# Flux and potential reconstructions for guaranteed error bounds in numerical approximations of model PDEs

**Martin Vohralík**

in collaboration with C. Cancès, E. Cancès, V. Dolejší, G. Dusson, A. Ern, Y. Maday, I. S. Pop, I. Smears, B. Stamm, & M. F. Wheeler

*Inria Paris & Ecole des Ponts*

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# Outline

# Optimal a posteriori error estimate

**Guaranteed upper bound**

- $\|u - u_h\|^2_{?,\Omega} \le \sum_{K\in\mathcal{T}_h} \eta_K(u_h)^2$
- no undetermined constant: **error control**

**Local efficiency**

- $\eta_K(u_h) \le C_{\mathrm{eff}} \|u - u_h\|_{?,\text{neighbors of } K}$
- **local** error lower bound (optimal space mesh refinement)

**Robustness**

- $C_{\mathrm{eff}}$ independent of data, domain $\Omega$, meshes, solution $u$, **polynomial degree** of $u_h$

**Asymptotic exactness**

- $\sum_{K\in\mathcal{T}_h} \eta_K(u_h)^2 / \|u - u_h\|^2_{?,\Omega} \searrow 1$
- overestimation factor goes to one with meshes size

**Small evaluation cost**

- estimators $\eta_K(u_h)$ can be evaluated cheaply (locally)

**Error components identification**

- $\eta_K(u_h)$ can distinguish the different error components

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# Laplace model problem

## Model problem

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

## Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

## Properties of the weak solution

- $u \in H_0^1(\Omega)$ (primal variable constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla \cdot \sigma = f$ (equilibrium)
- $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$ (dual variable constraint)

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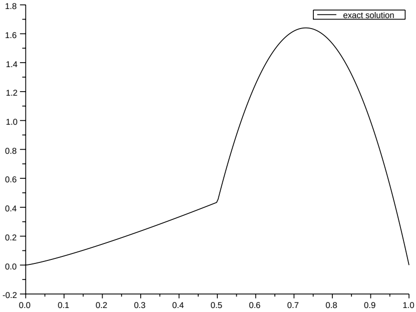
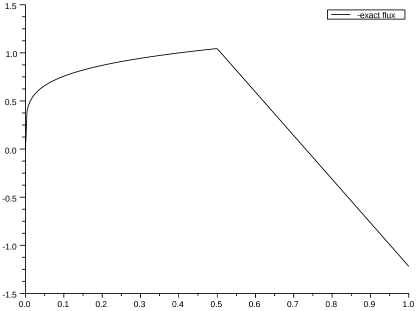
# Exact solution and flux

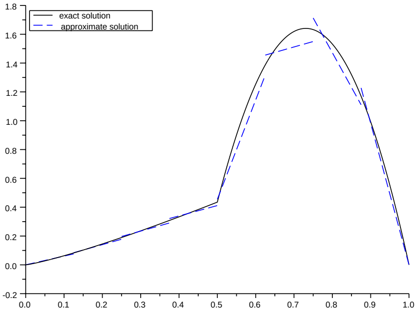


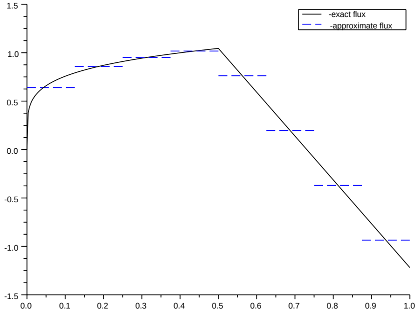




Solution $u$ is continuous

Flux $\boldsymbol{\sigma} := -\underline{\mathbf{K}}\nabla u$ is continuous

# Approximate solution and flux

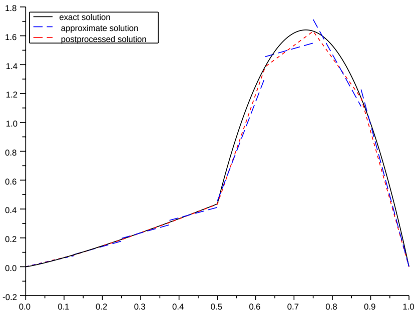


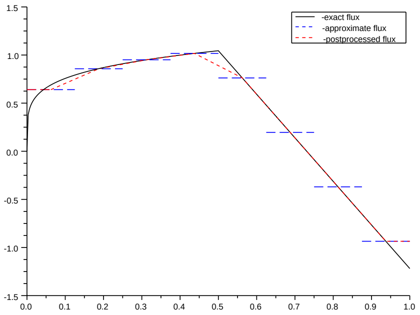

Approximate solution $u_h$ is not necessarily continuous

Approximate flux $-\underline{\mathbf{K}}\nabla u_h$ is not necessarily continuous

# Potential and flux reconstructions




Potential reconstruction




Flux reconstruction

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# Guaranteed a posteriori error estimate

**Theorem (A guaranteed a posteriori error estimate,** Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...**)**

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega), v|_K \in H^1(K) \, \forall K \in \mathcal{T}_h\}$ *be arbitrary*
- $s_h \in H_0^1(\Omega)$ *and* $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that*

$$(\nabla\cdot\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \text{ for all } K \in \mathcal{T}_h.$$

*Then*

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K\in\mathcal{T}_h} \Big( \underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K}_{\text{equilibrium}} \Big)^2$$

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# Proof I

*Proof.*

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of $s$:

$$\|\nabla(s - u_h)\| = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of $s$, definition of $u$:

$$\|\nabla(u - s)\| = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\ \|\nabla \varphi\| = 1}}_{\text{dual norm of the residual}}$$

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# Proof II

*Proof* (continuation).

- nonconformity upper bound:

$$\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\| \le \|\nabla(u_h - s_h)\|$$

- adding and subtracting equilibrated flux, Green theorem:

$$(f, \varphi) - (\nabla u_h, \nabla \varphi) = (f - \nabla\cdot\sigma_h, \varphi) - (\nabla u_h + \sigma_h, \nabla \varphi)$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \sigma_h, \nabla\varphi)$$

$$(f - \nabla\cdot\sigma_h, \varphi) = \sum_{K \in \mathcal{T}_h} (f - \nabla\cdot\sigma_h, \varphi \qquad)_K$$

$$\le \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla\cdot\sigma_h\|_K \|\nabla\varphi\|_K$$

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*Proof* (continuation).

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# Global potential and flux reconstructions

**Ideally**

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

✓ computable, discrete spaces $V_h \subset H_0^1(\Omega)$, $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $Q_h \subset L^2(\Omega)$

✗ too expensive, **global** minimization problems (the hypercircle method . . . )

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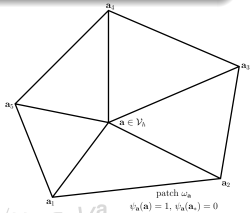
# Local potential reconstruction

**Definition (Construction of $s_h$, ≈ Carstensen and Merdon (2013), EV (2015))**

For each $\mathbf{a} \in \mathcal{V}_h$, solve the **local conforming FE problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}.$$




**Equivalent form**

Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

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**Key ideas**

- **local** minimizations
- cut-off by hat basis functions $\psi_{\mathbf{a}}$
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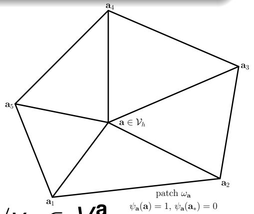


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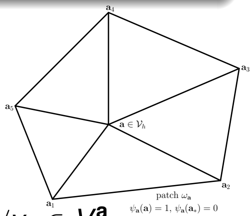


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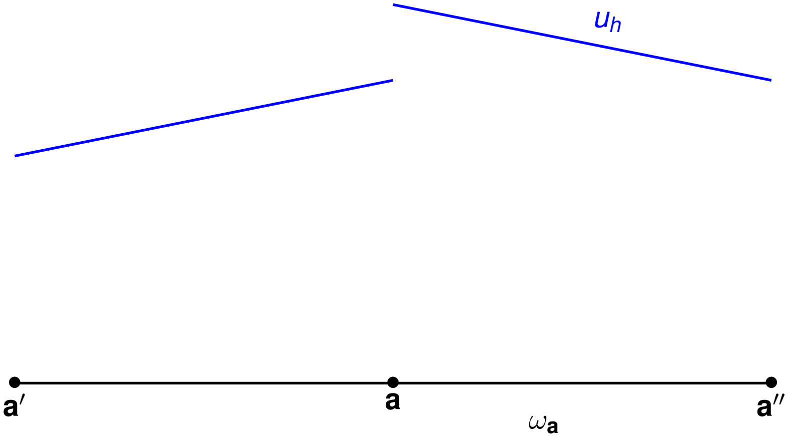

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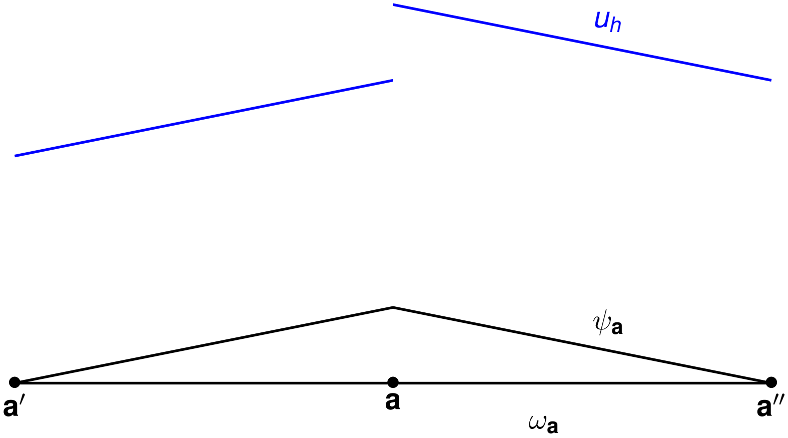

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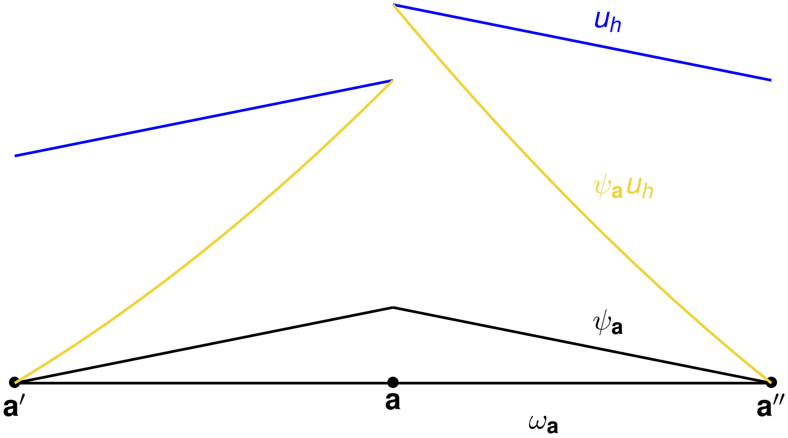

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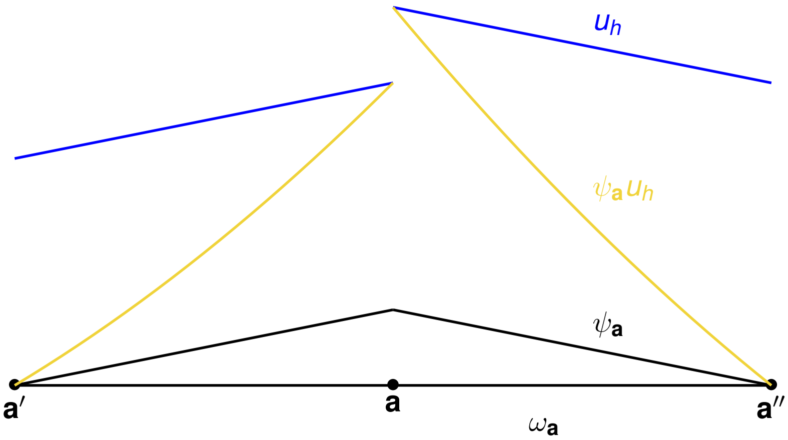

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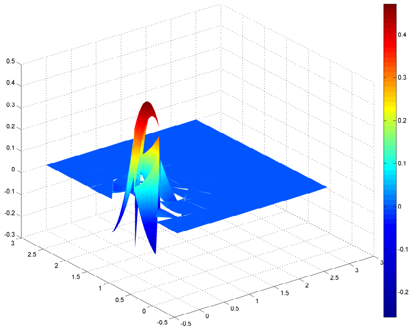


Potential $u_h$

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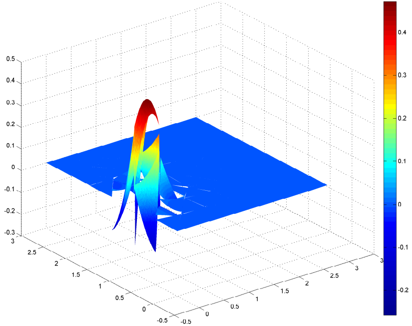


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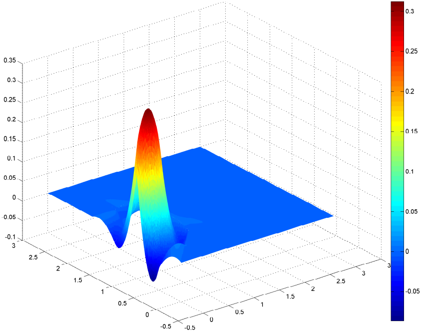


Potential reconstruction $s_h$

# Local flux reconstructions

**Assumption A (Galerkin orthogonality wrt hat functions)**

*There holds*

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**Key points**

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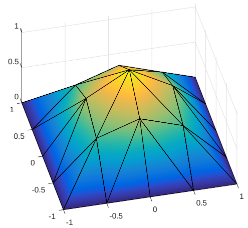

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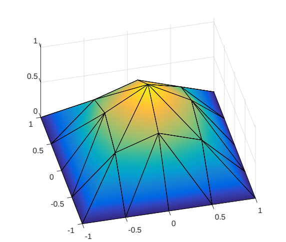


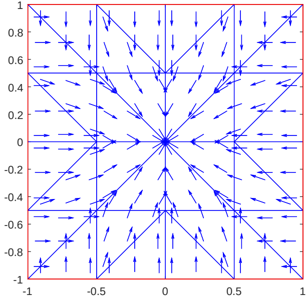


Flux $-\nabla u_h$

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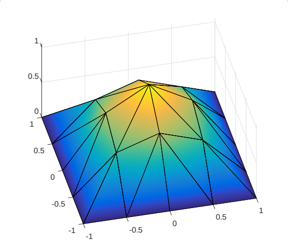

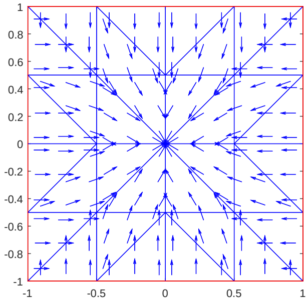

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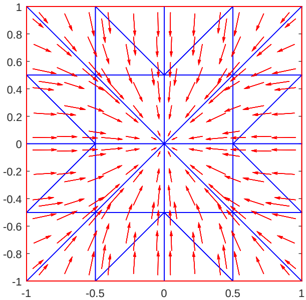

Flux reconstruction $\sigma_h$

# Outline

# Polynomial-degree-robust efficiency

**Assumption B (Piecewise polynomials, data, and meshes)**

*The approximation $u_h$ and the datum $f$ are piecewise polynomial. The degrees of the MFE reconstructions $\boldsymbol{\sigma}_h$ and $s_h$ are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.*

**Theorem (Polynomial-degree-robust efficiency** Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015)**)**

*Let $u$ be the weak solution and let Assumptions A and B hold. Then there exists constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that*

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**Remarks**

- equivalence error–estimate
- maximal overestimation factor guaranteed

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## Lemma ($H^1$ polynomial extension on a tetrahedron)

*Let $K \in \mathcal{T}_h$, $\mathcal{E}_K^{\mathrm{D}} \subset \mathcal{E}_K$. Let $r \in \mathbb{P}_p(\mathcal{E}_K^{\mathrm{D}})$ be continuous on $\mathcal{E}_K^{\mathrm{D}}$. Then for $C$ only depending on the shape regularity of $K$,*

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**Context**

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K,\\ \zeta_K &= r_e && \text{on all } e \in \mathcal{E}_K^{\mathrm{D}},\\ -\nabla \zeta_K \cdot \mathbf{n}_K &= 0 && \text{on all } e \in \mathcal{E}_K \setminus \mathcal{E}_K^{\mathrm{D}}. \end{aligned}$$

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*Let $K \in \mathcal{T}_h$, $\mathcal{E}_K^{\mathrm{D}} \subset \mathcal{E}_K$. Let $r \in \mathbb{P}_p(\mathcal{E}_K^{\mathrm{D}})$ be continuous on $\mathcal{E}_K^{\mathrm{D}}$. Then for $C$ only depending on the shape regularity of $K$,*

$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_e \text{ on all } e \in \mathcal{E}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \leq C \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_e \text{ on all } e \in \mathcal{E}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = C \|\nabla \zeta_K\|_K.$$

**Context**

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_e && \text{on all } e \in \mathcal{E}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= 0 && \text{on all } e \in \mathcal{E}_K \setminus \mathcal{E}_K^{\mathrm{D}}. \end{aligned}$$

# Fluxes (Costabel, McIntosh (2010), Demkowicz, Gopalakrishnan, Schöberl (2012), EV 2016)

## Lemma ($\mathbf{H}(\mathrm{div})$ polynomial extension on a tetrahedron)

*Let $K \in \mathcal{T}_h$, $\mathcal{E}_K^{\mathrm{N}} \subset \mathcal{E}_K$. Let $r \in \mathbb{P}_p(\mathcal{E}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying $\sum_{e \in \mathcal{E}_K} (r_e, 1)_e = (r_K, 1)_K$ if $\mathcal{E}_K^{\mathrm{N}} = \mathcal{E}_K$. Then for $C = C(\kappa_K) > 0$,*

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_e \ \forall e \in \mathcal{E}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_e \ \forall e \in \mathcal{E}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K .$$

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Set $\xi_K := -\nabla \zeta_K$.

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$$-\Delta \zeta_K = r_K \qquad \text{in } K,$$
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# Outline

1. Introduction
2. Laplace equation: potential & flux reconstructions
   - Guaranteed upper bound in a unified framework
   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Nonlinear Laplace equation: adaptive stopping criteria
   - Adaptive inexact Newton method
   - Applications & numerical results
4. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Applications & numerical results
5. Heat equation: robustness wrt final time & local efficiency
6. Two-phase flow in porous media: industrial application
7. Conclusions and outlook

# Conforming finite elements

**Conforming finite elements**

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$

✓ Assumption A: take $v_h = \psi_{\mathbf{a}}$

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# Discontinuous Galerkin finite elements

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Find $u_h \in V_h$ such that

$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

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$$\nabla_{\mathrm{d}} u_h := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

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- $\Rightarrow$ modified Galerkin orthogonality

$$(\nabla_{\mathrm{d}} u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

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# Numerics: smooth case

**Model problem**

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

**Exact solution**

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)$$

**Discretization**

- symmetric interior penalty discontinuous Galerkin method
- unstructured triangular grids
- uniform $h$ refinement

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# Uniform refinement: **asymptotic exactness**

| $h$ | $p$ | $\|\nabla_{\mathrm{d}}(u-u_h)\|$ | $\|\nabla_{\mathrm{d}} u_h+\sigma_h\|$ | $\eta_{\mathrm{osc}}$ | $\|\nabla_{\mathrm{d}}(u_h-s_h)\|$ | $\eta$ | $I^{\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|
| $h_0$ | 1 | 1.07E-00 | 1.12E-00 | 5.55E-02 | 4.16E-01 | 1.25E-00 | 1.17 |
| $\approx h_0/2$ | | 5.56E-01 | 5.71E-01 | 7.42E-03 | 1.82E-01 | 6.07E-01 | 1.09 |
| $\approx h_0/4$ | | 2.92E-01 | 2.96E-01 | 1.04E-03 | 8.77E-02 | 3.10E-01 | 1.06 |
| $\approx h_0/8$ | | 1.39E-01 | 1.40E-01 | 1.10E-04 | 3.85E-02 | 1.45E-01 | 1.04 |
| $h_0$ | 2 | 1.54E-01 | 1.55E-01 | 5.10E-03 | 3.05E-02 | 1.63E-01 | 1.06 |
| $\approx h_0/2$ | | 4.07E-02 | 4.13E-02 | 3.53E-04 | 7.55E-03 | 4.23E-02 | 1.04 |
| $\approx h_0/4$ | | 1.10E-02 | 1.12E-02 | 2.51E-05 | 1.97E-03 | 1.14E-02 | 1.03 |
| $\approx h_0/8$ | | 2.50E-03 | 2.54E-03 | 1.30E-06 | 4.21E-04 | 2.57E-03 | 1.03 |
| $h_0$ | 3 | 1.37E-02 | 1.37E-02 | 3.58E-04 | 1.74E-03 | 1.41E-02 | 1.03 |
| $\approx h_0/2$ | | 1.85E-03 | 1.85E-03 | 1.26E-05 | 2.10E-04 | 1.88E-03 | 1.01 |
| $\approx h_0/4$ | | 2.60E-04 | 2.60E-04 | 4.73E-07 | 2.54E-05 | 2.62E-04 | 1.01 |
| $\approx h_0/8$ | | 2.75E-05 | 2.75E-05 | 1.15E-08 | 2.55E-06 | 2.76E-05 | 1.01 |
| $h_0$ | 4 | 9.87E-04 | 9.84E-04 | 2.12E-05 | 1.11E-04 | 1.01E-03 | 1.02 |
| $\approx h_0/2$ | | 6.92E-05 | 6.92E-05 | 3.96E-07 | 7.44E-06 | 7.00E-05 | 1.01 |
| $\approx h_0/4$ | | 5.04E-06 | 5.04E-06 | 7.58E-09 | 4.98E-07 | 5.07E-06 | 1.01 |
| $\approx h_0/8$ | | 2.58E-07 | 2.58E-07 | 8.96E-11 | 2.47E-08 | 2.60E-07 | 1.01 |
| $h_0$ | 5 | 5.64E-05 | 5.63E-05 | 1.06E-06 | 4.50E-06 | 5.75E-05 | 1.02 |
| $\approx h_0/2$ | | 2.01E-06 | 2.01E-06 | 9.88E-09 | 1.46E-07 | 2.03E-06 | 1.01 |
| $\approx h_0/4$ | | 7.74E-08 | 7.73E-08 | 1.01E-10 | 4.35E-09 | 7.76E-08 | 1.00 |
| $\approx h_0/8$ | | 1.86E-09 | 1.86E-09 | 1.70E-12 | 1.00E-10 | 1.86E-09 | 1.00 |
| $h_0$ | 6 | 2.85E-06 | 2.85E-06 | 4.70E-08 | 2.18E-07 | 2.90E-06 | 1.02 |
| $\approx h_0/2$ | | 5.42E-08 | 5.42E-08 | 2.40E-10 | 4.02E-09 | 5.46E-08 | 1.01 |
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| $\approx h_0/8$ | | 1.86E-09 | 1.86E-09 | 1.70E-12 | 1.00E-10 | 1.86E-09 | 1.00 |
| $h_0$ | 6 | 2.85E-06 | 2.85E-06 | 4.70E-08 | 2.18E-07 | 2.90E-06 | 1.02 |
| $\approx h_0/2$ | | 5.42E-08 | 5.42E-08 | 2.40E-10 | 4.02E-09 | 5.46E-08 | 1.01 |
| $\approx h_0/4$ | | 1.07E-09 | 1.07E-09 | 1.03E-11 | 6.90E-11 | 1.08E-09 | 1.01 |

# Uniform refinement: **asymptotic exactness**

| $h$ | $p$ | $\|\nabla_d(u-u_h)\|$ | $\|\nabla_d u_h+\sigma_h\|$ | $\eta_{osc}$ | $\|\nabla_d(u_h-s_h)\|$ | $\eta$ | $I^{eff}$ |
|---|---|---|---|---|---|---|---|
| $h_0$ | **1** | 1.07E-00 | 1.12E-00 | 5.55E-02 | 4.16E-01 | 1.25E-00 | **1.17** |
| $\approx h_0/2$ | | 5.56E-01 | 5.71E-01 | 7.42E-03 | 1.82E-01 | 6.07E-01 | **1.09** |
| $\approx h_0/4$ | | 2.92E-01 | 2.96E-01 | 1.04E-03 | 8.77E-02 | 3.10E-01 | **1.06** |
| $\approx h_0/8$ | | 1.39E-01 | 1.40E-01 | 1.10E-04 | 3.85E-02 | 1.45E-01 | **1.04** |
| $h_0$ | **2** | 1.54E-01 | 1.55E-01 | 5.10E-03 | 3.05E-02 | 1.63E-01 | **1.06** |
| $\approx h_0/2$ | | 4.07E-02 | 4.13E-02 | 3.53E-04 | 7.55E-03 | 4.23E-02 | **1.04** |
| $\approx h_0/4$ | | 1.10E-02 | 1.12E-02 | 2.51E-05 | 1.97E-03 | 1.14E-02 | **1.03** |
| $\approx h_0/8$ | | 2.50E-03 | 2.54E-03 | 1.30E-06 | 4.21E-04 | 2.57E-03 | **1.03** |
| $h_0$ | **3** | 1.37E-02 | 1.37E-02 | 3.58E-04 | 1.74E-03 | 1.41E-02 | **1.03** |
| $\approx h_0/2$ | | 1.85E-03 | 1.85E-03 | 1.26E-05 | 2.10E-04 | 1.88E-03 | **1.01** |
| $\approx h_0/4$ | | 2.60E-04 | 2.60E-04 | 4.73E-07 | 2.54E-05 | 2.62E-04 | **1.01** |
| $\approx h_0/8$ | | 2.75E-05 | 2.75E-05 | 1.15E-08 | 2.55E-06 | 2.76E-05 | **1.01** |
| $h_0$ | **4** | 9.87E-04 | 9.84E-04 | 2.12E-05 | 1.11E-04 | 1.01E-03 | **1.02** |
| $\approx h_0/2$ | | 6.92E-05 | 6.92E-05 | 3.96E-07 | 7.44E-06 | 7.00E-05 | **1.01** |
| $\approx h_0/4$ | | 5.04E-06 | 5.04E-06 | 7.58E-09 | 4.98E-07 | 5.07E-06 | **1.01** |
| $\approx h_0/8$ | | 2.58E-07 | 2.58E-07 | 8.96E-11 | 2.47E-08 | 2.60E-07 | **1.01** |
| $h_0$ | **5** | 5.64E-05 | 5.63E-05 | 1.06E-06 | 4.50E-06 | 5.75E-05 | **1.02** |
| $\approx h_0/2$ | | 2.01E-06 | 2.01E-06 | 9.88E-09 | 1.46E-07 | 2.03E-06 | **1.01** |
| $\approx h_0/4$ | | 7.74E-08 | 7.73E-08 | 1.01E-10 | 4.35E-09 | 7.76E-08 | **1.00** |
| $\approx h_0/8$ | | 1.86E-09 | 1.86E-09 | 1.70E-12 | 1.00E-10 | 1.86E-09 | **1.00** |
| $h_0$ | **6** | 2.85E-06 | 2.85E-06 | 4.70E-08 | 2.18E-07 | 2.90E-06 | **1.02** |
| $\approx h_0/2$ | | 5.42E-08 | 5.42E-08 | 2.40E-10 | 4.02E-09 | 5.46E-08 | **1.01** |
| $\approx h_0/4$ | | 1.07E-09 | 1.07E-09 | 1.03E-11 | 6.90E-11 | 1.08E-09 | **1.01** |

# Uniform refinement: **asymptotic exactness**

| $h$ | $p$ | $\|\nabla_{\mathrm{d}}(u-u_h)\|$ | $\|\nabla_{\mathrm{d}}u_h+\sigma_h\|$ | $\eta_{\mathrm{osc}}$ | $\|\nabla_{\mathrm{d}}(u_h-s_h)\|$ | $\eta$ | $I^{\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|
| $h_0$ | **1** | 1.07E-00 | 1.12E-00 | 5.55E-02 | 4.16E-01 | 1.25E-00 | **1.17** |
| $\approx h_0/2$ | | 5.56E-01 | 5.71E-01 | 7.42E-03 | 1.82E-01 | 6.07E-01 | **1.09** |
| $\approx h_0/4$ | | 2.92E-01 | 2.96E-01 | 1.04E-03 | 8.77E-02 | 3.10E-01 | **1.06** |
| $\approx h_0/8$ | | 1.39E-01 | 1.40E-01 | 1.10E-04 | 3.85E-02 | 1.45E-01 | **1.04** |
| $h_0$ | **2** | 1.54E-01 | 1.55E-01 | 5.10E-03 | 3.05E-02 | 1.63E-01 | **1.06** |
| $\approx h_0/2$ | | 4.07E-02 | 4.13E-02 | 3.53E-04 | 7.55E-03 | 4.23E-02 | **1.04** |
| $\approx h_0/4$ | | 1.10E-02 | 1.12E-02 | 2.51E-05 | 1.97E-03 | 1.14E-02 | **1.03** |
| $\approx h_0/8$ | | 2.50E-03 | 2.54E-03 | 1.30E-06 | 4.21E-04 | 2.57E-03 | **1.03** |
| $h_0$ | **3** | 1.37E-02 | 1.37E-02 | 3.58E-04 | 1.74E-03 | 1.41E-02 | **1.03** |
| $\approx h_0/2$ | | 1.85E-03 | 1.85E-03 | 1.26E-05 | 2.10E-04 | 1.88E-03 | **1.01** |
| $\approx h_0/4$ | | 2.60E-04 | 2.60E-04 | 4.73E-07 | 2.54E-05 | 2.62E-04 | **1.01** |
| $\approx h_0/8$ | | 2.75E-05 | 2.75E-05 | 1.15E-08 | 2.55E-06 | 2.76E-05 | **1.01** |
| $h_0$ | **4** | 9.87E-04 | 9.84E-04 | 2.12E-05 | 1.11E-04 | 1.01E-03 | **1.02** |
| $\approx h_0/2$ | | 6.92E-05 | 6.92E-05 | 3.96E-07 | 7.44E-06 | 7.00E-05 | **1.01** |
| $\approx h_0/4$ | | 5.04E-06 | 5.04E-06 | 7.58E-09 | 4.98E-07 | 5.07E-06 | **1.01** |
| $\approx h_0/8$ | | 2.58E-07 | 2.58E-07 | 8.96E-11 | 2.47E-08 | 2.60E-07 | **1.01** |
| $h_0$ | **5** | 5.64E-05 | 5.63E-05 | 1.06E-06 | 4.50E-06 | 5.75E-05 | **1.02** |
| $\approx h_0/2$ | | 2.01E-06 | 2.01E-06 | 9.88E-09 | 1.46E-07 | 2.03E-06 | **1.01** |
| $\approx h_0/4$ | | 7.74E-08 | 7.73E-08 | 1.01E-10 | 4.35E-09 | 7.76E-08 | **1.00** |
| $\approx h_0/8$ | | 1.86E-09 | 1.86E-09 | 1.70E-12 | 1.00E-10 | 1.86E-09 | **1.00** |
| $h_0$ | **6** | 2.85E-06 | 2.85E-06 | 4.70E-08 | 2.18E-07 | 2.90E-06 | **1.02** |
| $\approx h_0/2$ | | 5.42E-08 | 5.42E-08 | 2.40E-10 | 4.02E-09 | 5.46E-08 | **1.01** |
| $\approx h_0/4$ | | 1.07E-09 | 1.07E-09 | 1.03E-11 | 6.90E-11 | 1.08E-09 | **1.01** |

# Uniform refinement: **asymptotic exactness**

| $h$ | $p$ | $\|\nabla_d(u-u_h)\|$ | $\|u-u_h\|_{DG}$ | $\|\nabla_d u_h+\sigma_h\|$ | $\eta_{osc}$ | $\|\nabla_d(u_h-s_h)\|$ | $\eta$ | $\eta_{DG}$ | $I^{eff}$ | $I^{eff}_{DG}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h_0$ | 1 | 1.07E-00 | 1.09E-00 | 1.12E-00 | 5.55E-02 | 4.16E-01 | 1.25E-00 | 1.26E-00 | 1.17 | 1.16 |
| $\approx h_0/2$ | | 5.56E-01 | 5.61E-01 | 5.71E-01 | 7.42E-03 | 1.82E-01 | 6.07E-01 | 6.11E-01 | 1.09 | 1.09 |
| $\approx h_0/4$ | | 2.92E-01 | 2.93E-01 | 2.96E-01 | 1.04E-03 | 8.77E-02 | 3.10E-01 | 3.11E-01 | 1.06 | 1.06 |
| $\approx h_0/8$ | | 1.39E-01 | 1.39E-01 | 1.40E-01 | 1.10E-04 | 3.85E-02 | 1.45E-01 | 1.45E-01 | 1.04 | 1.04 |
| $h_0$ | 2 | 1.54E-01 | 1.55E-01 | 1.55E-01 | 5.10E-03 | 3.05E-02 | 1.63E-01 | 1.64E-01 | 1.06 | 1.06 |
| $\approx h_0/2$ | | 4.07E-02 | 4.09E-02 | 4.13E-02 | 3.53E-04 | 7.55E-03 | 4.23E-02 | 4.26E-02 | 1.04 | 1.04 |
| $\approx h_0/4$ | | 1.10E-02 | 1.11E-02 | 1.12E-02 | 2.51E-05 | 1.97E-03 | 1.14E-02 | 1.15E-02 | 1.03 | 1.03 |
| $\approx h_0/8$ | | 2.50E-03 | 2.52E-03 | 2.54E-03 | 1.30E-06 | 4.21E-04 | 2.57E-03 | 2.59E-03 | 1.03 | 1.03 |
| $h_0$ | 3 | 1.37E-02 | 1.37E-02 | 1.37E-02 | 3.58E-04 | 1.74E-03 | 1.41E-02 | 1.41E-02 | 1.03 | 1.03 |
| $\approx h_0/2$ | | 1.85E-03 | 1.85E-03 | 1.85E-03 | 1.26E-05 | 2.10E-04 | 1.88E-03 | 1.88E-03 | 1.01 | 1.01 |
| $\approx h_0/4$ | | 2.60E-04 | 2.60E-04 | 2.60E-04 | 4.73E-07 | 2.54E-05 | 2.62E-04 | 2.62E-04 | 1.01 | 1.01 |
| $\approx h_0/8$ | | 2.75E-05 | 2.75E-05 | 2.75E-05 | 1.15E-08 | 2.55E-06 | 2.76E-05 | 2.76E-05 | 1.01 | 1.01 |
| $h_0$ | 4 | 9.87E-04 | 9.87E-04 | 9.84E-04 | 2.12E-05 | 1.11E-04 | 1.01E-03 | 1.01E-03 | 1.02 | 1.02 |
| $\approx h_0/2$ | | 6.92E-05 | 6.93E-05 | 6.92E-05 | 3.96E-07 | 7.44E-06 | 7.00E-05 | 7.00E-05 | 1.01 | 1.01 |
| $\approx h_0/4$ | | 5.04E-06 | 5.04E-06 | 5.04E-06 | 7.58E-09 | 4.98E-07 | 5.07E-06 | 5.07E-06 | 1.01 | 1.01 |
| $\approx h_0/8$ | | 2.58E-07 | 2.59E-07 | 2.58E-07 | 8.96E-11 | 2.47E-08 | 2.60E-07 | 2.60E-07 | 1.01 | 1.01 |
| $h_0$ | 5 | 5.64E-05 | 5.64E-05 | 5.63E-05 | 1.06E-06 | 4.50E-06 | 5.75E-05 | 5.75E-05 | 1.02 | 1.02 |
| $\approx h_0/2$ | | 2.01E-06 | 2.01E-06 | 2.01E-06 | 9.88E-09 | 1.46E-07 | 2.03E-06 | 2.03E-06 | 1.01 | 1.01 |
| $\approx h_0/4$ | | 7.74E-08 | 7.74E-08 | 7.73E-08 | 1.01E-10 | 4.35E-09 | 7.76E-08 | 7.76E-08 | 1.00 | 1.00 |
| $\approx h_0/8$ | | 1.86E-09 | 1.86E-09 | 1.86E-09 | 1.70E-12 | 1.00E-10 | 1.86E-09 | 1.86E-09 | 1.00 | 1.00 |
| $h_0$ | 6 | 2.85E-06 | 2.85E-06 | 2.85E-06 | 4.70E-08 | 2.18E-07 | 2.90E-06 | 2.90E-06 | 1.02 | 1.02 |
| $\approx h_0/2$ | | 5.42E-08 | 5.42E-08 | 5.42E-08 | 2.40E-10 | 4.02E-09 | 5.46E-08 | 5.46E-08 | 1.01 | 1.01 |
| $\approx h_0/4$ | | 1.07E-09 | 1.07E-09 | 1.07E-09 | 1.03E-11 | 6.90E-11 | 1.08E-09 | 1.08E-09 | 1.01 | 1.01 |

# Numerics: singular case

## Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

## Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

## Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

# Numerics: singular case

## Model problem

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \\
u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

## Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

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- incomplete interior penalty discontinuous Galerkin method
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$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

## Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
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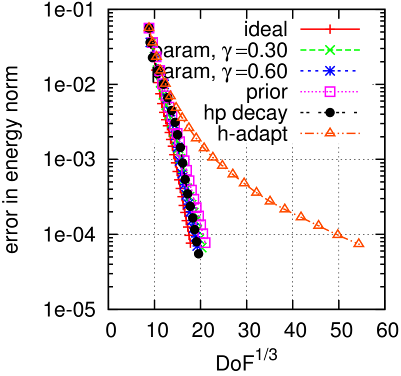
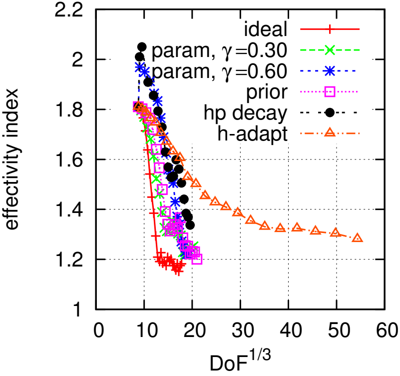
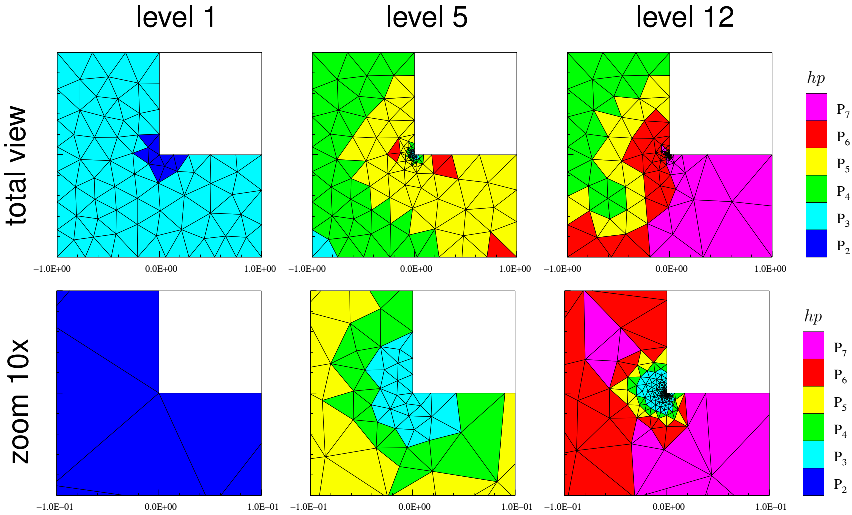
# *hp*-adaptive refinement: **exponential convergence**



# *hp*-refinement grids



# Outline

# Inexact iterative linearization

**System of nonlinear algebraic equations**

Nonlinear operator $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

**Algorithm (Inexact iterative linearization)**

1. *Choose initial vector $U^0$. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector $F^{k-1}$: find $U^k$ s.t.*

$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do an algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*

   $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow k := k + 1$, back to 2.*

# Inexact iterative linearization

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$$\mathcal{A}(U)=F$$

## Algorithm (Inexact iterative linearization)

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$$\mathbb{A}^{k-1}U^k\approx F^{k-1}.$$

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   1. *Set $U^{k,0}:=U^{k-1}$ and $i:=1$.*
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   $$\mathbb{A}^{k-1}U^{k,i}=F^{k-1}-R^{k,i}.$$

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# Inexact iterative linearization

**System of nonlinear algebraic equations**

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

## Algorithm (Inexact iterative linearization)

1. *Choose initial vector $U^0$. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector $F^{k-1}$: find $U^k$ s.t.*

$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3. 
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do an algebraic solver step $\Rightarrow$ $U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*

   $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK $\Rightarrow$ $U^k := U^{k,i}$. KO $\Rightarrow$ $i := i + 1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow$ $k := k + 1$, back to 2.*

# Context and questions

**Approximate solution**

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

**Numerical method**

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

**Partial differential equation**

- underlying PDE, $u$ its weak solution: $A(u) = f$

**Question (Stopping criteria** Eisenstat and Walker (1990's), Becker, Johnson, and Rannacher (1995), Deuflhard (2004 book), Arioli (2000's)**)**

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

**Question (Error** Verfürth (1994), Carstensen and Klose (2003), Chaillou and Suri (2006), Kim (2007)**)**

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**Approximate solution**

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

**Numerical method**

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

**Partial differential equation**

- underlying PDE, $u$ its weak solution: $A(u) = f$

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# Outline

1. Introduction
2. Laplace equation: potential & flux reconstructions
   - Guaranteed upper bound in a unified framework
   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Nonlinear Laplace equation: adaptive stopping criteria
   - Adaptive inexact Newton method
   - Applications & numerical results
4. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Applications & numerical results
5. Heat equation: robustness wrt final time & local efficiency
6. Two-phase flow in porous media: industrial application
7. Conclusions and outlook

# Model steady problem, discretization

**Quasi-linear elliptic problem**

$$\begin{aligned} -\nabla\cdot\overline{\boldsymbol{\sigma}}(u,\nabla u) &= f \qquad &&\text{in } \Omega,\\ u &= 0 \qquad &&\text{on } \partial\Omega \end{aligned}$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$
- example: $p$-Laplacian with $\overline{\boldsymbol{\sigma}}(u,\nabla u) = |\nabla u|^{p-2}\nabla u$
- weak solution: $u \in V := W_0^{1,p}(\Omega)$ such that

$$(\overline{\boldsymbol{\sigma}}(u,\nabla u),\nabla v) = (f,v) \qquad \forall v \in V$$

**Numerical approximation**

- simplicial mesh $\mathcal{T}_h$, linearization step $k$, algebraic step $i$
- $u_h^{k,i} \in V(\mathcal{T}_h) := \{v \in L^p(\Omega),\, v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\} \not\subset V$

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# Intrinsic error measure

## Energy error in the Laplace case

$$\|\nabla(u-u_h)\|^2 = \underbrace{\sup_{\varphi\in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla(u-u_h),\nabla\varphi)^2}_{\text{dual norm of the residual}} + \underbrace{\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

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$$+ \underbrace{\left\{\sum_{K\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_K} h_e^{1-q}\|[\![u-u_h^{k,i}]\!]\|_{q,e}^q\right\}^{1/q}}_{\text{distance of } u_h \text{ to } V}$$

✓ there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$

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# Abstract assumptions

**Assumption A (Total flux reconstruction)**

*There exists* $\boldsymbol{\sigma}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *such that*

$$\nabla \cdot \boldsymbol{\sigma}_h^{k,i} = f.$$

**Assumption B (Discretization, linearization, and alg. fluxes)**

*There exist fluxes* $\boldsymbol{\sigma}_{h,\mathrm{dis}}^{k,i}, \boldsymbol{\sigma}_{h,\mathrm{lin}}^{k,i}, \boldsymbol{\sigma}_{h,\mathrm{alg}}^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\boldsymbol{\sigma}_h^{k,i} = \boldsymbol{\sigma}_{h,\mathrm{dis}}^{k,i} + \boldsymbol{\sigma}_{h,\mathrm{lin}}^{k,i} + \boldsymbol{\sigma}_{h,\mathrm{alg}}^{k,i}$;

(ii) *as the linear solver converges,* $\|\boldsymbol{\sigma}_{h,\mathrm{alg}}^{k,i}\|_q \to 0$;

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# Estimate distinguishing error components

**Theorem (Estimate distinguishing different error components)**

*Let*

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

*Then there holds (up to quadrature and data oscillation)*

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i}.$$

# Estimators

- *discretization* estimator

$$\eta_{\mathrm{disc},K}^{k,i} := 2^{\frac{1}{p}} \left( \| \overline{\boldsymbol{\sigma}}(u_h^{k,i}, \nabla u_h^{k,i}) + \boldsymbol{\sigma}_{h,\mathrm{dis}}^{k,i} \|_{q,K} + \left\{ \sum_{e \in \mathcal{E}_K} h_e^{1-q} \| [\![ u_h^{k,i} ]\!] \|_{q,e}^q \right\}^{1/q} \right)$$

- *linearization* estimator

$$\eta_{\mathrm{lin},K}^{k,i} := \| \boldsymbol{\sigma}_{h,\mathrm{lin}}^{k,i} \|_{q,K}$$

- *algebraic* estimator

$$\eta_{\mathrm{alg},K}^{k,i} := \| \boldsymbol{\sigma}_{h,\mathrm{alg}}^{k,i} \|_{q,K}$$

- $\eta_{\cdot}^{k,i} := \left\{ \sum_{K \in \mathcal{T}_h} (\eta_{\cdot,K}^{k,i})^q \right\}^{1/q}$

# Stopping criteria: error components of similar size

## Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{alg}}^{k,i} \le \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \le \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

## Local stopping criteria

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## Comments

- ✓ same physical units (fluxes)
- ✓ naturally relative
- ✓ proper $[L^q(\Omega)]^d$ framework $\times$ $l_2$ norms of algebraic vectors

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# Global efficiency

**Theorem (Global efficiency)**

*Let the Assumptions C and D be satisfied. Let the global stopping criteria hold. Then,*

$$\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} \leq C \mathcal{J}_u(u_h^{k,i}),$$

*where C is independent of $\overline{\sigma}$ and q.*

**Theorem (Local efficiency)**

*Let the Assumptions C and D be satisfied. Let the **local** stopping criteria hold. Then*

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# Outline

# Applications

## Discretization methods

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin
- ✓ various finite volumes
- ✓ mixed finite elements

## Linearizations

- ✓ fixed point
- ✓ Newton

## Linear solvers

- ✓ independent of the linear solver

…all Assumptions A to D verified

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# Numerical experiment I

**Model problem**

- $p$-Laplacian

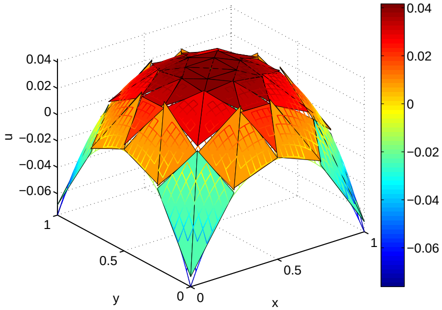
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f && \text{in } \Omega, \\ u &= u_{\mathrm{D}} && \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose the Dirichlet BC)

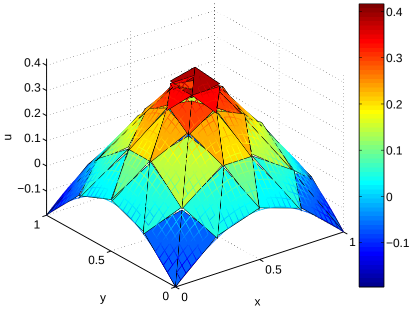
$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10
- Crouzeix–Raviart nonconforming finite elements

# Analytical and approximate solutions

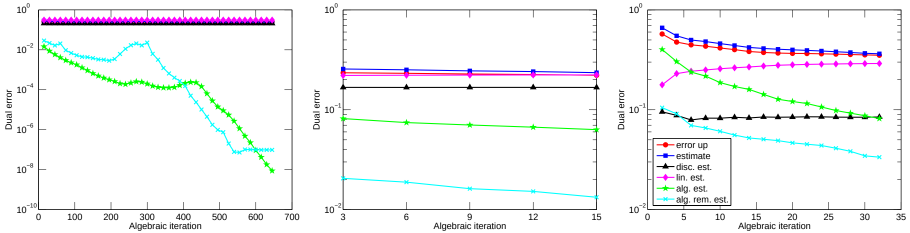



Case $p = 1.5$



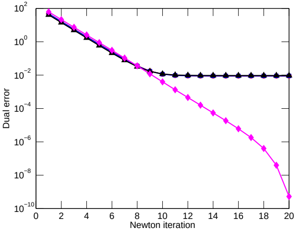

Case $p = 10$

# Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step

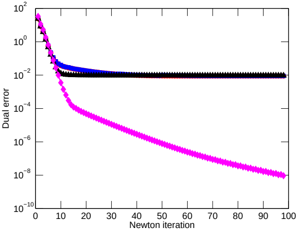



Newton inexact Newton ad. inexact Newton

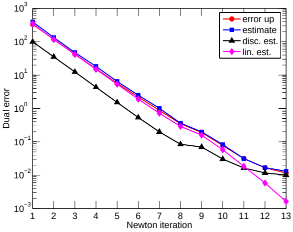
# Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh




Newton

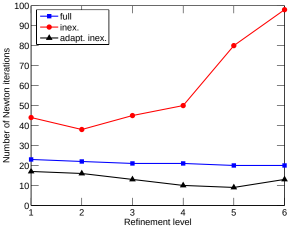



inexact Newton

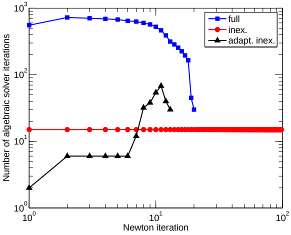



ad. inexact Newton

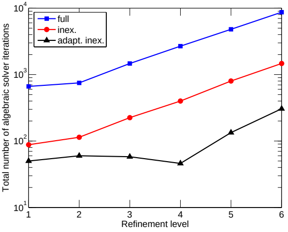
# Newton and algebraic iterations, $p = 10$




Newton it. / refinement

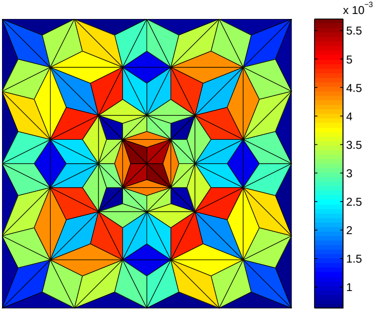



alg. it. / Newton step

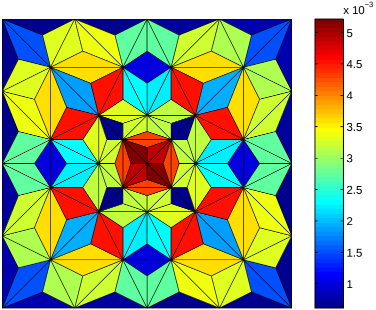



alg. it. / refinement

# Error distribution, $p = 10$

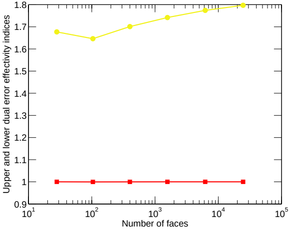



Estimated error distribution

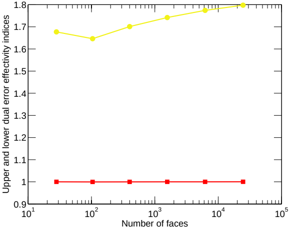



Exact error distribution

# Effectivity indices, $p = 10$




Newton

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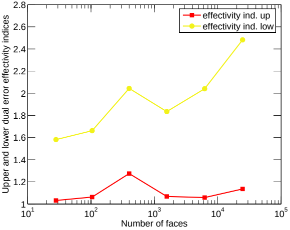
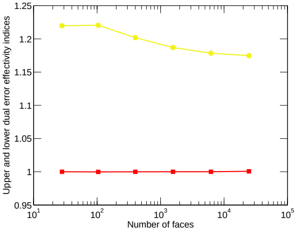
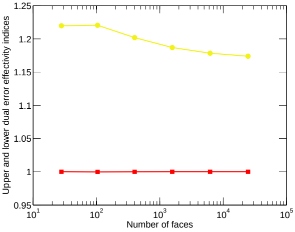



ad. inexact Newton

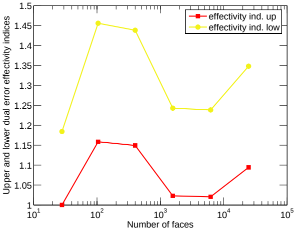
# Effectivity indices, $p = 1.5$




Newton




inexact Newton




ad. inexact Newton

# Numerical experiment II

**Model problem**

- $p$-Laplacian

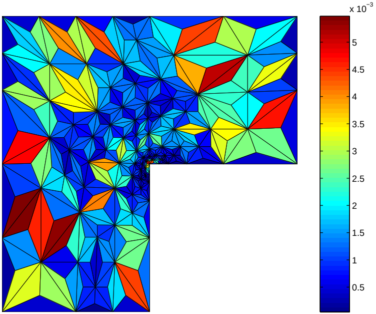
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f && \text{in } \Omega, \\ u &= u_{\mathrm{D}} && \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose the Dirichlet BC)

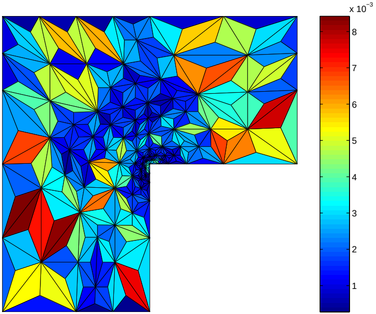
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- Crouzeix–Raviart nonconforming finite elements

# Error distribution on an adaptively refined mesh

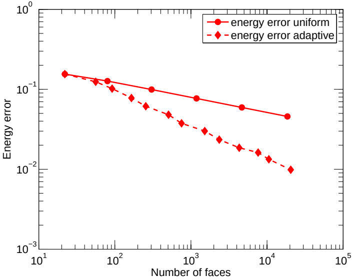



Estimated error distribution

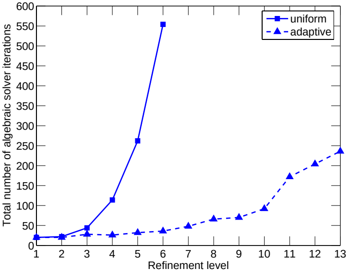



Exact error distribution

# Energy error and overall performance




Energy error




Overall performance

# Outline

# Laplace eigenvalue problem

**Problem**

Find eigenvector & eigenvalue pair $(u, \lambda)$ such that

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

**Weak formulation**

Find $(u_i, \lambda_i) \in V \times \mathbb{R}^+$, $i \geq 1$, with $\|u_i\| = 1$, such that

$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \qquad \forall v \in V.$$

# Laplace eigenvalue problem

**Problem**

Find eigenvector & eigenvalue pair $(u, \lambda)$ such that

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

**Weak formulation**

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$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \qquad \forall v \in V.$$

# Main results (conforming setting)

## Assumption A (Conforming variational solution)

*There holds*

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $\qquad (\Rightarrow \lambda_{1h} \geq \lambda_1)$

**We bound**

2. *i*-th eigenvector energy error

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**Assumption A (Conforming variational solution)**

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- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $\quad (\Rightarrow \lambda_{1h} \geq \lambda_1)$

**We bound**

2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih})$$

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- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $\quad(\Rightarrow \lambda_{1h} \geq \lambda_1)$

**We bound**

1. *i*-th eigenvalue error

$$\lambda_{ih} - \lambda_i \leq \eta_i(u_{ih}, \lambda_{ih})^2$$

2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih})$$

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**Assumption A (Conforming variational solution)**

*There holds*

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
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**We bound**

1. *i*-th eigenvalue error

$$\lambda_{ih} - \lambda_i \leq \eta_i(u_{ih}, \lambda_{ih})^2$$

2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih}) \leq C_{\mathrm{eff},i} \|\nabla(u_i - u_{ih})\|$$

# Main results (conforming setting)

## Assumption A (Conforming variational solution)

*There holds*

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $(\Rightarrow \lambda_{1h} \geq \lambda_1)$

**We bound**

1. *i*-th eigenvalue error

$$\lambda_{ih} - \lambda_i \leq \eta_i(u_{ih}, \lambda_{ih})^2$$

2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih}) \leq C_{\mathrm{eff},i} \|\nabla(u_i - u_{ih})\|$$

✓ $C_{\mathrm{eff},i}$ only depends on mesh shape regularity and on $\max\left\{\left(\frac{\lambda_i}{\lambda_{i-1}} - 1\right)^{-1}, \left(1 - \frac{\lambda_i}{\lambda_{i+1}}\right)^{-1}\right\} \frac{\lambda_i}{\lambda_1}$

✓ we give computable upper bounds on $C_{\mathrm{eff},i}$

# Main results (conforming setting)

**Assumption A (Conforming variational solution)**

*There holds*

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $\quad (\Rightarrow \lambda_{1h} \geq \lambda_1)$

**We bound**

1. *i*-th eigenvalue upper and lower bounds

$$\lambda_{ih} - \eta_i(u_{ih}, \lambda_{ih})^2 \leq \lambda_i \leq \lambda_{ih} - \tilde{\eta}_i(u_{ih}, \lambda_{ih})^2$$

2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih})$$

# Outline

# Unit square

## Setting

- $\Omega = (0,1)^2$
- $\lambda_1 = 2\pi^2$, $\lambda_2 = 5\pi^2$ known explicitly
- $u_1(x,y) = \sin(\pi x)\sin(\pi y)$ known explicitly

## Effectivity indices

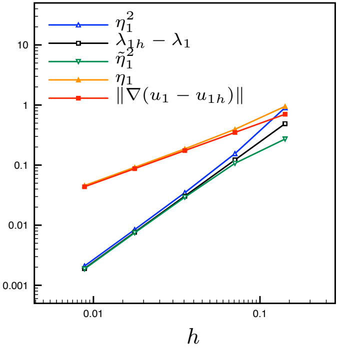
- recall $\tilde{\eta}_i^2 \leq \lambda_{ih} - \lambda_i \leq \eta_i^2$

$$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}} := \frac{\lambda_{ih} - \lambda_i}{\tilde{\eta}_i^2}, \qquad I^{\mathrm{ub}}_{\lambda,\mathrm{eff}} := \frac{\eta_i^2}{\lambda_{ih} - \lambda_i}$$

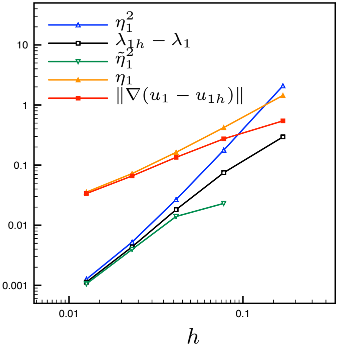
- recall $\|\nabla(u_i - u_{ih})\| \leq \eta_i$

$$I^{\mathrm{ub}}_{u,\mathrm{eff}} := \frac{\eta_i}{\|\nabla(u_i - u_{ih})\|}$$

# Conforming finite elements




Structured meshes




Unstructured meshes

# Conforming finite elements

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\lambda_{1h} - \eta_1^2$ | $\lambda_{1h} - \tilde{\eta}_1^2$ | $I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$ | $I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$ | $E_{\lambda,\mathrm{rel}}$ | $I^{\mathrm{ub}}_{u,\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.1414 | 121 | 19.7392 | 20.2284 | 19.5054 | 19.8667 | 1.35 | 1.48 | 1.84E-02 | 1.21 |
| 20 | 0.0707 | 441 | 19.7392 | 19.8611 | 19.7164 | 19.7486 | 1.08 | 1.19 | 1.63E-03 | 1.09 |
| 40 | 0.0354 | 1,681 | 19.7392 | 19.7696 | 19.7356 | 19.7401 | 1.03 | 1.12 | 2.28E-04 | 1.06 |
| 80 | 0.0177 | 6,561 | 19.7392 | 19.7468 | 19.7384 | 19.7393 | 1.02 | 1.10 | 4.56E-05 | 1.05 |
| 160 | 0.0088 | 25,921 | 19.7392 | 19.7411 | 19.7390 | 19.7392 | 1.02 | 1.10 | 1.01E-05 | 1.05 |

Structured meshes

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\lambda_{1h} - \eta_1^2$ | $\lambda_{1h} - \tilde{\eta}_1^2$ | $I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$ | $I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$ | $E_{\lambda,\mathrm{rel}}$ | $I^{\mathrm{ub}}_{u,\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.1698 | 143 | 19.7392 | 20.0336 | 18.8265 | – | – | 4.10 | – | 2.02 |
| 20 | 0.0776 | 523 | 19.7392 | 19.8139 | 19.6820 | 19.7682 | 1.63 | 1.77 | 4.37E-03 | 1.33 |
| 40 | 0.0413 | 1,975 | 19.7392 | 19.7573 | 19.7342 | 19.7416 | 1.15 | 1.28 | 3.75E-04 | 1.13 |
| 80 | 0.0230 | 7,704 | 19.7392 | 19.7436 | 19.7386 | 19.7395 | 1.07 | 1.14 | 4.56E-05 | 1.07 |
| 160 | 0.0126 | 30,666 | 19.7392 | 19.7403 | 19.7391 | 19.7393 | 1.06 | 1.10 | 1.01E-05 | 1.05 |

Unstructured meshes

# Conforming finite elements

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\lambda_{1h} - \eta_1^2$ | $\lambda_{1h} - \tilde{\eta}_1^2$ | $I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$ | $I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$ | $E_{\lambda,\mathrm{rel}}$ | $I^{\mathrm{ub}}_{u,\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.1414 | 121 | 19.7392 | 20.2284 | 19.5054 | 19.8667 | 1.35 | 1.48 | 1.84E-02 | 1.21 |
| 20 | 0.0707 | 441 | 19.7392 | 19.8611 | 19.7164 | 19.7486 | 1.08 | 1.19 | 1.63E-03 | 1.09 |
| 40 | 0.0354 | 1,681 | 19.7392 | 19.7696 | 19.7356 | 19.7401 | 1.03 | 1.12 | 2.28E-04 | 1.06 |
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Structured meshes

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\lambda_{1h} - \eta_1^2$ | $\lambda_{1h} - \tilde{\eta}_1^2$ | $I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$ | $I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$ | $E_{\lambda,\mathrm{rel}}$ | $I^{\mathrm{ub}}_{u,\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.1698 | 143 | 19.7392 | 20.0336 | 18.8265 | – | – | 4.10 | – | 2.02 |
| 20 | 0.0776 | 523 | 19.7392 | 19.8139 | 19.6820 | 19.7682 | 1.63 | 1.77 | 4.37E-03 | 1.33 |
| 40 | 0.0413 | 1,975 | 19.7392 | 19.7573 | 19.7342 | 19.7416 | 1.15 | 1.28 | 3.75E-04 | 1.13 |
| 80 | 0.0230 | 7,704 | 19.7392 | 19.7436 | 19.7386 | 19.7395 | 1.07 | 1.14 | 4.56E-05 | 1.07 |
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Unstructured meshes

# Conforming finite elements

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\lambda_{1h} - \eta_1^2$ | $\lambda_{1h} - \tilde{\eta}_1^2$ | $I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$ | $I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$ | $E_{\lambda,\mathrm{rel}}$ | $I^{\mathrm{ub}}_{u,\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
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| 160 | 0.0088 | 25,921 | 19.7392 | 19.7411 | 19.7390 | 19.7392 | 1.02 | 1.10 | 1.01E-05 | 1.05 |

Structured meshes

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\lambda_{1h} - \eta_1^2$ | $\lambda_{1h} - \tilde{\eta}_1^2$ | $I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$ | $I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$ | $E_{\lambda,\mathrm{rel}}$ | $I^{\mathrm{ub}}_{u,\mathrm{eff}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.1698 | 143 | 19.7392 | 20.0336 | 18.8265 | – | – | 4.10 | – | 2.02 |
| 20 | 0.0776 | 523 | 19.7392 | 19.8139 | 19.6820 | 19.7682 | 1.63 | 1.77 | 4.37E-03 | 1.33 |
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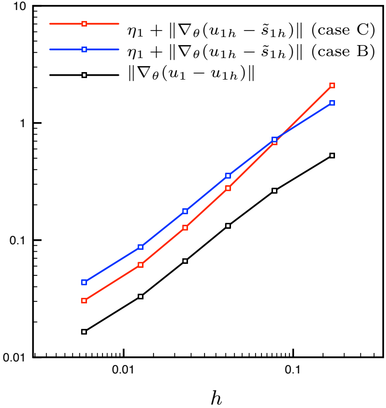
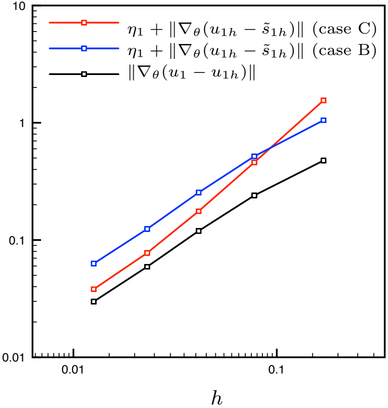
| 80 | 0.0230 | 7,704 | 19.7392 | 19.7436 | 19.7386 | 19.7395 | 1.07 | 1.14 | 4.56E-05 | 1.07 |
| 160 | 0.0126 | 30,666 | 19.7392 | 19.7403 | 19.7391 | 19.7393 | 1.06 | 1.10 | 1.01E-05 | 1.05 |

Unstructured meshes

# Nonconforming finite elements & DG's




Nonconforming finite elements




Discontinuous Galerkin

# Nonconforming finite elements & DG's

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$ | $E_{\lambda,\mathrm{rel}}$ | $I_{u,\mathrm{eff}}^{\mathrm{ub}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.1414 | 320 | 19.7392 | 19.6850 | 18.8966 | 19.8262 | 4.80e-02 | 2.68 |
| 20 | 0.0707 | 1240 | 19.7392 | 19.7257 | 19.6495 | 19.7616 | 5.69e-03 | 2.11 |
| 40 | 0.0354 | 4880 | 19.7392 | 19.7358 | 19.7246 | 19.7448 | 1.02e-03 | 1.91 |
| 80 | 0.0177 | 19360 | 19.7392 | 19.7384 | 19.7361 | 19.7406 | 2.29e-04 | 1.85 |
| 160 | 0.0088 | 77120 | 19.7392 | 19.7390 | 19.7385 | 19.7396 | 5.53e-05 | 1.83 |
| 320 | 0.0044 | 307840 | 19.7392 | 19.7392 | 19.7390 | 19.7393 | 1.37e-05 | 1.83 |

Nonconforming finite elements

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$ | $E_{\lambda,\mathrm{rel}}$ | $I_{u,\mathrm{eff}}^{\mathrm{ub}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.1698 | 732 | 19.7392 | 19.9432 | 17.8788 | 19.9501 | 1.10e-01 | 3.26 |
| 20 | 0.0776 | 2892 | 19.7392 | 19.7928 | 19.6264 | 19.7939 | 8.50e-03 | 1.91 |
| 40 | 0.0413 | 11364 | 19.7392 | 19.7526 | 19.7295 | 19.7529 | 1.18e-03 | 1.47 |
| 80 | 0.0230 | 45258 | 19.7392 | 19.7425 | 19.7381 | 19.7426 | 2.28e-04 | 1.31 |
| 160 | 0.0126 | 182070 | 19.7392 | 19.7400 | 19.7390 | 19.7401 | 5.35e-05 | 1.28 |

SIP discontinuous Galerkin

# Nonconforming finite elements & DG's

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$ | $E_{\lambda,\mathrm{rel}}$ | $I_{u,\mathrm{eff}}^{\mathrm{ub}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.1414 | 320 | 19.7392 | 19.6850 | 18.8966 | 19.8262 | 4.80e-02 | 2.68 |
| 20 | 0.0707 | 1240 | 19.7392 | 19.7257 | 19.6495 | 19.7616 | 5.69e-03 | 2.11 |
| 40 | 0.0354 | 4880 | 19.7392 | 19.7358 | 19.7246 | 19.7448 | 1.02e-03 | 1.91 |
| 80 | 0.0177 | 19360 | 19.7392 | 19.7384 | 19.7361 | 19.7406 | 2.29e-04 | 1.85 |
| 160 | 0.0088 | 77120 | 19.7392 | 19.7390 | 19.7385 | 19.7396 | 5.53e-05 | 1.83 |
| 320 | 0.0044 | 307840 | 19.7392 | 19.7392 | 19.7390 | 19.7393 | 1.37e-05 | 1.83 |

Nonconforming finite elements

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$ | $E_{\lambda,\mathrm{rel}}$ | $I_{u,\mathrm{eff}}^{\mathrm{ub}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.1698 | 732 | 19.7392 | 19.9432 | 17.8788 | 19.9501 | 1.10e-01 | 3.26 |
| 20 | 0.0776 | 2892 | 19.7392 | 19.7928 | 19.6264 | 19.7939 | 8.50e-03 | 1.91 |
| 40 | 0.0413 | 11364 | 19.7392 | 19.7526 | 19.7295 | 19.7529 | 1.18e-03 | 1.47 |
| 80 | 0.0230 | 45258 | 19.7392 | 19.7425 | 19.7381 | 19.7426 | 2.28e-04 | 1.31 |
| 160 | 0.0126 | 182070 | 19.7392 | 19.7400 | 19.7390 | 19.7401 | 5.35e-05 | 1.28 |

SIP discontinuous Galerkin

# Nonconforming finite elements & DG's

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$ | $E_{\lambda,\mathrm{rel}}$ | $I_{u,\mathrm{eff}}^{\mathrm{ub}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.1414 | 320 | 19.7392 | 19.6850 | 18.8966 | 19.8262 | 4.80e-02 | 2.68 |
| 20 | 0.0707 | 1240 | 19.7392 | 19.7257 | 19.6495 | 19.7616 | 5.69e-03 | 2.11 |
| 40 | 0.0354 | 4880 | 19.7392 | 19.7358 | 19.7246 | 19.7448 | 1.02e-03 | 1.91 |
| 80 | 0.0177 | 19360 | 19.7392 | 19.7384 | 19.7361 | 19.7406 | 2.29e-04 | 1.85 |
| 160 | 0.0088 | 77120 | 19.7392 | 19.7390 | 19.7385 | 19.7396 | 5.53e-05 | 1.83 |
| 320 | 0.0044 | 307840 | 19.7392 | 19.7392 | 19.7390 | 19.7393 | 1.37e-05 | 1.83 |

Nonconforming finite elements

| $N$ | $h$ | ndof | $\lambda_1$ | $\lambda_{1h}$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$ | $\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$ | $E_{\lambda,\mathrm{rel}}$ | $I_{u,\mathrm{eff}}^{\mathrm{ub}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.1698 | 732 | 19.7392 | 19.9432 | 17.8788 | 19.9501 | 1.10e-01 | 3.26 |
| 20 | 0.0776 | 2892 | 19.7392 | 19.7928 | 19.6264 | 19.7939 | 8.50e-03 | 1.91 |
| 40 | 0.0413 | 11364 | 19.7392 | 19.7526 | 19.7295 | 19.7529 | 1.18e-03 | 1.47 |
| 80 | 0.0230 | 45258 | 19.7392 | 19.7425 | 19.7381 | 19.7426 | 2.28e-04 | 1.31 |
| 160 | 0.0126 | 182070 | 19.7392 | 19.7400 | 19.7390 | 19.7401 | 5.35e-05 | 1.28 |

SIP discontinuous Galerkin

# Outline

# Model parabolic problem

**The heat equation**

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

**Spaces**

$$\begin{aligned} X &:= L^2(0,T;H_0^1(\Omega)), \\ \|v\|_X^2 &:= \int_0^T \|\nabla v\|^2 \,\mathrm{d}t, \\ Y &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ \|v\|_Y^2 &:= \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2 \end{aligned}$$

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Find $u \in Y$ with $u(0) = u_0$ such that

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# Model parabolic problem

**The heat equation**

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# Error and residual in the unsteady case

**Theorem (Parabolic inf–sup identity)**

*For every $\varphi \in Y$, we have*

$$\|\varphi\|_Y^2 = \left[ \sup_{v \in X,\, \|v\|_X = 1} \int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) \,\mathrm{d}t \right]^2 + \|\varphi(0)\|^2.$$

**Residual of $u_{h\tau} \in Y$**

- $\mathcal{R}(u_{h\tau}) \in X'$, the misfit of $u_{h\tau}$ in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v \rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v) \,\mathrm{d}t$$

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$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(u_{h\tau}), v \rangle$$

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# A posteriori estimate

**Guaranteed upper bound**

- ✓ $\|u - u_{h\tau}\|^2_{\mathcal{E}_Y,\Omega\times(0,T)} \le \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2$
- ✓ no undetermined constant: **error control**

**Local space-time efficiency**

- ✓ $\eta_K^n(u_{h\tau}) \le C_{\text{eff}} \|u - u_{h\tau}\|_{\mathcal{E}_Y,\text{neighbors of } K\times(t^{n-1},t^n)}$
- ✓ optimal space-time mesh refinement
- ✓ **local** in **time** and in **space** error lower bound

**Robustness**

- ✓ $C_{\text{eff}}$ independent of data, domain $\Omega$, **final time** $T$, meshes, solution $u$, **polynomial degrees** of $u_{h\tau}$ in space and in time

**Asymptotic exactness**

- ✓ $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|^2_{\mathcal{E}_Y,\Omega\times(0,T)} \searrow 1$
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# Multiphase, multi-compositional flows

**Two-phase immiscible incompressible flow**

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\begin{aligned}
\partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha, && \alpha \in \{\mathrm{o}, \mathrm{w}\},\\
-\lambda_\alpha(s_\mathrm{w})\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z) &= \mathbf{u}_\alpha, && \alpha \in \{\mathrm{o}, \mathrm{w}\},\\
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+ boundary & initial conditions

**Mathematical issues**

- coupled system
- unsteady, nonlinear
- elliptic–degenerate parabolic type
- dominant advection

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**Theorem (Distinguishing the error components)**

*Let*

- $n$ *be the time step,*
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*with the approximations* $(s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w},h\tau}^{n,k,i})$. *Then*

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- only a **necessary number** of all **solver iterations**
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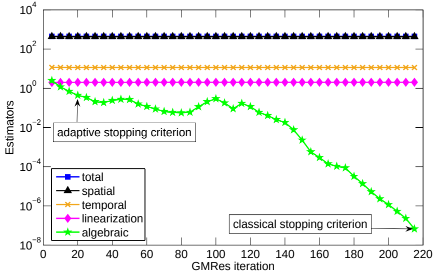
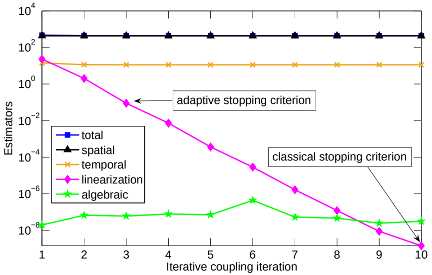
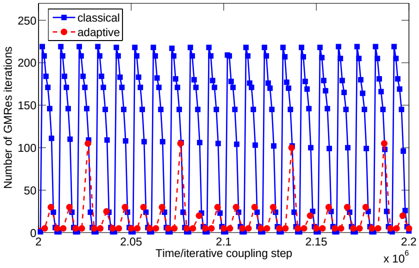



Estimators in function of GMRes iterations

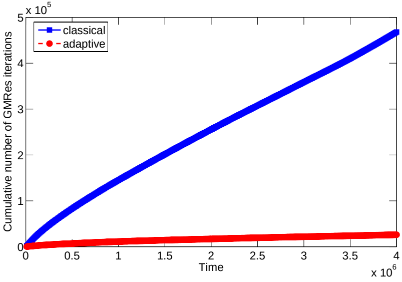



Estimators in function of iterative coupling iterations

# GMRes iterations

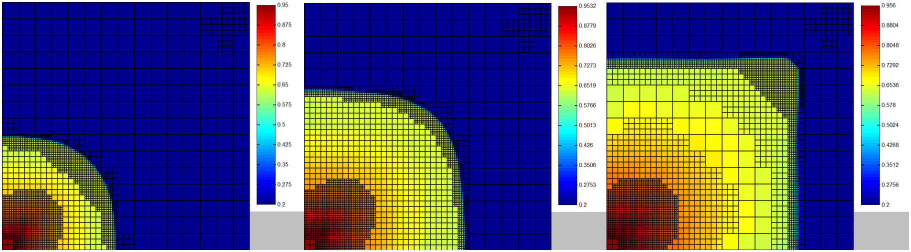



Per time and iterative coupling step




Cumulated

# Space/time/nonlinear solver/linear solver adaptivity




Fully adaptive computation

# Outline

# Conclusions and outlook

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- ✓ guaranteed energy error estimates
- ✓ robustness (polynomial degree, final time)
- ✓ local (space-time) efficiency
- ✓ unified framework for all classical numerical schemes
- ✓ cover the set of basic model problems (also variational inequalities, Stokes, changing coefficients, $H^{-1}$ source terms. . . )

## Ongoing work

- guaranteed reduction factor for *hp* refinement strategies
- convergence and optimality

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**Thank you for your attention!**