A posteriori error estimates for linear and nonlinear evolution problems using space-time dual norms

Vít Dolejší, Alexandre Ern, and Martin Vohralík

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Outline

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Literature: a posteriori estimates for the heat equation

- *Bieterman and Babuška (1982), Eriksson and Johnson (1991), Picasso (1998), Repin (2002), Makridakis and Nochetto (2003), Lakkis and Makridakis (2006), Demlow, Lakkis, and Makridakis (2009)*: upper bound for FEMs ($L^\infty(L^2)$, $L^\infty(L^\infty)$, or $L^2(H^1)$ (energy error)), possibly efficiency but under a restriction on the relative size of space and time steps or just order OK
- *Verfürth (2003), Bergam, Bernardi, and Mghazli (2004)*: local-in-time but global-in-space efficiency for FEM, robustness w.r.t. final time (energy error + $L^2(H^{-1})$ dual norm of the time derivative)
- *Ern and V. (2010):* extension to a unified framework for spatial discretizations (DGs, NC FEMs, MFEMs, FVs)
- *Schötzau and Wihler (2010)*: energy error, higher-order time stepping, semi-discretizations in time
- *Georgoulis, Lakkis, and Virtanen (2011)*: DGs
- ...

Literature: advection and nonlinear problems

Linear advection-diffusion problems

- *Verfürth (2005):* efficiency and robustness w.r.t. advection dominance (energy error + $L^2(H^{-1})$ dual norm of the material derivative), reaction-diffusion solve on each time step

Nonlinear problems

- *Verfürth (1998)*: efficiency under a restriction on the relative size of space and time steps
- *Verfürth (2004)*: efficiency (no restriction) but need to solve a linear diffusion problem on each time step
- *Makridakis and Nochetto (2006)*: higher-order time stepping, semi-discretizations in time
- *Kreuzer (2013)*: efficiency using a dual quasi-norm (parabolic p-Laplacian)
- *Bernardi, Dakroub, Mansour, and Sayah (2016), Amrein and Wihler (2016)*: adaptive linearization
- ...

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Literature: degenerate problems

- *Nochetto, Schmidt, and Verdi (2000), Ohlberger (2001)*: degenerate problems, upper bound
- *Di Pietro, V., and Yousef (2015)* – Stefan problem (degenerate diffusion), *Cancès, Pop, and V. (2014)* – two-phase porous media flow (coupled, advection, degenerate diffusion): adaptive regularization, linearization, discretization, and linear algebra; upper bound and efficiency

Problem

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$$\begin{aligned} \partial_t u - \nabla\cdot\boldsymbol{\sigma}(u,\nabla u) &= f && \text{in } Q := \Omega \times (0, t_{\mathrm{F}}), \\ u &= 0 && \text{on } \partial\Omega \times (0, t_{\mathrm{F}}), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega \end{aligned}$$

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$: polytopal domain
- $t_{\mathrm{F}} > 0$: final simulation time
- $f \in L^2(Q)$: source term
- $u_0 \in L^2(\Omega)$: initial datum
- $\boldsymbol{\sigma}(u,\nabla u) \in [L^2(Q)]^d$: nonlinear (diffusive-advective) flux function, $\boldsymbol{\sigma}(u,\nabla u) := \underline{\mathbf{K}}(u)\nabla u - \phi(u)$

Overview of main results

Main result

$$\mathcal{J}_u(u_{h\tau}) \leq \left\{ \sum_{n=1}^{N} \sum_{T \in \mathcal{T}^{n-1,n}} (\eta_T^n)^2 \right\}^{\frac{1}{2}} =: \eta,$$

$$\eta_T^n \leq C \mathcal{J}_u(u_{h\tau})|_{\omega_T \times I_n} \qquad (f \text{ pw pol., quadrature err. small}).$$

Comments

- $\mathcal{J}_u(u_{h\tau}) = \mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$ + jump terms, $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$: space-time mesh-dependent dual norm stemming from the problem and meshes at hand
- $u_{h\tau}$: piecewise space-time polynomial approximation (nonconforming, general framework: verify 2 assumptions)
- η_T^n are easily and locally computable from $u_{h\tau}$
- guaranteed upper bound (constant one)
- η_T^n can be decomposed into error components (spatial, temporal, regularization, linearization, algebraic)

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Comments

- **local space-time efficiency** and **robustness**:
  - C is independent of the final time t_{F}, domain Ω, nonlinear function σ, diffusion jumps, degenerate diffusion, advection dominance, and absolute and relative sizes of the space and time steps
  - C only depends on the space dimension d, shape-regularity of the meshes, maximal polynomial degree of $u_{h\tau}$, and maximal coarsening/refinement ratio (the last two dependencies removed while proceeding as in the talk by I. Smears - ESV HAL Preprint 01377086 (2016))
- estimates directly for the dual norm of the residual
- equivalence error–residual for nonlinear problems: a priori bounds of the linearized differential operator needed, difficult to trace the influence of nonlinearities & advection
- heat equation, conf. approximations: dual norm of the residual in $L^2(H^1) \cap H^1(H^{-1})$ = error in $L^2(H^1)$ ... energy estimates, results so far only under restrictions on h and τ

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Easily computable upper bound on $\mathcal{J}_u(u_{h\tau})$

Weighted $L^2(Q)$ norm + jump terms

$$\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}) \le e_{\mathrm{FR}} := \left\{ \sum_{n=1}^{N} \sum_{T\in\mathcal{T}^{n-1,n}} (e^n_{\mathrm{FR},T})^2 \right\}^{\frac{1}{2}}$$

$$e^n_{\mathrm{FR},T} := C_{T,n}^{-\frac{1}{2}} \{ (\tau^n)^{-2} \|u_{h\tau} - u\|^2_{T\times I_n} + h_T^{-2} \|\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_{h\tau},\nabla u_{h\tau})\|^2_{T\times I_n} \}^{\frac{1}{2}}$$

- typically $C_{T,n} := \left((\tau^n)^{-2} t_{\mathrm{F}} + h_T^{-2} C_{\phi,T,n} + h_T^{-2} C_{\underline{\mathbf{K}},T,n} \right)$
- $C_{\phi,T,n} := h_\Omega \|\phi'(u_{h\tau})\|_{\infty,Q}$, h_Ω is the diameter of Ω
- $C_{\underline{\mathbf{K}},T,n} := (h_\Omega / h_T) \|\underline{\mathbf{K}}(u_{h\tau})\|_{\infty,Q}$
- the first two addends in $C_{T,n}$ are of similar size if the Courant numbers $\tau^n \|\phi'(u_{h\tau})\|_{\infty,Q}/h_T$ are of order unity, as well as the ratios $t_{\mathrm{F}} \|\phi'(u_{h\tau})\|_{\infty,Q}/h_\Omega$ (the final time allows particles to be advected across a relevant part of Ω)
- the ratio of the second to the third addend is of the order of the Péclet numbers $h_T \|\phi'(u_{h\tau})\|_{\infty,Q} / \|\underline{\mathbf{K}}(u_{h\tau})\|_{\infty,Q}$.

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- incomplete interior penalty discontinuous Galerkin space discretization with polynomial degrees $p = 1, 2, 3$
- Crank–Nicolson in time
- space and time meshes both uniformly refined: $m = 1, 2, 3$

Effectivity indices

- dual norm

$$i_{\mathrm{e}} := \frac{\eta}{\mathcal{J}_u(u_{h\tau})} = \frac{\eta}{\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}) + \text{jumps}} \geq 1$$

- weighted L^2 norm:

$$i_{\mathrm{e,FR}} := \frac{\eta}{e_{\mathrm{FR}} + \text{jumps}} (< 1 \text{ possible})$$

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Viscous Burgers equation

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$$\partial_t u - \nabla\cdot(\varepsilon \nabla u - \phi(u)) = 0 \quad \text{in } Q$$

- $\varepsilon = 10^{-2}$ or $\varepsilon = 10^{-4}$
- $\phi(u) = (u^2/2, u^2/2)^{\mathrm{T}}$
- $\Omega = (-1,1) \times (-1,1)$
- $t_{\mathrm{F}} = 1$

Exact solution

- $$u(x,y,t) = \left(1 + \exp\left(\frac{x+y+1-t}{2\varepsilon}\right)\right)^{-1}$$

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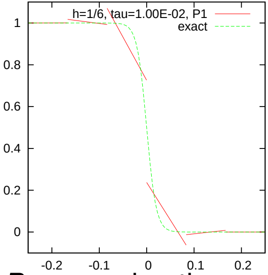
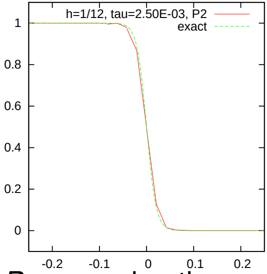
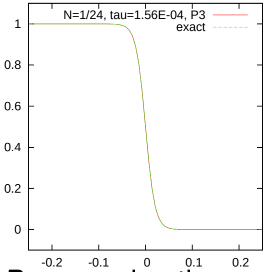
Exact and approximate solutions, $\varepsilon = 10^{-2}$



P_1 approximation on $\{h_1, \tau_1\}$



P_2 approximation on $\{h_2, \tau_2\}$



P_3 approximation on $\{h_3, \tau_3\}$

Errors, estimators, and effectivity indices, $\varepsilon = 10^{-2}$, $(h_0, \tau_0) = (1/6, 0.05)$

m	p	$J_{u,\mathrm{FR}}(u_{h\tau})$	η_{F}	η_{R}	η_{NC}	η_{IC}	η_{qd}	η	i_{e}	$i_{\mathrm{e,FR}}$
1	1	1.50E-02	1.11E-02	2.28E-02	4.11E-02	2.94E-02	3.82E-03	1.04E-01	1.85	1.15
2	1	1.17E-02	8.30E-03	1.52E-02	2.29E-02	1.31E-02	1.92E-03	5.94E-02	1.71	1.35
		(0.36)	(0.43)	(0.59)	(0.84)	(1.16)	(0.99)	(0.81)		
3	1	1.02E-02	5.16E-03	7.78E-03	1.16E-02	2.69E-03	7.49E-04	2.72E-02	1.25	1.36
		(0.20)	(0.69)	(0.96)	(0.98)	(2.29)	(1.36)	(1.13)		
1	2	4.97E-03	3.78E-03	8.23E-03	1.23E-02	1.32E-02	9.38E-04	3.72E-02	2.15	1.01
2	2	1.74E-03	1.36E-03	2.52E-03	4.02E-03	1.76E-03	2.34E-04	9.54E-03	1.65	0.94
		(1.52)	(1.47)	(1.71)	(1.61)	(2.90)	(2.00)	(1.96)		
3	2	4.63E-04	4.00E-04	7.36E-04	1.26E-03	3.01E-04	3.97E-05	2.63E-03	1.53	1.08
		(1.91)	(1.77)	(1.77)	(1.67)	(2.55)	(2.56)	(1.86)		
1	3	1.78E-03	9.11E-04	1.69E-03	3.41E-03	3.01E-03	2.20E-04	8.88E-03	1.71	0.59
2	3	3.47E-04	1.57E-04	3.26E-04	6.06E-04	6.20E-04	2.50E-05	1.67E-03	1.75	0.73
		(2.35)	(2.54)	(2.38)	(2.49)	(2.28)	(3.14)	(2.41)		
3	3	1.33E-05	1.80E-05	3.81E-05	6.97E-05	8.88E-05	1.64E-06	2.10E-04	2.54	0.97
		(4.71)	(3.12)	(3.10)	(3.12)	(2.80)	(3.93)	(2.99)		

Effectivity indices for varying ε and (h_0, τ_0)

ε		10^{-2}		10^{-2}		10^{-2}		10^{-4}	
(h_0, τ_0)		$(1/6, 0.05)$		$(1/6, 0.2)$		$(1/6, 0.0125)$		$(1/6, 0.05)$	
m	p	i_{e}	$i_{\mathrm{e,FR}}$	i_{e}	$i_{\mathrm{e,FR}}$	i_{e}	$i_{\mathrm{e,FR}}$	i_{e}	$i_{\mathrm{e,FR}}$
1	1	1.85	1.15	2.21	1.28	3.00	0.81	1.45	0.71
2	1	1.71	1.35	2.38	1.12	2.45	1.03	1.68	1.06
3	1	1.25	1.36	2.15	0.90	1.33	1.03	1.82	1.34
1	2	2.15	1.01	3.13	1.71	3.69	0.67	1.38	0.62
2	2	1.65	0.94	2.74	1.58	2.16	0.49	1.41	0.62
3	2	1.53	1.08	2.38	1.52	1.83	0.58	1.54	0.69
1	3	1.71	0.59	2.74	1.47	3.00	0.34	1.26	0.31
2	3	1.75	0.73	2.63	1.67	3.15	0.46	1.13	0.21
3	3	2.54	0.97	2.77	1.73	—	0.69	1.03	0.15

Degenerate advection-diffusion equation

Degenerate advection-diffusion problem (Kačur 2001)

$$\partial_t u - \nabla \cdot (2\varepsilon u \nabla u - \phi(u)) = 0 \quad \text{in } Q$$

- $\varepsilon = 10^{-2}$
- $\phi(u) = 0.5(u^2, 0)^{\mathrm{T}}$
- $\Omega = (0,1) \times (0,1)$
- $t_{\mathrm{F}} = 1$

Exact solution

- $$u(x,y,t) = \begin{cases} 1 - \exp\left(\frac{v(x-vt-x_0)}{2\varepsilon}\right) & \text{for } x \le vt + x_0, \\ 0 & \text{for } x > vt + x_0 \end{cases}$$
- $x_0 = 1/4$ is the initial position of the front

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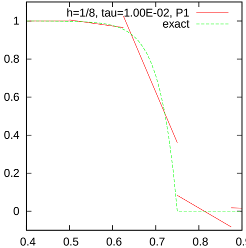
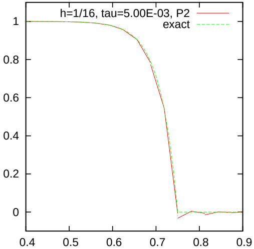
Exact and approximate solutions



P_1 approximation on $\{h_1, \tau_1\}$



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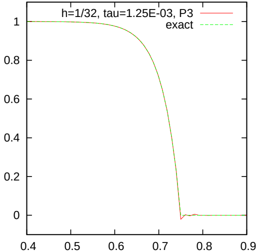

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Errors, estimators, and effectivity indices, $(h_0, \tau_0) = (1/8, 0.05)$

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1	1	9.91E-03	1.00E-02	6.02E-03	2.77E-02	2.31E-02	2.17E-03	6.62E-02	1.76	0.97
2	1	7.39E-03	7.71E-03	5.68E-03	1.62E-02	7.71E-03	1.23E-03	3.66E-02	1.55	1.02
		(0.42)	(0.37)	(0.08)	(0.78)	(1.59)	(0.82)	(0.86)		
3	1	4.58E-03	4.52E-03	4.95E-03	8.33E-03	1.86E-03	5.22E-04	1.89E-02	1.47	1.16
		(0.69)	(0.77)	(0.20)	(0.96)	(2.05)	(1.23)	(0.95)		
1	2	2.62E-03	3.30E-03	5.40E-03	9.33E-03	6.27E-03	6.74E-04	2.35E-02	1.97	0.73
2	2	1.11E-03	1.43E-03	1.93E-03	4.22E-03	1.09E-03	2.67E-04	8.34E-03	1.56	0.62
		(1.23)	(1.21)	(1.48)	(1.14)	(2.52)	(1.34)	(1.50)		
3	2	4.26E-04	5.63E-04	6.13E-04	1.84E-03	1.51E-04	1.00E-04	3.06E-03	1.35	0.57
		(1.38)	(1.34)	(1.65)	(1.20)	(2.85)	(1.42)	(1.45)		
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Porous medium equation

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$$\partial_t u - \nabla\cdot(\underline{\mathbf{K}}(u)\nabla u) = 0 \quad \text{in } Q$$

- $\underline{\mathbf{K}}(u) = \kappa|u|^{\kappa-1}\mathbb{I}$,
- $\kappa = 2$ or $\kappa = 4$
- $\Omega = (-6, 6) \times (-6, 6)$
- $t_{\mathrm{F}} = 1$

Barenblatt solution

- $$u(x, y, t) = \left\{ \frac{1}{t+1} \left[1 - \frac{\kappa - 1}{4\kappa^2} \frac{x^2 + y^2}{(t+1)^{1/\kappa}} \right]_+^{\frac{\kappa}{\kappa-1}} \right\}^{\frac{1}{\kappa}}$$

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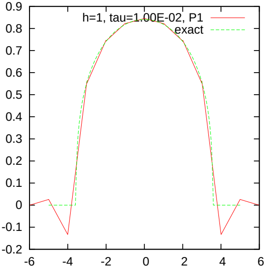
Exact and approximate solutions, $\kappa = 4$



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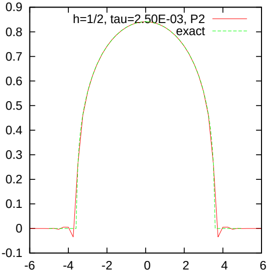

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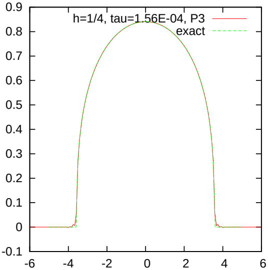

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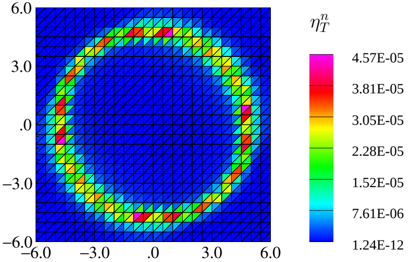
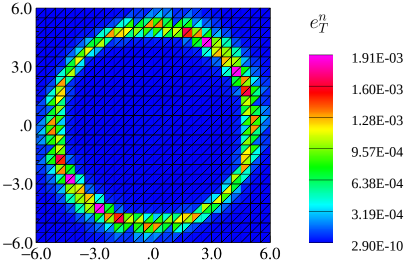
Errors, estimators, and effectivity indices, $(h_0, \tau_0) = (0.5, 0.02)$

					$\kappa = 2$						$\kappa = 4$	
m	$\mathbb{P}_p$	$J_{u,\mathrm{FR}}(u_{h\tau})$	η_{F}	η_{R}	η_{NC}	η_{IC}	η_{qd}	η	i_{e}	$i_{\mathrm{e,FR}}$	i_{e}	$i_{\mathrm{e,FR}}$
1	1	7.90E-03	5.90E-03	1.32E-02	9.10E-03	3.23E-02	7.08E-05	5.88E-02	3.46	0.92	4.68	0.98
2	1	8.36E-03	4.64E-03	1.71E-02	8.46E-03	1.11E-02	3.99E-05	4.03E-02	2.40	1.46	3.72	1.62
		(-0.08)	(0.35)	(-0.38)	(0.10)	(1.54)	(0.83)	(0.54)				
3	1	8.91E-03	4.38E-03	2.18E-02	9.56E-03	3.44E-03	1.83E-05	3.87E-02	2.09	2.49	3.38	2.68
		(-0.09)	(0.08)	(-0.35)	(-0.18)	(1.69)	(1.13)	(0.06)				
1	2	1.09E-03	1.06E-02	1.06E-01	3.12E-02	1.35E-02	1.74E-04	1.61E-01	4.99	3.22	5.13	3.18
2	2	4.02E-04	8.04E-03	8.12E-02	2.37E-02	5.16E-03	6.40E-05	1.18E-01	4.90	3.89	5.05	3.84
		(1.43)	(0.40)	(0.39)	(0.40)	(1.39)	(1.45)	(0.45)				
3	2	1.28E-04	5.22E-03	5.33E-02	1.55E-02	1.69E-03	2.23E-05	7.57E-02	4.84	4.26	4.97	4.30
		(1.65)	(0.62)	(0.61)	(0.61)	(1.61)	(1.52)	(0.64)				
1	3	6.53E-04	2.26E-02	3.27E-01	7.58E-02	8.39E-03	1.36E-04	4.33E-01	5.67	5.01	5.67	4.88
2	3	1.78E-04	9.26E-03	1.38E-01	3.13E-02	3.14E-03	3.51E-05	1.82E-01	5.76	5.17	5.78	5.03
		(1.87)	(1.29)	(1.24)	(1.27)	(1.42)	(1.95)	(1.25)				
3	3	3.83E-05	3.41E-03	5.08E-02	1.15E-02	1.14E-03	8.89E-06	6.68E-02	5.80	5.21	5.85	5.10
		(2.22)	(1.44)	(1.44)	(1.45)	(1.46)	(1.98)	(1.44)				

Exact and approximate error, $\kappa = 4$, $t = t_{\mathrm{F}}$, $p = 2$, $m = 2$



Estimators η_T^n



Error upper bounds e_T^n

Outline

Weak solution

Weak solution

Find $u \in X$ such that, for all $\varphi \in Y$,

$$\int_0^{t_{\mathrm{F}}} \{(f,\varphi) + (u,\partial_t\varphi) - (\boldsymbol{\sigma}(u,\nabla u),\nabla\varphi)\}(t)\,\mathrm{d}t + (u_0,\varphi(\cdot,0)) = 0$$

- $X := L^2(0,t_{\mathrm{F}};H_0^1(\Omega))$
- $Y := \{\varphi \in X;\ \partial_t\varphi \in L^2(Q);\ \varphi(\cdot,t_{\mathrm{F}}) = 0\}$

Comments

- ultra-weak spirit (time derivatives on the test functions)
- $\partial_t\varphi \in L^2(Q)$ on purpose: we can easily evaluate the norm on the test space in contrast to $\partial_t\varphi \in L^2(0,t_{\mathrm{F}};H^{-1}(\Omega))$

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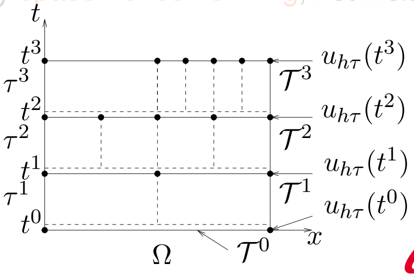
Discrete setting

Discrete setting

- discrete times $\{t^n\}_{0 \le n \le N}$, $t^0 = 0$ and $t^N = t_{\mathrm{F}}$
- time intervals $I_n := (t^{n-1}, t^n]$ and time steps $\tau^n := t^n - t^{n-1}$
- a different simplicial mesh $\mathcal{T}^n$ on all $0 \le n \le N$
  - $\overline{\mathcal{T}}^{n-1,n}$: the coarsest common refinement of $\mathcal{T}^{n-1}$ and $\mathcal{T}^n$
  - $\underline{\mathcal{T}}^{n-1,n}$: the finest common coarsening of $\mathcal{T}^{n-1}$ and $\mathcal{T}^n$

Approximate solution

- $u_{h\tau} \in X_h := \{\varphi \in L^2(0, t_{\mathrm{F}}; H^1(\mathcal{T})); \partial_t \varphi \in L^2(Q)\}$ (time-conf.)
- $u_{h\tau}$ possibly space-nonconforming, not included in X



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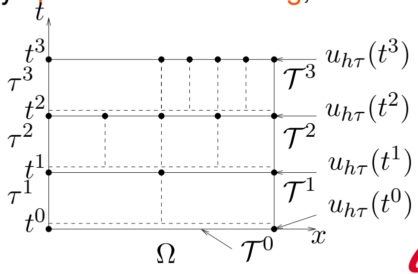

Space-time mesh-dependent dual norm

Residual

For $v \in L^2(0, t_{\mathrm{F}}; H^1(\mathcal{T}))$, $R(v) \in Y'$: for all $\varphi \in Y$,

$$\langle R(v), \varphi \rangle_{Y',Y} := \int_0^{t_{\mathrm{F}}} \{(f,\varphi) + (v, \partial_t \varphi) - (\boldsymbol{\sigma}(v, \nabla v), \nabla \varphi)\}(t)\,\mathrm{d}t + (u_0, \varphi(\cdot, 0))$$

Dual norm of the residual

$$\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}) := \sup_{\varphi \in Y,\, \|\varphi\|_Y = 1} \langle R(u_{h\tau}), \varphi \rangle_{Y',Y}$$

$$\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}) = \sup_{\varphi \in Y,\, \|\varphi\|_Y = 1} \int_0^{t_{\mathrm{F}}} \{(u_{h\tau} - u, \partial_t \varphi) + (\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_{h\tau}, \nabla u_{h\tau}), \nabla \varphi)\}(t)\mathrm{d}t$$

Space-time mesh-dependent norm on Y

$$\|\varphi\|^2_{Y,T\times I_n} := C_{T,n}\left({h_T}^2 \|\nabla \varphi\|^2_{T\times I_n} + (\tau^n)^2 \|\partial_t \varphi\|^2_{T\times I_n} \right),$$

$$\|\varphi\|^2_Y := \sum_{n=1}^{N} \sum_{T \in \mathcal{T}^{n-1,n}} \|\varphi\|^2_{Y,T\times I_n}$$

- $C_{T,n}$: user-given weights (no impact on a posteriori results)

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Evaluating $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$

- $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$ cannot be computed easily in practice (in the test cases where the exact solution u is available)
- solve the infinite-dimensional space-time dual problem: find $\psi \in Y$ such that

$$(\psi, \varphi)_Y = \langle R(u_{h\tau}), \varphi \rangle_{Y', Y} \qquad \forall \varphi \in Y,$$

  where $(\cdot,\cdot)_Y$ denotes the inner product corresponding to the $\|\cdot\|_Y$-norm
- then

$$\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}) = \|\psi\|_Y$$

- evaluate $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$ = solve a space-time problem in $H^1(Q)$ (0 at the final time)
- for numerical experiments, approximated on a finer mesh

Total error measure

Properties of the dual norm of the residual $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$

- for $u_{h\tau} \in X$, $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}) = 0$ if and only if $u_{h\tau} = u$

Nonconformity evaluation

$$\mathcal{J}_{u,\mathrm{NC}}(u_{h\tau}) := \left\{ \sum_{n=1}^{N} \sum_{T \in \mathcal{T}^{n-1,n}} \sum_{F \in \mathcal{F}_T} C_{T,n}^{-1} h_T^{-2} C_{\underline{\mathbf{K}},\phi,T,F,n} \| [\![u - u_{h\tau}]\!] \|_{F \times I_n}^2 \right\}^{\frac{1}{2}}$$

- $\mathcal{J}_{u,\mathrm{NC}}(u_{h\tau}) = 0$ if and only if $u_{h\tau} \in X$
- easily computable
- $C_{\underline{\mathbf{K}},\phi,T,F,n}$: weights (problem- and scheme-given)

Total error measure

$$\mathcal{J}_u(u_{h\tau}) := \mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}) + \mathcal{J}_{u,\mathrm{NC}}(u_{h\tau})$$

- $\mathcal{J}_u(u_{h\tau}) = 0$ if and only if $u_{h\tau} = u$

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Outline

Equilibrated flux reconstruction

Assumption (Space-time equilibrated flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_{h\tau}$ *such that*

$$\mathbf{t}_{h\tau} \in \mathbf{L}^2(0, t_{\mathrm{F}}; \mathbf{H}(\mathrm{div}, \Omega))$$

and

$$(f - \partial_t u_{h\tau} - \nabla\cdot\mathbf{t}_{h\tau}, 1)_{T\times I_n} = 0 \qquad \forall 1 \le n \le N, \forall T \in \mathcal{T}^{n-1,n}.$$

Comments

- the equilibration assumption expresses local mass conservation over the space-time element $T \times I_n$
- equilibrium not requested everywhere in time as in the talk by I Smears - ESV HAL Preprint 01377086 (2016)
- seems suitable for space-time methods
- both $u_{h\tau}$ and $\mathbf{t}_{h\tau}$ of same polynomial degree in time
- practical construction of $\mathbf{t}_{h\tau}$: spatial discretization at hand
  - implicit constructions by patchwise MFE problems (sharper, p-robust) (talk by I. Smears)
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Guaranteed upper bound

Theorem (Guaranteed a posteriori error estimate)

Let u be the weak solution. Let $u_{h\tau} \in X_h$ be arbitrary. Let the equilibration assumption hold true. Then

$$\mathcal{J}_u(u_{h\tau}) \leq \eta_{\mathrm{FR}} + \eta_{\mathrm{NC}} + \eta_{\mathrm{IC}}.$$

Comments

- no definition of any numerical scheme needed
- hinges only on the equilibration assumption

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Estimators

Estimators

- local: for all $1 \leq n \leq N$ and all $T \in \mathcal{T}^{n-1,n}$

$$\eta^n_{\mathrm{R},T} := C_{T,n}^{-\frac{1}{2}} \frac{1}{\pi} \|f - \partial_t u_{h\tau} - \nabla \cdot \mathbf{t}_{h\tau}\|_{T \times I_n}, \quad \text{equilibrium (time)}$$

$$\eta^n_{\mathrm{F},T} := C_{T,n}^{-\frac{1}{2}} h_T^{-1} \|\boldsymbol{\sigma}(u_{h\tau}, \nabla u_{h\tau}) + \mathbf{t}_{h\tau}\|_{T \times I_n}, \quad \text{const. law (space)}$$

$$\eta^n_{\mathrm{NC},T} := \left\{ \sum_{T' \in \overline{\mathcal{T}}^{n-1,n},\, T' \subset T} \sum_{F \in \mathcal{F}_{T'}} C_{T',n}^{-1} h_{T'}^{-2} C_{\underline{\mathbf{K}},\phi,T',F,n} \|[\![u_{h\tau}]\!]\|^2_{F \times I_n} \right\}^{\frac{1}{2}},$$

constraint (space)

$$\eta^n_{\mathrm{IC},T} := C_{T,n}^{-\frac{1}{2}} (\tau^n)^{-\frac{1}{2}} \|u_0 - u_{h\tau}(\cdot, 0)\|_T \quad \text{initial condition}$$

- global

$$\eta_\bullet := \left\{ \sum_{n=1}^{N} \sum_{T \in \mathcal{T}^{n-1,n}} (\eta^n_{\bullet,T})^2 \right\}^{\frac{1}{2}}$$

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Proof idea (bound on $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$)

- $\mathbf{t}_{h\tau} \in \mathbf{L}^2(0, t_\mathrm{F}; \mathbf{H}(\mathrm{div}, \Omega))$ and $\varphi \in Y$ & Green theorem; assumption $\partial_t u_{h\tau} \in L^2(Q)$ and $\varphi \in Y$ & IPP in time; space–time equilibration:

$$\langle R(u_{h\tau}),\varphi\rangle_{Y',Y} = \int_0^{t_\mathrm{F}} \{(f,\varphi) + (u_{h\tau}, \partial_t\varphi) - (\boldsymbol{\sigma}(u_{h\tau},\nabla u_{h\tau}),\nabla\varphi)\}(t)\,\mathrm{d}t$$
$$+ (u_0, \varphi(\cdot,0)) = \sum_{n=1}^{N} \sum_{T\in\mathcal{T}^{n-1,n}} \{(f - \partial_t u_{h\tau} - \nabla\cdot\mathbf{t}_{h\tau}, \varphi - \Pi_0\varphi)_{T\times I_n}$$
$$+ (u_{h\tau}(\cdot,0) - u_0, \partial_t\varphi)_{T\times I_n} - (\boldsymbol{\sigma}(u_{h\tau},\nabla u_{h\tau}) + \mathbf{t}_{h\tau}, \nabla\varphi)_{T\times I_n}\}$$

- local space-time Poincaré inequality:

$$\|\varphi - \Pi_0\varphi\|_{T\times I_n} \le C_\mathrm{P}\left(h_T^2\|\nabla\varphi\|^2_{T\times I_n} + (\tau^n)^2\|\partial_t\varphi\|^2_{T\times I_n}\right)^{\frac{1}{2}},$$

with $C_\mathrm{P} = \frac{1}{\pi}$ and $\Pi_0\varphi$ the mean value of φ over $T\times I_n$

Proof idea (bound on $\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau})$)

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- Cauchy–Schwarz inequality

$$
\begin{aligned}
&(u_{h\tau}(\cdot,0)-u_0,\partial_t\varphi)_{T\times I_n}-(\boldsymbol{\sigma}(u_{h\tau},\nabla u_{h\tau})+\mathbf{t}_{h\tau},\nabla\varphi)_{T\times I_n}\\
&\leq\left((\tau^n)^{-2}\|u_{h\tau}(\cdot,0)-u_0\|^2_{T\times I_n}+h_T^{-2}\|\boldsymbol{\sigma}(u_{h\tau},\nabla u_{h\tau})+\mathbf{t}_{h\tau}\|^2_{T\times I_n}\right)^{\frac{1}{2}}\\
&\quad\times\left((\tau^n)^2\|\partial_t\varphi\|^2_{T\times I_n}+h_T^2\|\nabla\varphi\|^2_{T\times I_n}\right)^{\frac{1}{2}}
\end{aligned}
$$

- altogether

$$\langle R(u_{h\tau}),\varphi\rangle_{Y',Y}\leq\sum_{n=1}^{N}\sum_{T\in\mathcal{T}^{n-1,n}}\left(\eta^n_{\mathrm{R},T}+\left((\eta^n_{\mathrm{F},T})^2+(\eta^n_{\mathrm{IC},T})^2\right)^{\frac{1}{2}}\right)\|\varphi\|_{Y,T\times I_n}$$

- conclude by

$$\mathcal{J}_{u,\mathrm{FR}}(u_{h\tau}):=\sup_{\varphi\in Y,\,\|\varphi\|_Y=1}\langle R(u_{h\tau}),\varphi\rangle_{Y',Y}$$

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Outline

Approximation property

Residual-based estimator

$$\eta^n_{\mathrm{clas},T} := h_T \| f - \partial_t u_{h\tau} + \nabla\cdot\big(\boldsymbol{\sigma}(u_{h\tau}, \nabla u_{h\tau})\big) \|_{T\times I_n} + \left\{ \sum_{F\in\mathcal{F}_T^{\mathrm{int}}} h_F \| [\![\boldsymbol{\sigma}(u_{h\tau}, \nabla u_{h\tau})]\!] \cdot \mathbf{n}_F \|^2_{F\times I_n} \right\}^{\frac{1}{2}} + \left\{ \sum_{F\in\mathcal{F}_T} C_{\underline{\mathbf{K}},\phi,T,F,n} \| [\![u_{h\tau}]\!] \|^2_{F\times I_n} \right\}^{\frac{1}{2}}$$

Assumption (Flux approximation property)

For all $1 \le n \le N$ *and all* $T \in \underline{\mathcal{T}}^{n-1,n}$, *there holds*

$$\| \boldsymbol{\sigma}(u_{h\tau}, \nabla u_{h\tau}) + \mathbf{t}_{h\tau} \|^2_{T\times I_n} \lesssim \sum_{T'\in\mathcal{T}^{n-1,n},\, T'\subset T} (\eta^n_{\mathrm{clas},T'})^2.$$

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Local efficiency

Theorem (Local-in-space and in-time efficiency)

Let a time step $1 \leq n \leq N$ and a mesh element $T \in \mathcal{T}^{n-1,n}$ be fixed. *Let the approximation assumption hold true. Let f be a piecewise space-time polynomial and let the quadrature errors be small enough. Then, there holds*

$$\eta^n_{\mathrm{FR},T} + \eta^n_{\mathrm{NC},T} \lesssim \mathcal{J}_u(u_{h\tau})|_{\omega_T \times I_n} + \mathcal{J}_{u,\mathrm{NC},T}(u_{h\tau}).$$

Comments

- $\mathcal{J}_{u,\mathrm{NC},T}(u_{h\tau})$ local nonconformity term
- local in space and in time
- full robustness w.r.t. nonlinearity, final time, advection dominance, degenerate diffusion, and discretization parameters – thanks to the choice of the dual residual norm

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Proof idea

Bounding the element residual

- Verfürth's bubble function technique
- $v_{T,n} := (f - \partial_t u_{h\tau} + \nabla \cdot \boldsymbol{\sigma}(u_{h\tau}, \nabla u_{h\tau}))|_{T \times I_n}$
- space-time bubble $\psi_{T,n}$, product of the barycentric coordinates on T and of the barycentric coordinates on I_n
- norm equivalence in finite-dimensional spaces:

$$(v_{T,n}, v_{T,n})_{T \times I_n} \lesssim (v_{T,n}, \psi_{T,n} v_{T,n})_{T \times I_n}$$

- inverse inequality separately in space and in time:

$$h_T \|\nabla(\psi_{T,n} v_{T,n})\|_{T \times I_n} \lesssim \|\psi_{T,n} v_{T,n}\|_{T \times I_n},$$
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Proof idea

- definition of $v_{T,n}$, bubble inequality, weak solution definition, and the Green theorem ($\boldsymbol{\sigma}_{\mathrm{d}} := \boldsymbol{\sigma}(u, \nabla u) - \overline{\boldsymbol{\sigma}}(u_{h\tau}, \nabla u_{h\tau})$),

$$C_{T,n}^{-1}\|v_{T,n}\|_{T\times I_n}^2 \lesssim C_{T,n}^{-1}(f - \partial_t u_{h\tau} + \nabla{\cdot}\overline{\boldsymbol{\sigma}}(u_{h\tau}, \nabla u_{h\tau}), \psi_{T,n}v_{T,n})_{T\times I_n}$$
$$= C_{T,n}^{-1}(u_{h\tau} - u, \partial_t(\psi_{T,n}v_{T,n}))_{T\times I_n} + (\boldsymbol{\sigma}_{\mathrm{d}}, \nabla(\psi_{T,n}v_{T,n}))_{T\times I_n}$$

- thus $C_{T,n}^{-\frac{1}{2}}\|v_{T,n}\|_{T\times I_n} \lesssim$

$$\frac{(u_{h\tau} - u, \partial_t(\psi_{T,n}v_{T,n}))_{T\times I_n} + (\boldsymbol{\sigma}_{\mathrm{d}}, \nabla(\psi_{T,n}v_{T,n}))_{T\times I_n}}{\|\psi_{T,n}v_{T,n}\|_{Y,T\times I_n}} \frac{C_{T,n}^{-\frac{1}{2}}\|\psi_{T,n}v_{T,n}\|_{Y,T\times I_n}}{\|v_{T,n}\|_{T\times I_n}}$$

- definition of the $\|\cdot\|_{Y,T\times I_n}$ norm, inverse ineq., $\|\psi_{T,n}v_{T,n}\|_{\infty,T\times I_n} \le 1$

$$C_{T,n}^{-1}\|\psi_{T,n}v_{T,n}\|_{Y,T\times I_n}^2 = \left(h_T^2\|\nabla(\psi_{T,n}v_{T,n})\|_{T\times I_n}^2 + (\tau^n)^2\|\partial_t(\psi_{T,n}v_{T,n})\|_{T\times I_n}^2\right)$$
$$\lesssim \|\psi_{T,n}v_{T,n}\|_{T\times I_n}^2 \le \|v_{T,n}\|_{T\times I_n}^2$$

- thus:

$$C_{T,n}^{-\frac{1}{2}}\|v_{T,n}\|_{T\times I_n} \lesssim \frac{(u_{h\tau} - u, \partial_t(\psi_{T,n}v_{T,n}))_{T\times I_n} + (\boldsymbol{\sigma}_{\mathrm{d}}, \nabla(\psi_{T,n}v_{T,n}))_{T\times I_n}}{\|\psi_{T,n}v_{T,n}\|_{Y,T\times I_n}},$$

Outline

Discontinuous Galerkin method

Discontinuous Galerkin method with CN time stepping

For all $1 \le n \le N$, find $u_h^n \in \mathbb{P}_p(\mathcal{T}^n)$ ($u_{h\tau}$ pw affine in time) s.t.

$$(\partial_t u_h^n, v_h) + \frac{1}{2} \sum_{m=n-1}^{n} \Bigg\{ (\boldsymbol{\sigma}(u_h^m, \nabla u_h^m), \nabla v_h) + \sum_{F \in \mathcal{F}^m} \alpha_{\underline{\mathbf{K}},F}^m h_F^{-1} ([\![u_h^m]\!], [\![v_h]\!])_F$$

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Flux reconstruction

- $\mathbf{t}_{h\tau}$ continuous and piecewise affine in time
- $\mathbf{t}_h^n$ prescribed in the Raviart–Thomas–Nédélec finite element spaces on $\mathcal{T}^n$ following Ainsworth (2007), Kim (2007), and Ern, Nicaise, and Vohralík (2007)
- both assumptions easily verified

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Conclusions and future work

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- space-time mesh-dependent dual norm stemming from the problem and meshes at hand
- guaranteed estimates
- **robustness** w.r.t. nonlinearity, final time, advection dominance, degenerate diffusion, discretization parameters
- **unified framework** (two conditions to verify for application)

Future work

- robustness in other norms
- extension to more complex problems

Thank you for your attention!

DOLEJŠÍ V., ERN A., VOHRALÍK M., A framework for robust a posteriori error control in unsteady nonlinear advection-diffusion problems, *SIAM J. Numer. Anal.* **51** (2013), 773–793

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- **unified framework** (two conditions to verify for application)

Future work

- robustness in other norms
- extension to more complex problems

Thank you for your attention!

DOLEJŠÍ V., ERN A., VOHRALÍK M., A framework for robust a posteriori error control in unsteady nonlinear advection-diffusion problems, *SIAM J. Numer. Anal.* **51** (2013), 773–793

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A Posteriori Error Estimates Adaptivity and Advanced Applications

Thursday 4th and Friday 5th May 2017

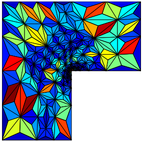
Speakers

- Sören Bartels
- Roland Becker
- Claudio Canuto
- Carsten Carstensen
- Emmanuel Creusé
- Lars Diening
- Christian Kreuzer
- Charalambos Makridakis
- Markus Melenk
- Pedro Morin
- Dirk Praetorius
- Iain Smears
- Rob Stevenson
- Andreas Veeser
- Thomas Wihler

Venue

Inria Paris,
2 Rue Simone Iff,
12th Arrondissement,
Paris 75012

Jacques-Louis Lions room, ground floor, building C



Organisers *Alexandre Ern, Iain Smears & Martin Vohralík*

Free registration by email
organisers_aposterioriworkshop2017@inria.fr
Website
https://project.inria.fr/gatipor/events/workshop/