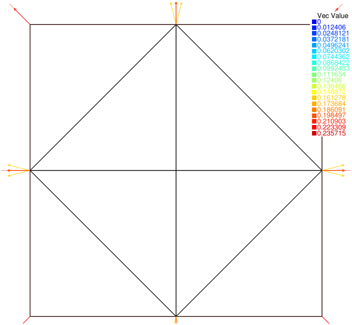


Coarse mesh $\mathcal{T}_H$: subdomains Ω_i

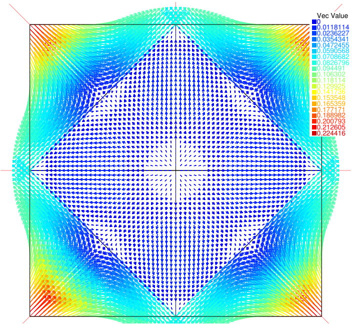


Fine meshes $\mathcal{T}_{i,h}$ forming $\mathcal{T}_h$

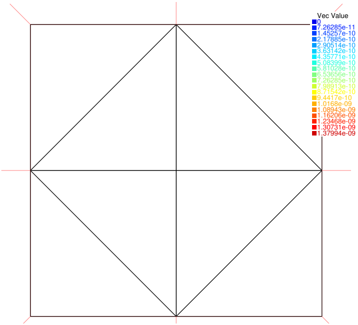
Initialization (equilibration): lifted residuals



$\delta_H^{0,2}$

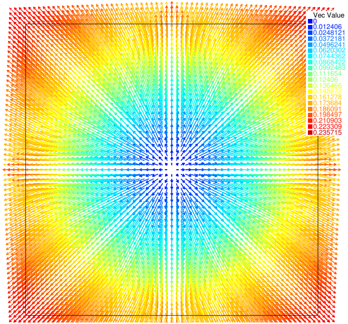


$\delta_h^{0,3}$

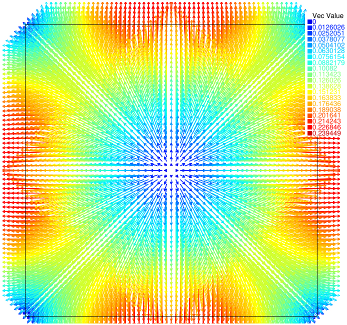


$\delta_H^{0,4}$

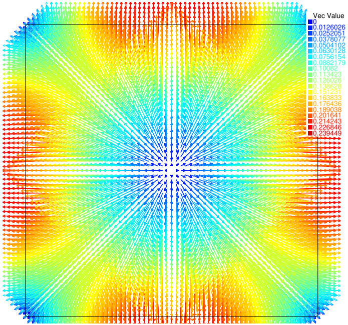
Initialization (equilibration): intermediate fluxes



$$\boldsymbol{u}_h^{0,2} = \delta_H^{0,2}$$

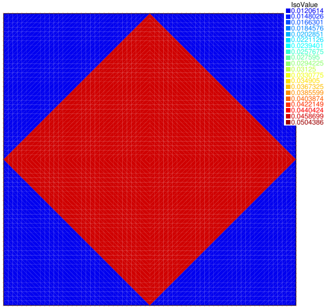


$$\boldsymbol{u}_h^{0,3} = \boldsymbol{u}_h^{0,2} + \delta_h^{0,3}$$

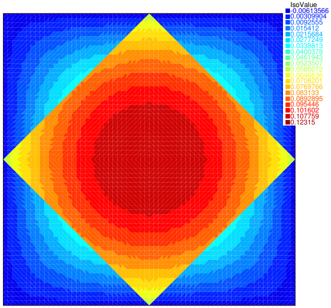


$$\boldsymbol{u}_h^{1} = \mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^0, p_h^0) = \boldsymbol{u}_h^{0,3} + \delta_H^{0,4}$$

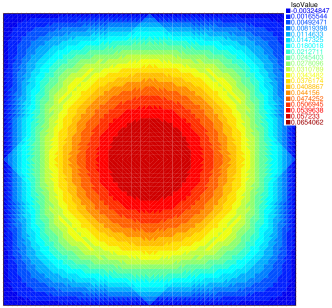
Initialization (equilibration): intermediate potentials



$p_h^{0,2} = r_H^{0,2}$

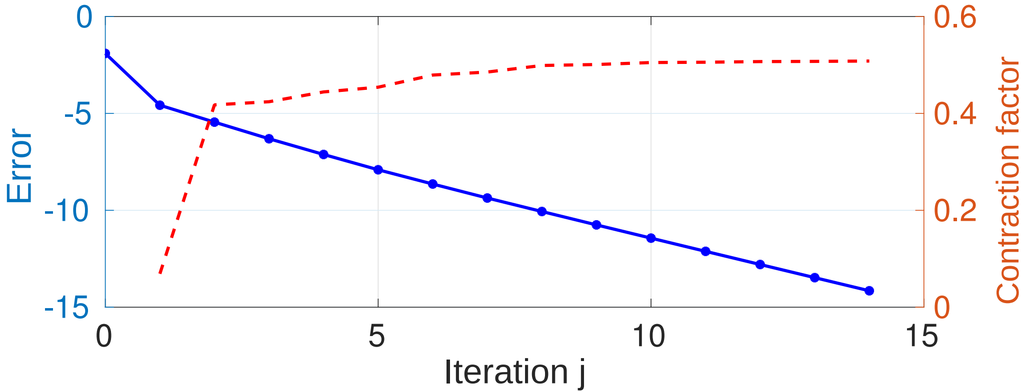


$p_h^{0,3} = p_h^{0,2} + r_h^{0,3}$



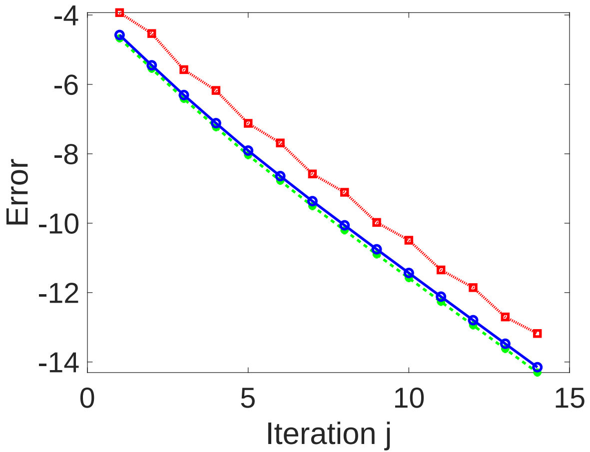
$p_h^1 = \mathcal{R}_{\mathrm{P}}(\boldsymbol{u}_h^0, p_h^0) = p_h^{0,3} + r_H^{0,4}$

Error decrease and contraction factor



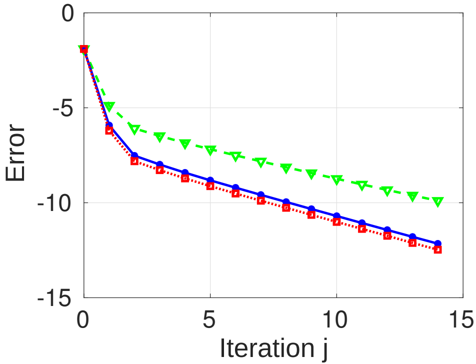
Error $|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$ and contraction factor $|||\boldsymbol{u}_h - \boldsymbol{u}_h^{j+1}|||/|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$

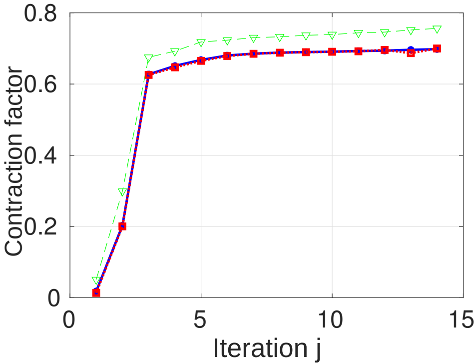
A posteriori estimates of the algebraic error



Error $|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$ (blue), upper bound η^j (red), and lower bound $\underline{\eta}^j$ (green)

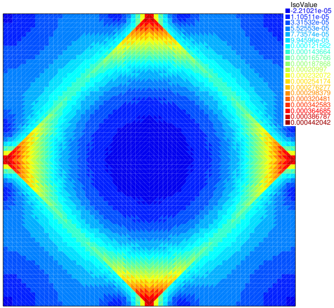
Scalability





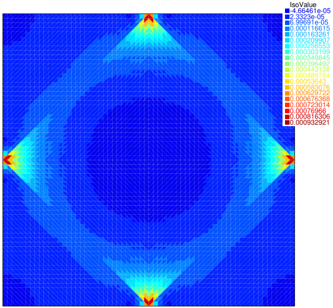
Errors $|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$ and contraction factors $|||\boldsymbol{u}_h - \boldsymbol{u}_h^{j+1}|||/|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$: 648 subdomains & 209952 triangles (green), 5832 subdomains & 472392 triangles (blue), and 10368 subdomains & 839808 triangles (red) (respectively 525528, 1181952, and 2100816 unknowns)

Elementwise errors and a posteriori error estimators, iteration 1



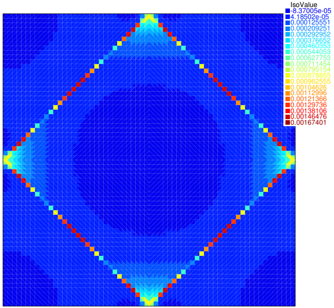
Errors

$|||\boldsymbol{u}_h - \boldsymbol{u}_h^1|||_K$



Lower estimators

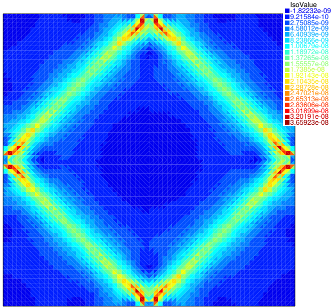
$\alpha^1 |||\hat{\boldsymbol{u}}_h^1 - \boldsymbol{u}_h^1|||_K$



Upper estimators

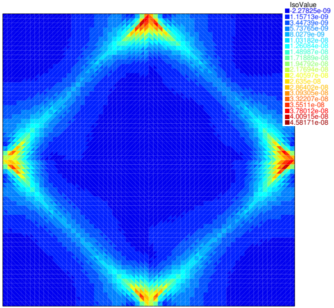
$|||\boldsymbol{u}_h^1 + \boldsymbol{\Pi}_k^{\mathcal{RTN}}(\nabla \tilde{p}_h^2)|||_K$

Elementwise errors and a posteriori error estimators, iteration 14



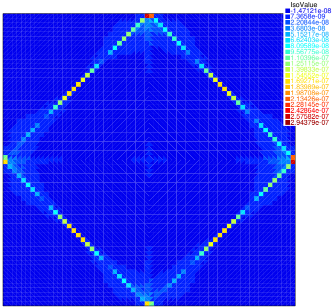
Errors

$$|||\boldsymbol{u}_h - \boldsymbol{u}_h^{14}|||_K$$



Lower estimators

$$\alpha^{14}|||\hat{\boldsymbol{u}}_h^{14} - \boldsymbol{u}_h^{14}|||_K$$



Upper estimators

$$|||\boldsymbol{u}_h^{14} + \boldsymbol{\Pi}_K^{\mathcal{RTN}}(\nabla \tilde{p}_h^{15})|||_K$$

Jumping diffusion

Setting ($k = 0$)

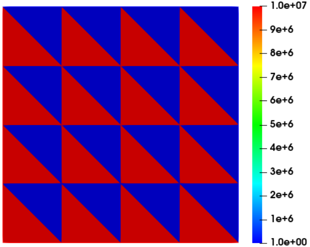
- $\Omega = (0, 1) \times (0, 1)$
- $\boldsymbol{S} = c(x, y)\mathrm{Id}$
- $f(x, y) = 1$
- $\Gamma_{\mathrm{D}} = \partial\Omega$
- zero initial guess $(\boldsymbol{u}_h^0, p_h^0) = (0, 0)$

Jumping diffusion

Setting $(k = 0)$

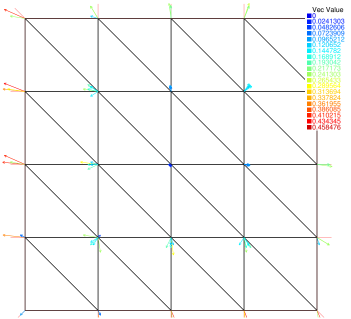
- $\Omega = (0, 1) \times (0, 1)$
- $\boldsymbol{S} = c(x, y)\mathrm{Id}$
- $f(x, y) = 1$
- $\Gamma_{\mathrm{D}} = \partial\Omega$
- zero initial guess $(\boldsymbol{u}_h^0, p_h^0) = (0, 0)$

Coarse mesh and diffusion coefficient

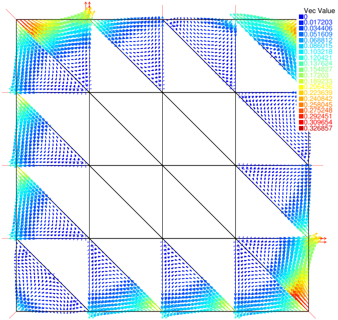


Coarse mesh $\mathcal{T}_H$ and variations of the coefficient $c(x, y)$

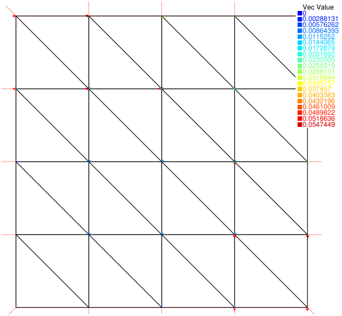
Initialization (equilibration): lifted residuals



$\delta_H^{0,2}$

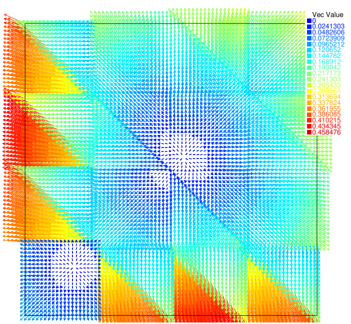


$\delta_h^{0,3}$

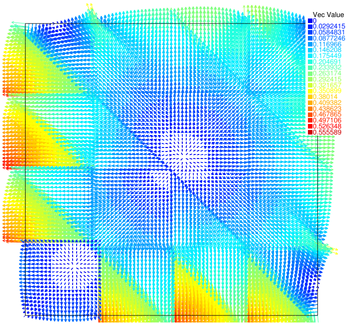


$\delta_H^{0,4}$

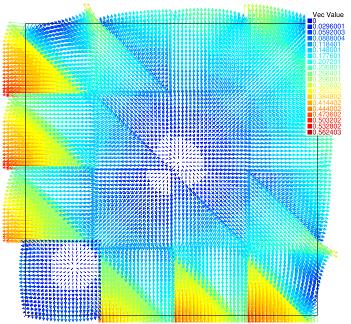
Initialization (equilibration): intermediate fluxes



$$\boldsymbol{u}_h^{0,2} = \delta_H^{0,2}$$

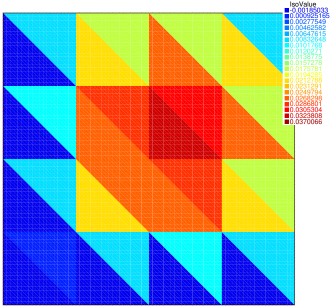


$$\boldsymbol{u}_h^{0,3} = \boldsymbol{u}_h^{0,2} + \delta_h^{0,3}$$

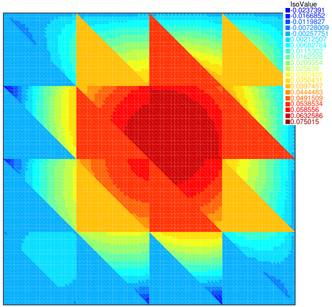


$$\boldsymbol{u}_h^{1} = \mathcal{R}_{\mathrm{F}}(\boldsymbol{u}_h^{0}, p_h^{0}) = \boldsymbol{u}_h^{0,3} + \delta_H^{0,4}$$

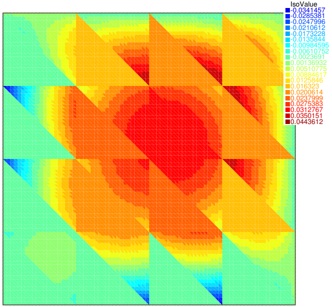
Initialization (equilibration): intermediate potentials



$$p_h^{0,2} = r_H^{0,2}$$

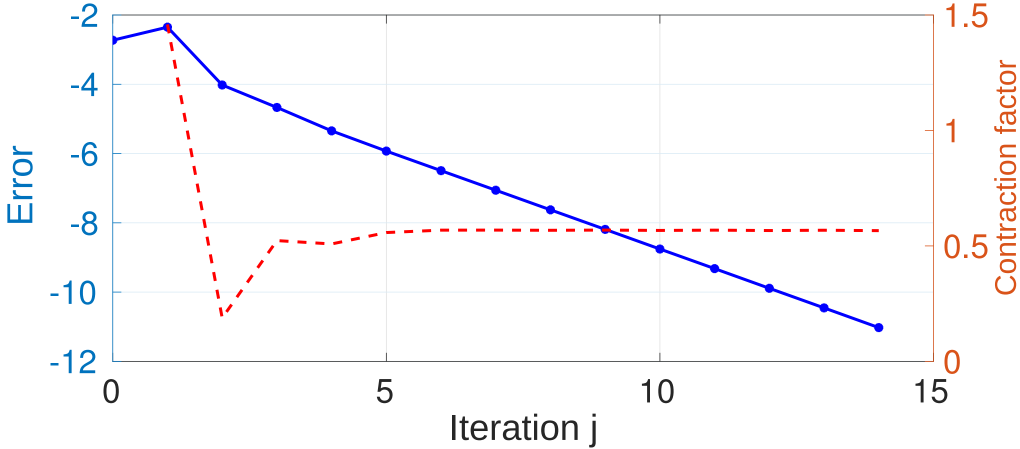


$$p_h^{0,3} = p_h^{0,2} + r_h^{0,3}$$



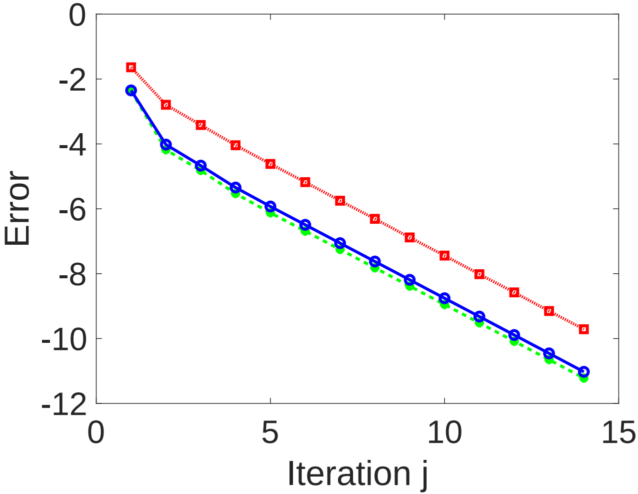
$$p_h^1 = \mathcal{R}_{\mathrm{P}}(\boldsymbol{u}_h^0, p_h^0) = p_h^{0,3} + r_H^{0,4}$$

Error decrease and contraction factor



Error $|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$ and contraction factor $|||\boldsymbol{u}_h - \boldsymbol{u}_h^{j+1}|||/|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$

A posteriori estimates of the algebraic error



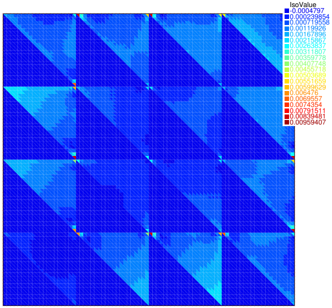
Error $|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$ (blue), upper bound η^j (red), and lower bound $\underline{\eta}^j$ (green)

Robustness

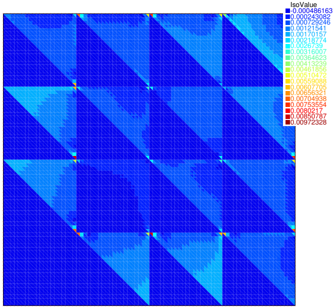
Diffusion contrast	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8
Number of iterations	19	16	15	15	15	15	15	15

Number of iterations needed to reduce the initial algebraic error estimator $\underline{\eta}^1$ by the factor 10^5

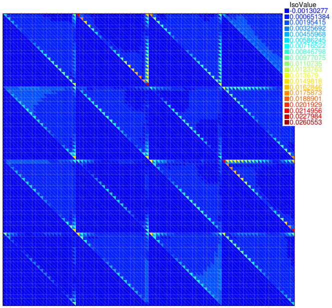
Elementwise errors and a posteriori error estimators, iteration 1



Errors
$|||\boldsymbol{u}_h - \boldsymbol{u}_h^1|||_K$

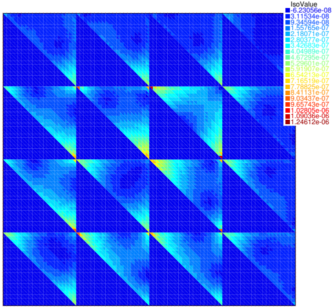


Lower estimators
$\alpha^1 |||\hat{\boldsymbol{u}}_h^1 - \boldsymbol{u}_h^1|||_K$

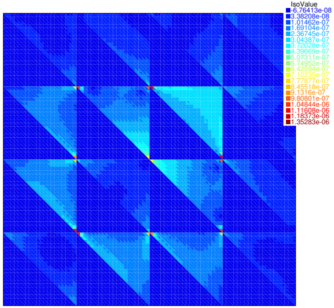


Upper estimators
$|||\boldsymbol{u}_h^1 + \boldsymbol{\Pi}_k^{\mathcal{RTN}}(\nabla \tilde{p}_h^2)|||_K$

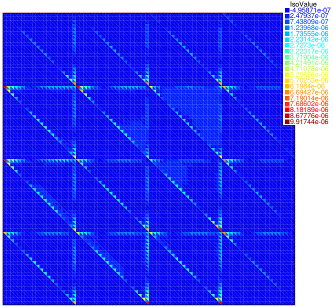
Elementwise errors and a posteriori error estimators, iteration 14



Errors
$$|||\boldsymbol{u}_h - \boldsymbol{u}_h^{14}|||_K$$

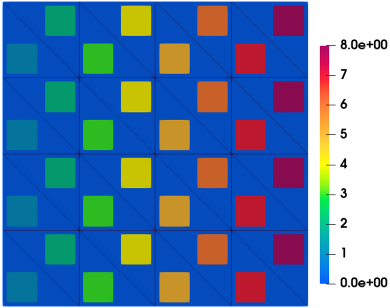


Lower estimators
$$\alpha^{14}|||\hat{\boldsymbol{u}}_h^{14} - \boldsymbol{u}_h^{14}|||_K$$

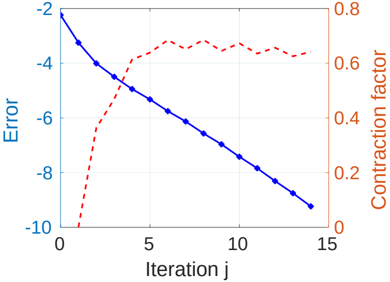


Upper estimators
$$|||\boldsymbol{u}_h^{14} + \boldsymbol{\Pi}_k^{\mathcal{RTN}}(\nabla \tilde{p}_h^{15})|||_K$$

Discontinuities not corresponding to the coarse mesh



Coefficient $c(x, y)$



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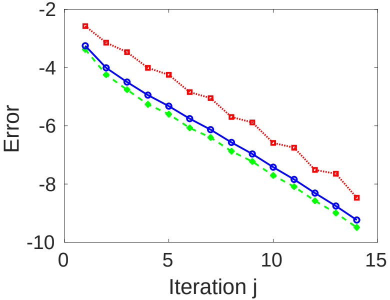
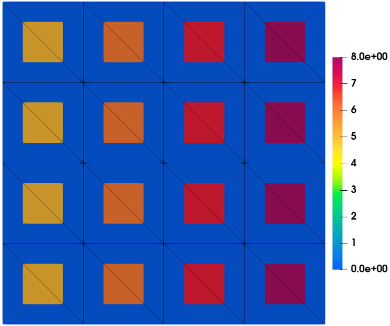
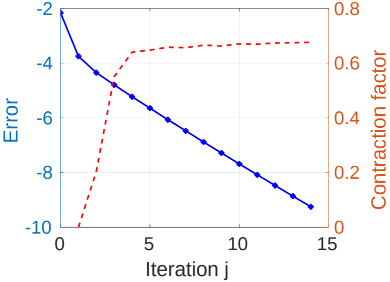


Error $|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$ (blue), upper bound η^j (red), and lower bound $\underline{\eta}^j$ (green)

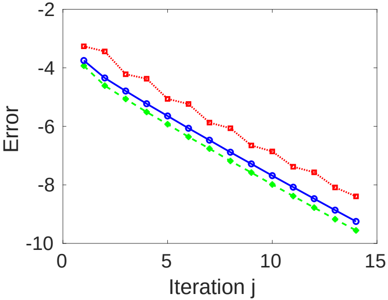
Discontinuities crossing the interfaces of the coarse mesh



Coefficient $c(x, y)$



Error $|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$ and contraction factor $|||\boldsymbol{u}_h - \boldsymbol{u}_h^{j+1}|||/|||\boldsymbol{u}_h - \boldsymbol{u}_h^j|||$



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Outline

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Taylored domain decomposition method for saddle-point mixed finite elements

✓ **flux equilibration** (balancing) by **coarse mesh constrained energy minimization**

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Thank you for your attention!