Mixed finite element methods: reduction to one unknown per element

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Outline

1. Primal and dual formulations, mixed finite elements
2. Stokes flow with implicit constitutive laws, motivations
3. MFEs reduced to one unknown per element
   - Local problems definition and a link to the MPFA method
   - Global problems definition
4. Numerical experiments
5. General polygonal meshes
6. Conclusions and future work

Outline

Model problem and different weak formulations

A model second-order elliptic problem

$$-\nabla\cdot(\mathbf{S}\nabla p) = g \quad \text{in } \Omega,$$
$$p = 0 \quad \text{on } \partial\Omega$$

Decomposition to two first-order systems

$$\mathbf{u} = -\mathbf{S}\nabla p \quad \text{in } \Omega,$$
$$\nabla\cdot\mathbf{u} = g \quad \text{in } \Omega,$$
$$p = 0 \quad \text{on } \partial\Omega$$

Primal weak formulation

Find $p \in H_0^1(\Omega)$ such that

$$(\mathbf{S}\nabla p, \nabla\varphi) = (g, \varphi) \quad \forall\varphi \in H_0^1(\Omega)$$

Dual mixed weak formulation

Find $p \in L^2(\Omega)$ & $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$ s. that

$$(\mathbf{S}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = 0 \quad \forall\,\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),$$
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Mixed finite element method

Find $p_h \in \Phi_h \subset L^2(\Omega)$ and $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ such that

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- Φ_h, $\mathbf{V}_h$: Raviart–Thomas–Nédélec MFE spaces
- high precision

Matrix form

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

- indefinite, saddle-point-type
- both fluxes U and potentials P involved $\Rightarrow$ expensive
- $U = \mathbb{A}^{-1}(F - \mathbb{B}^t P)$: only global flux expression

Main goal

Rewrite **equivalently** as

$$\mathbb{S}P = H$$

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Extension to unsteady nonlinear problems

Unsteady nonlinear advection–diffusion–reaction problem

$$\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} + F(p) = q \quad \text{in} \quad \Omega,$$

$$\mathbf{u} = -\mathbf{S}\nabla\varphi(p) + \psi(p)\mathbf{w} \quad \text{in} \quad \Omega,$$

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Define p_h^0 by p_0. On each t_n, find $\mathbf{u}_h^n \in \mathbf{V}_h$ & $p_h^n \in \Phi_h$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h^n, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, \varphi(p_h^n)) - (\psi(p_h^n)\mathbf{w}, \mathbf{S}^{-1}\mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

$$\left(\frac{p_h^n - p_h^{n-1}}{\tau_n}, \phi_h\right) + (\nabla \cdot \mathbf{u}_h^n, \phi_h) + (F(p_h^n), \phi_h) = (q, \phi_h) \quad \forall \phi_h \in \Phi_h.$$

Properties

- works $\Leftrightarrow$ the steady linear diffusion case
- assemblage and inversion of local condensation matrices only once; linearization and time steps – only p_h as in FVs

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Stokes flow with implicit constitutive laws

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\begin{aligned}
-\nabla\cdot \mathbb{s} + \nabla p &= \mathbf{f} && \text{in } \Omega,\\
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g(\mathbb{s}, d(\mathbf{u})) &= \mathbb{0} && \text{in } \Omega.
\end{aligned}
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Nomenclature

- $\mathbf{u}$: velocity, p: pressure, $\mathbb{s}$: shear stress
- $d(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$ symmetric velocity gradient
- $\mathbf{f}$: volume forces, μ: viscosity, τ_*: yield stress
- $g(\cdot,\cdot)$: nonlinear **implicit constitutive law**
  - $g(\mathbb{s}, d(\mathbf{u})) = 2\mu(\tau_* + (|\mathbb{s}| - \tau_*)^+)d(\mathbf{u}) - (|\mathbb{s}| - \tau_*)^+\mathbb{s}$: *Bingham* fluid
  - $g(\mathbb{s}, d(\mathbf{u})) = 2\mu|d(\mathbf{u})|^{r-2}(\tau_* + (|\mathbb{s}| - \tau_*)^+)d(\mathbf{u}) - (|\mathbb{s}| - \tau_*)^+\mathbb{s}$, $r \in (1,\infty)$: *Herschel–Bulkley* fluid
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Velocity (explicit law) formulation

Weak formulation

For $\mathbf{f} \in [L^s(\Omega)]^d$, find $\mathbf{u} \in \mathbf{V}_0$ such that

$$(\mathbb{s}(d(\mathbf{u})), \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_0.$$

Function spaces

- $\mathbf{V} := [W_0^{1,r}(\Omega)]^d$
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Velocity–pressure (explicit law) formulation

Weak formulation

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned}(\mathbb{s}(d(\mathbf{u})), \nabla \mathbf{v}) - (\nabla\cdot \mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V},\\ (\nabla\cdot \mathbf{u}, q) &= 0 \qquad \forall q \in Q.\end{aligned}$$

Function spaces

- $Q := L^s_0(\Omega) := \{q \in L^s(\Omega);\ (q, 1) = 0\}$; $\frac{1}{r} + \frac{1}{s} = 1$

inf–sup **condition**

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(q, \nabla\cdot \mathbf{v})}{\|\nabla \mathbf{v}\|_r \|q\|_s} = \beta > 0$$

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Velocity–pressure–stress implicit law formulation

Weak formulation

Find $(\mathbf{u}, p, \mathbb{s}) \in \mathbf{V} \times Q \times \mathbb{T}$ such that

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Function spaces

- $\mathbb{T} := [L^s_{\mathrm{sym}}(\Omega)]^{d \times d}$

Second inf–sup **condition**

$$\inf_{\mathbf{v} \in \mathbf{V}} \sup_{\mathbb{t} \in \mathbb{T}} \frac{(\mathbb{t}, \nabla \mathbf{v})}{\|\nabla \mathbf{v}\|_r \, \|\mathbb{t}\|_s} = \gamma > 0$$

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Motivations

Motivations of the present work

- reduce the unknowns back to one per element in various situations
- exemplify local flux expressions
- present a unified framework in which MFEs with one unknown/element can be derived/studied/used
- show closeness in building principles of MFE and FD/FV/MFD/MPFA, even on general polygonal meshes
- give **hints** for the **numerical treatment** of **implicit constitutive laws**

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Previous results

Links to nonconforming finite elements

- Arnold & Brezzi 1985, Marini 1985, Arbogast & Chen 1995, Chen 1996

Links to finite volumes

- Younès, Mose, Ackerer, & Chavent 1999–2004

Links to mimetic finite differences and and multi-point flux-approximations (using approximate numerical integration)

- Klausen & Winther, 2006
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- Arnold & Brezzi 1985, Marini 1985, Arbogast & Chen 1995, Chen 1996

Links to finite volumes

- Younès, Mose, Ackerer, & Chavent 1999–2004

Links to mimetic finite differences and and multi-point flux-approximations (using approximate numerical integration)

- Klausen & Winther, 2006
- Wheeler & Yotov, 2006
- Aavatsmark, Eigestad, Klausen, Wheeler, & Yotov, 2007
- Droniou, Eymard, Gallouët, & Herbin, 2010
- Bause Hoffmann, & Knabner, 2010
- . . . Brezzi, da Veiga, Lipnikov, Manzini, Shashkov . . .

Outline

Local flux expression from the Lagrange multipliers

Mixed finite element method

Find $p_h \in \Phi_h$ and $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla\cdot\mathbf{u}_h, \phi_h) = (g, \phi_h) \qquad \forall \phi_h \in \Phi_h$$

Nonconforming finite element method

Find $\tilde{\lambda}_h \in \tilde{\Psi}_h$ such that

$$(\mathbf{S}\nabla\tilde{\lambda}_h, \nabla\tilde{\psi}_h) = (g, \tilde{\psi}_h) \qquad \forall\tilde{\psi}_h \in \tilde{\Psi}_h$$

Local flux expression from the Lagrange multipliers

There holds (Marini 1985)

$$\mathbf{u}_h|_K = -\mathbf{S}_K\nabla\tilde{\lambda}_h|_K + \frac{g_K}{d}(\mathbf{x} - \mathbf{x}_K) \qquad \forall K \in \mathcal{T}_h$$

- $\mathbf{x}_K$: barycenter of K
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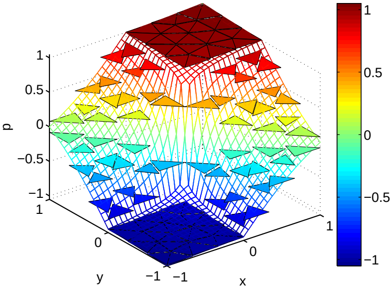
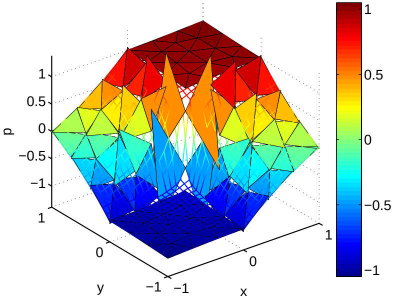
Different representations of the MFE solution



p_h



$\tilde{\lambda}_h$

Outline

1. Primal and dual formulations, mixed finite elements
2. Stokes flow with implicit constitutive laws, motivations
3. MFEs reduced to one unknown per element
   - Local problems definition and a link to the MPFA method
   - Global problems definition
4. Numerical experiments
5. General polygonal meshes
6. Conclusions and future work

A new element value

A new element value in $K \in \mathcal{T}_h$

- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
- new element value: $\bar{p}_K = \tilde{\lambda}_h(\mathbf{z}_K)$
- $\tilde{\lambda}_h$ expressed in the three points $\mathbf{x}_\sigma$, $\mathbf{x}_\gamma$, and $\mathbf{z}_K$ ($d = 2$)
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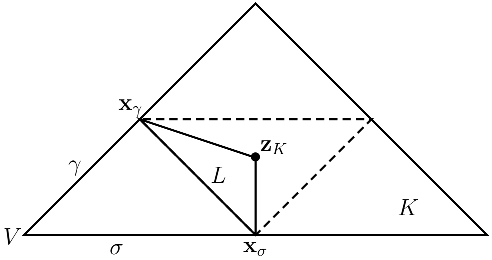

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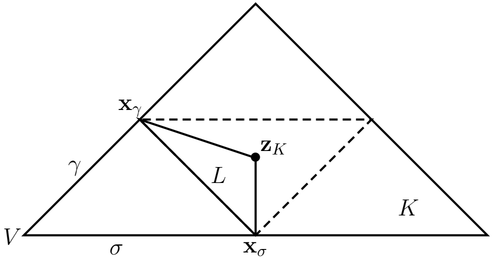

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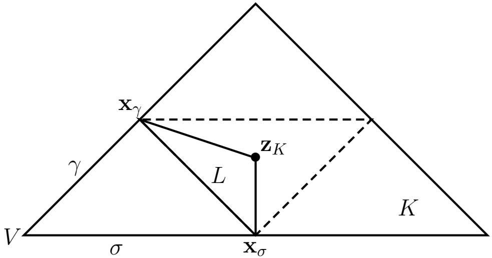

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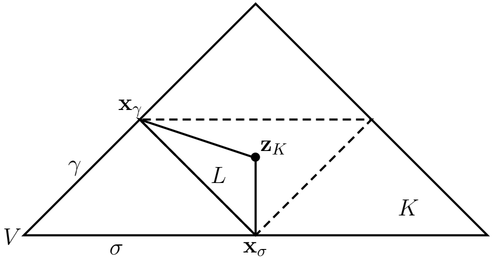

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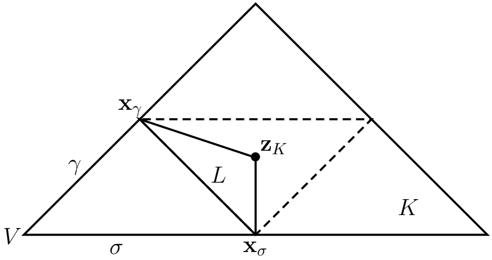

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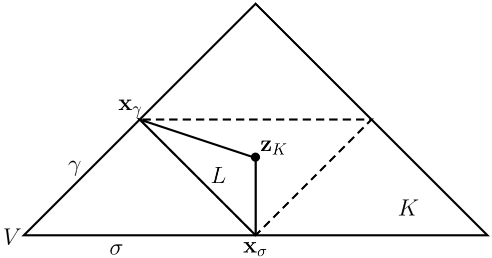

Definition of a local problem

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- consider a patch $\mathcal{T}_V$ of the elements around a vertex V
- given the new element values $\bar{p}_K$ and λ_σ, $\sigma \in \mathcal{E}_V^{\rm int}$, in the patch, express the fluxes $\mathbf{u}_h$ in the patch
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$$\sum_{K \in \mathcal{T}_V;\, \sigma \in \mathcal{E}_K} \langle \mathbf{u}_h \cdot \mathbf{n}_K, 1 \rangle_\sigma = 0 \qquad \forall \sigma \in \mathcal{E}_V^{\rm int}$$
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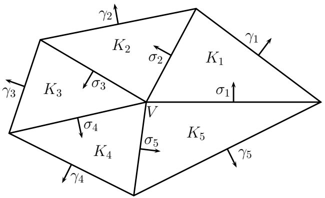

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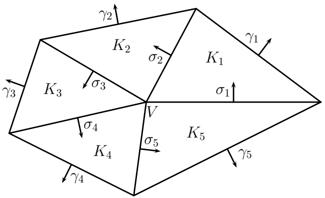

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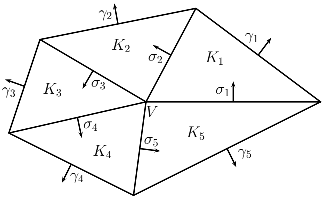

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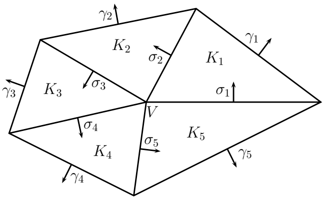

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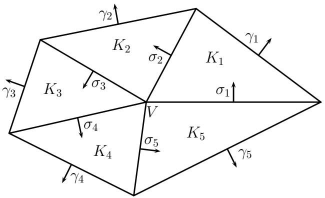

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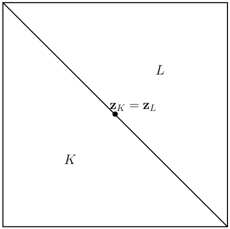
S-circumcenter as the evaluation point

S-circumcenter as the point $\mathbf{z}_K$

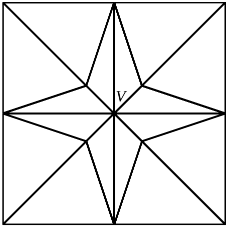
- circumcenter when $\mathbf{S}_K = \mathbb{I}s_K$
- the approach of Younès, Mose, Ackerer, & Chavent, 1999
- $\mathbb{M}_V$ gets diagonal
- no local linear system needs to be solved
- two-point flux expression (on arbitrary triangular grids and full-matrix piecewise constant $\mathbf{S}$)
- impossible in 3D (except particular cases)
- $\mathbb{M}_V$ can explode (modifications necessary):



Barycenter as the evaluation point

Barycenter as the point $\mathbf{z}_K$

- this is the approach of Vohralík, 2004/2006
- $\mathbb{M}_V$ is not diagonal (unless barycenter = circumcenter)
- a local linear system needs to be solved
- multi-point flux expression
- works generally in *d* space dimensions
- $\mathbb{M}_V$ can get singular (modifications necessary):



Changing adaptively the evaluation point

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- change $\mathbf{z}_K$ according to the local geometry and diffusion tensor
- ensure the well-posedness of the local problems
- influence the properties of the local matrices $\mathbb{M}_V$
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Examples of the different evaluation points

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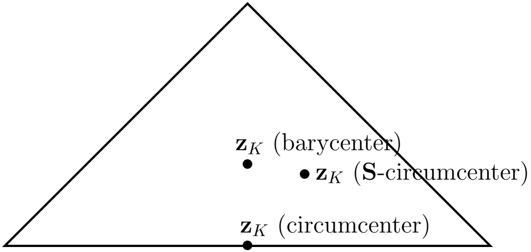
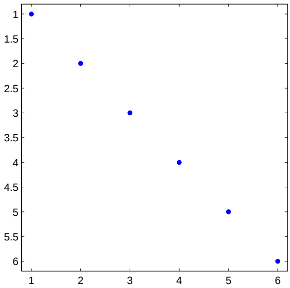
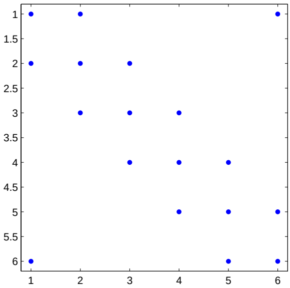
- $\mathbf{S} = \begin{pmatrix} 0.7236 & 0.3804 \\ 0.3804 & 0.4764 \end{pmatrix}$



Examples of the local matrices

Examples of the local matrices $\mathbb{M}_V$





S-circumcenter barycenter/opt. evaluation point

Outline

Expressing the Lagrange multipliers Λ or the fluxes U

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- local problems give $\Lambda_V^{\rm int} = (\mathbb{M}_V)^{-1}(\widetilde{G}_V - \mathbb{J}_V \bar{P}_V)$
- for every vertex V, we have one expression for $\Lambda_V^{\rm int}$
- run through all vertices and combine the (weighted) inverses of the local condensation matrices
- this gives

$$\Lambda = \widetilde{\mathbb{M}}^{\rm inv}\widetilde{G} - \mathbb{M}^{\rm inv}\bar{P}$$

- similarly

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Prescribing the final system by a flux equilibrium

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- recall $U = \widetilde{\mathbb{O}}^{\mathrm{inv}} G - \mathbb{O}^{\mathrm{inv}} \bar{P}$
- put this into $\mathbb{B}U = G$
- this gives

$$\bar{\mathbb{S}}\bar{P} = \bar{H}$$

with

$$\bar{\mathbb{S}} = -\mathbb{B}\mathbb{O}^{\mathrm{inv}}, \quad \bar{H} = G - \mathbb{B}\widetilde{\mathbb{O}}^{\mathrm{inv}} G$$

- $\mathbf{z}_K$ = **S**-circumcenter gives the FV method (Younès, Mose, Ackerer, & Chavent, 1999)
- $\mathbf{z}_K$ = barycenter gives the CMFE method (Vohralík, 2004/2006) (fully equivalent to the MPFA-O method when $g = 0$ (Hoffmann, 2008))

Prescribing the final system by a flux equilibrium

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- recall $U = \widetilde{\mathbb{O}}^{\mathrm{inv}} G - \mathbb{O}^{\mathrm{inv}} \bar{P}$
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- recall $\Lambda = \widetilde{\mathbb{M}}^{\mathrm{inv}}\widetilde{G} - \mathbb{M}^{\mathrm{inv}}\bar{P}$
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$$\bar{\mathbb{S}} = \mathbb{N}\mathbb{M}^{\mathrm{inv}} + \mathbb{I}, \quad \bar{H} = \mathbb{N}\widetilde{\mathbb{M}}^{\mathrm{inv}}\widetilde{G}$$

- using $\mathbf{z}_K$ = **S**-circumcenter, we name it the MFEC method
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Outline

Model problem

Model problem

- $\Omega = (0,1) \times (0,1)$
- inhomogeneous Dirichlet boundary condition given by $p(x,y) = 0.1y + 0.9$
- $K \in \mathcal{T}_h$:

$$\mathbf{S}|_K = \begin{pmatrix} \cos(\theta_K) & -\sin(\theta_K) \\ \sin(\theta_K) & \cos(\theta_K) \end{pmatrix} \begin{pmatrix} s_K & 0 \\ 0 & \nu s_K \end{pmatrix} \begin{pmatrix} \cos(\theta_K) & \sin(\theta_K) \\ -\sin(\theta_K) & \cos(\theta_K) \end{pmatrix}$$

- homogeneous isotropic case, $s_K = 1$ for all $K \in \mathcal{T}_h$, $\nu = 1$
- anisotropic case, $s_K = 1$ for all $K \in \mathcal{T}_h$, $\theta_K \in \left\{\frac{\pi}{5}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{5}, \frac{\pi}{3}\right\}$, $\nu = 0.2$
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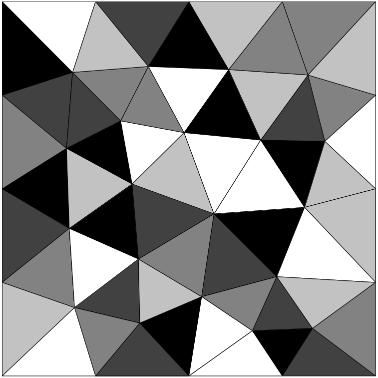
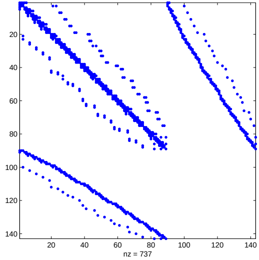
Initial mesh

Initial mesh and the distribution of the inhomogeneities and anisotropies

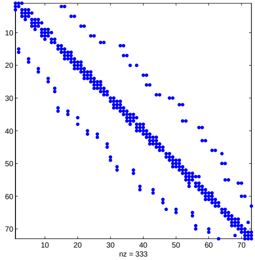


Matrices of the different methods

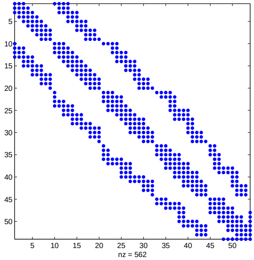
System matrix sparsity patterns



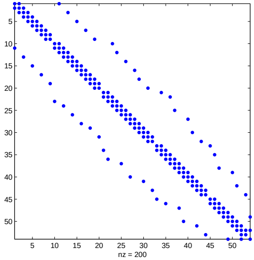
MFE



NCFE



MFEB
MFEO
CMFE



FV
MFEO

Results, homogeneous isotropic case

							DS	CG/ Bi-CGStab		PCG/ PBi-CGStab		
Meth.	Un.	Mat.	St.	Nonz.	CN	CNS	CPU	CPU	Iter.	CPU	IC/ ILU	Iter.
MFEB	13824	NPD	14	177652	7564	7580	0.27	4.86	324.5	0.81	0.36	9.0
MFEC	13824	NNS	4	55040	11256	11056	0.09	2.23	372.0	0.42	0.19	6.5
MFEO	13824	NPD	14	177652	7531	7558	0.28	4.08	270.0	0.80	0.41	7.5
CMFE	13824	NPD	14	177652	7397	7380	0.27	4.70	312.0	0.83	0.39	8.5
FV	13824	SPD	4	55040	65722	8898	0.07	3.09	1098.0	0.42	0.17	17.0
NCFE	20608	SPD	5	102528	14064	9944	0.14	2.92	620.0	1.11	0.56	19.0

Results, anisotropic case

							DS	CG/ Bi-CGStab		PCG/ PBi-CGStab		
Meth.	Un.	Mat.	St.	Nonz.	CN	CNS	CPU	CPU	Iter.	CPU	IC/ ILU	Iter.
MFEB	13824	NPD	14	177652	14489	11203	0.28	6.61	448.0	0.98	0.59	6.5
MFEC	13824	NID	4	55040	2401279	416769	0.08	—	—	0.45	0.20	7.0
MFEO	13824	NPD	14	177652	13401	10767	0.27	6.51	440.5	0.95	0.41	10.0
CMFE	13824	NPD	14	177652	9276	7758	0.28	5.27	350.5	0.84	0.38	9.0
FV	13824	SID	4	55040	247055	239934	0.09	—	—	0.45	0.20	7.0
NCFE	20608	SPD	5	102528	25393	16969	0.18	4.03	850.0	1.12	0.41	30.0

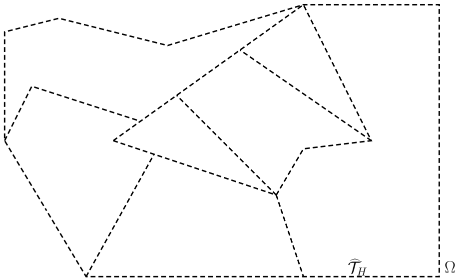
Results, inhomogeneous case

							DS	CG/ Bi-CGStab		PCG/ PBi-CGStab		
Meth.	Un.	Mat.	St.	Nonz.	CN	CNS	CPU	CPU	Iter.	CPU	IC/ ILU	Iter.
MFEB	13824	NPD	14	177652	819248	740706	0.28	13.33	897.5	1.05	0.62	6.5
MFEC	13824	NNS	4	55040	903789	763849	0.09	5.34	947.5	0.47	0.20	7.5
MFEO	13824	NPD	14	177652	820367	739957	0.28	12.45	790.5	1.05	0.56	8.0
CMFE	13824	NPD	14	177652	2500730	478974	0.28	102.27	6842.5	1.01	0.41	10.5
FV	13824	SPD	4	55040	16387758	497974	0.07	39.41	14101.0	0.44	0.17	16.0
NCFE	20608	SPD	5	102528	4797335	670623	0.18	52.42	11226.0	1.22	0.64	16.0

Outline

General polygonal meshes

A general polygonal mesh $\widehat{\mathcal{T}}_H$



- nonconvex and non star-shaped elements in $\widehat{\mathcal{T}}_H$
- $\widehat{\mathcal{T}}_H$ can be nonmatching
- maximal number of sides of $K \in \widehat{\mathcal{T}}_H$ is not limited
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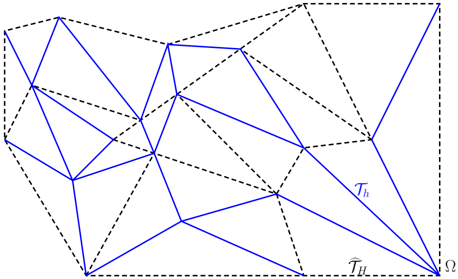

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MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\begin{pmatrix} \widehat{\mathbb{A}} & \widehat{\mathbb{B}}^t \\ \widehat{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{P} \end{pmatrix} = \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix}$$

- $\widehat{U}$: flux unknowns related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{P}$: potential unknowns related to the elements of $\widehat{\mathcal{T}}_H$ only
- indefinite, saddle point system, well-posed
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of loc. matrices corresponding to local Neumann problems)
- works for arbitrary order
- **equivalent** to **formulation on** $\mathcal{T}_h$ (a priori and a posteriori error estimates, discrete maximum principle, . . .)

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$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

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$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

- $\widehat{\Lambda}$: Lagrange multipliers related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{\mathbb{Z}}$ is symmetric and positive definite
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of local matrices corresponding to local Dirichlet problems)
- works for arbitrary order
- **equivalent** to **formulation on** $\mathcal{T}_h$ (a priori and a posteriori error estimates, discrete maximum principle . . .)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\mathbb{Z}\Lambda = E$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

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Outline

Conclusions and future work

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- mixed finite elements: one method with
  - saddle point / symmetric pos. definite / nonsymmetric pos. definite / symmetric indefinite / nonsymmetric indef. matrix
  - U and P unknowns / Λ unknowns / P unknowns
  - narrow stencil and two-point flux expressions / wider stencil and multi-point flux expressions
  - discrete maximum principle for values in some points but not in some others
- no free parameter to choose, no stabilization, the best method if your criterion is min. complementary energy
- **paradigm**: **decompose into a system** in order to better understand, describe, & analyze and **reduce back** in order to solve cheaply

Work in progress

- extensions to all order MFE schemes
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Bibliography

Bibliography

- VOHRALÍK M., Equivalence between lowest-order mixed finite element and multi-point finite volume methods on simplicial meshes, *M2AN Math. Model. Numer. Anal.* **40** (2006), 367–391.
- VOHRALÍK M., WOHLMUTH B., Mixed finite element methods: implementation with one unknown per element, local flux expressions, positivity, polygonal meshes, and relations to other methods, *Math. Models Methods Appl. Sci.* **23** (2013), 803–838.
- VOHRALÍK M., WOHLMUTH B., From face to element unknowns by local static condensation with application to nonconforming finite elements, Comput. Methods Appl. Mech. Engrg. **253** (2013), 517–529.

Thank you for your attention!