Polynomial-degree-robust a posteriori error estimation for the curl-curl problem

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Outline

1. Introduction
2. Reminder on the H^1-case
3. The $\boldsymbol{H}$(curl)-case
4. $\boldsymbol{H}$(curl) patchwise equilibration
5. Stable (broken) $\boldsymbol{H}$(curl) polynomial extensions
6. Numerical experiments
7. Conclusions

The curl-curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{j}=0$)

The curl–curl problem

Find the magnetic vector potential $\boldsymbol{A}:\Omega\to\mathbb{R}^3$ *such that*

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{j}, \quad \nabla\cdot\boldsymbol{A}=0 \qquad & \text{in } \Omega,\\ \boldsymbol{A}\times\boldsymbol{n}_\Omega = \boldsymbol{0}, \qquad & \text{on } \Gamma_{\mathrm{D}},\\ (\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega = \boldsymbol{0}, \quad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega=0 \qquad & \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

Weak formulation (consequence)

$\boldsymbol{A}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ *satisfies*

$$(\nabla\times\boldsymbol{A},\nabla\times\boldsymbol{v}) = (\boldsymbol{j},\boldsymbol{v}) \qquad \forall \boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega).$$

Nédélec finite element discretization (consequence)

$\boldsymbol{V}_h := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, $p\ge 0$; $\boldsymbol{A}_h\in\boldsymbol{V}_h$ *satisfies*

$$(\nabla\times\boldsymbol{A}_h,\nabla\times\boldsymbol{v}_h) = (\boldsymbol{j},\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h.$$

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$$(\nabla\times\boldsymbol{A},\nabla\times\boldsymbol{v})=(\boldsymbol{j},\boldsymbol{v}) \qquad \forall\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega).$$

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Nédélec finite element discretization (consequence)

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, $p\ge 0$; $\boldsymbol{A}_h\in\boldsymbol{V}_h$ *satisfies*

$$(\nabla\times\boldsymbol{A}_h,\nabla\times\boldsymbol{v}_h) = (\boldsymbol{j},\boldsymbol{v}_h) \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h.$$

Bibliography

Residual estimates

- Monk (1998)
- Beck, Hiptmair, Hoppe, & Wohlmuth (2000)
- Nicaise & Creusé (2003)

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- Repin (2007)
- Hannukainen (2008)
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Equlibrated estimates

- Braess & Schöberl (2008): patchwise minimizations (lowest-order case $p = 0$)
- Licht (2019): a conceptual discussion
- Gedicke, Geevers, & Perugia (2020): equilibrated-residual-style construction
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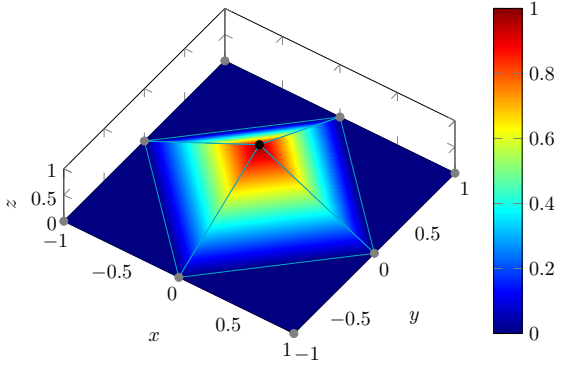
The hat function and the partition of unity, $\Omega \subset \mathbb{R}^d$



The hat function $\psi^{\boldsymbol{a}}$, $d = 2$

Partition of unity

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi^{\boldsymbol{a}} = 1|_{\Omega}$$

The Laplacian $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Weak solution $u \in H^1_0(\Omega)$ is such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_0(\Omega)$$

Approximation $u_h \in H^1_0(\Omega)$ satisfies

$$(\nabla u_h, \nabla \psi^{\boldsymbol{a}}) = (f, \psi^{\boldsymbol{a}}) \qquad \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$ is defined by

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v)$$

Norm characterization

$$\|\nabla(u - u_h)\| = \|\mathcal{R}(u_h)\|_{-1} = \sup_{\substack{v \in H^1_0(\Omega) \\ \|\nabla v\| = 1}} \langle \mathcal{R}(u_h), v \rangle$$

$$H^1_*(\omega_{\boldsymbol{a}}) := \begin{cases} \{v \in H^1(\omega_{\boldsymbol{a}});\ (v, 1)_{\omega_{\boldsymbol{a}}} = 0\} \text{ for interior vertex } \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}} \\ \{v \in H^1(\omega_{\boldsymbol{a}});\ v = 0 \text{ on faces sharing } \boldsymbol{a}\} \text{ for boundary vertex } \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{ext}} \end{cases}$$

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \right]^2 \right\}^{1/2}$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} C_{\mathrm{weight,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

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Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u-u_h)\| \le (d+1)^{1/2} C_{\mathrm{cont,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

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Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u-u_h)\| \le (d+1)^{1/2} C_{\mathrm{cont,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

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Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u-u_h)\| \le (d+1)^{1/2} C_{\mathrm{cont,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

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$$\|\nabla(u-u_h)\| \le (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \right]^2 \right\}^{1/2}$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u-u_h)\| \le (d+1)^{1/2} C_{\mathrm{cont,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

Bound by $\psi^{\boldsymbol{a}}$-weighted residuals on $H^1_*(\omega_{\boldsymbol{a}})'$

$v \in H^1_0(\Omega)$, $\|\nabla v\| = 1$:

$$\langle \mathcal{R}(u_h), v\rangle \overset{\text{PU}}{=} \left\langle \mathcal{R}(u_h), \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\psi^{\boldsymbol{a}} v)\right\rangle \overset{\text{GO}}{=} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}}(v - \tilde{\Pi}_0 v)\rangle$$

$$= \sum_{\boldsymbol{a}\in\mathcal{V}_h} \frac{\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}}(v - \tilde{\Pi}_0 v)\rangle}{\|\nabla v\|_{\omega_{\boldsymbol{a}}}} \|\nabla v\|_{\omega_{\boldsymbol{a}}}$$

$$\overset{\text{CS}}{\leq} \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left[\sup_{\substack{w\in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla w\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} w\rangle \right]^2 \right\}^{1/2} \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla v\|^2_{\omega_{\boldsymbol{a}}} \right\}^{1/2}$$

$$\overset{\text{overlaps}}{\leq} (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left[\sup_{\substack{w\in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla w\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} w\rangle \right]^2 \right\}^{1/2}$$

Bound by unweighted residuals on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$:

$$\|\nabla(\psi^{\boldsymbol{a}} v)\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla\psi^{\boldsymbol{a}} v\|_{\omega_{\boldsymbol{a}}} + \|\psi^{\boldsymbol{a}}\nabla v\|_{\omega_{\boldsymbol{a}}} \leq \underbrace{(1 + C_{\mathrm{PF}} h_{\omega_{\boldsymbol{a}}} \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}})}_{\leq C_{\mathrm{cont,PF}}} \|\nabla v\|_{\omega_{\boldsymbol{a}}}$$

bound for the $\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$, since $\psi^{\boldsymbol{a}} v \in H^1_0(\omega_{\boldsymbol{a}})$ for $v \in H^1_*(\omega_{\boldsymbol{a}})$:

$$\sup_{\substack{v\in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle = \sup_{v\in H^1_*(\omega_{\boldsymbol{a}})} \frac{\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle}{\|\nabla v\|_{\omega_{\boldsymbol{a}}}} = \sup_{v\in H^1_*(\omega_{\boldsymbol{a}})} \frac{\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle}{\|\nabla(\psi^{\boldsymbol{a}} v)\|_{\omega_{\boldsymbol{a}}}} \frac{\|\nabla(\psi^{\boldsymbol{a}} v)\|_{\omega_{\boldsymbol{a}}}}{\|\nabla v\|_{\omega_{\boldsymbol{a}}}}$$

$$\leq C_{\mathrm{cont,PF}} \sup_{\substack{v\in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), v\rangle$$

Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\sigma}_h^{\boldsymbol{a}}\cdot\boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ on $\partial\omega_{\boldsymbol{a}}$ and $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h\cdot\nabla\psi^{\boldsymbol{a}}$,

$$
\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \\
&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
&= (f\psi^{\boldsymbol{a}} - \nabla u_h\cdot\nabla\psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\psi^{\boldsymbol{a}}\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\psi^{\boldsymbol{a}}\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}
$$

$$
\sup_{\substack{v\in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \leq \|\psi^{\boldsymbol{a}}\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

$$
\begin{aligned}
\langle \mathcal{R}(u_h), v\rangle &= (f, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}
$$

$$
\sup_{\substack{v\in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), v\rangle \leq \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
$$

Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ *on* $\partial\omega_{\boldsymbol{a}}$ *and* $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h\cdot\nabla\psi^{\boldsymbol{a}}$,

$$
\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \\
&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
&= (f\psi^{\boldsymbol{a}} - \nabla u_h\cdot\nabla\psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
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$$

$$
\sup_{\substack{v\in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \le \|\psi^{\boldsymbol{a}}\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

$$
\begin{aligned}
\langle \mathcal{R}(u_h), v\rangle &= (f, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\le} \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}
$$

$$
\sup_{\substack{v\in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), v\rangle \le \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
$$

Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\sigma_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\sigma_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ on $\partial\omega_{\boldsymbol{a}}$ and $\nabla \cdot \sigma_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}$,

$$
\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \\
&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
&= (f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla \cdot \sigma_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\psi^{\boldsymbol{a}} \nabla u_h + \sigma_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\psi^{\boldsymbol{a}} \nabla u_h + \sigma_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}
$$

$$
\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \leq \|\psi^{\boldsymbol{a}} \nabla u_h + \sigma_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\sigma_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla \cdot \sigma_h^{\boldsymbol{a}} = f$,

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Patchwise bounds by equilibrated fluxes

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\begin{aligned}
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&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
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Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

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Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ on $\partial\omega_{\boldsymbol{a}}$ and $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla\psi^{\boldsymbol{a}}$,

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$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

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\langle \mathcal{R}(u_h), v\rangle &= (f, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
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&\overset{\text{CS}}{\le} \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}
$$

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Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

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$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

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&\overset{\text{Green}}{=} -(\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
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$$

Patchwise bounds by equilibrated fluxes

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for $v \in H^1_*(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ *on* $\partial\omega_{\boldsymbol{a}}$ *and* $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla\psi^{\boldsymbol{a}}$,

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$$= (f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla\psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}}$$
$$= (\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}}$$
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$$\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \leq \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

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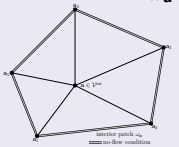
Discrete patchwise/broken patchwise equilibrated fluxes

Definition (Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \underset{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div};\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_{p+1}(f\psi^{\boldsymbol{a}} - \nabla u_h\cdot\nabla\psi^{\boldsymbol{a}})}}{\arg\min} \|\psi^{\boldsymbol{a}}\nabla u_h + \boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$



and combine $\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$.

Key points

- **homogeneous normal BC** on $\partial\omega_{\boldsymbol{a}}$:
  $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega)$
- **global equilibrium** $\nabla\cdot\boldsymbol{\sigma}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}$
  $= \sum_{\boldsymbol{a}\in\mathcal{V}_h} \Pi_{p+1}(f\psi^{\boldsymbol{a}} - \nabla u_h\cdot\nabla\psi^{\boldsymbol{a}}) = \Pi_{p+1} f$

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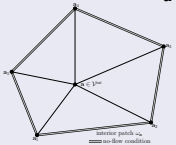

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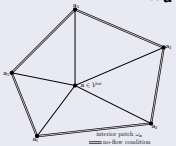

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Definition (~ Chaumont-Frelet, Ern, Vohralík (2021))

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Key points

- **no BC** on $\partial\omega_{\boldsymbol{a}}$:
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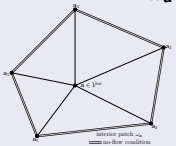

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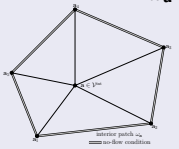

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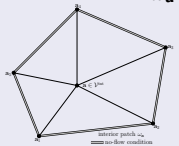

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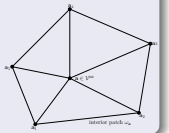

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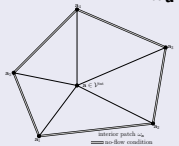

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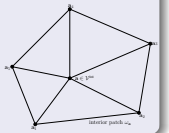

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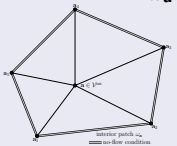

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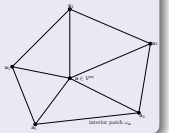

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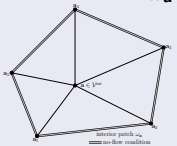

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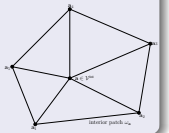

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Guaranteed upper bound

$$\underbrace{\|\nabla(u-u_h)\|}_{\text{unknown error}} \le \underbrace{\|\nabla u_h + \sigma_h\|}_{\text{computable estimator}}$$

$$\le (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2 \right\}^{1/2}$$

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Outline

1 Introduction

2 Reminder on the H^1-case

3 The $\boldsymbol{H}$(curl)-case

4 $\boldsymbol{H}$(curl) patchwise equilibration

5 Stable (broken) $\boldsymbol{H}$(curl) polynomial extensions

6 Numerical experiments

7 Conclusions

The curl-curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{j}=0$)

The curl–curl problem

Find the magnetic vector potential $\boldsymbol{A}:\Omega\to\mathbb{R}^3$ *such that*

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{A})=\boldsymbol{j}, \quad \nabla\cdot\boldsymbol{A}=0 \qquad & \text{in } \Omega,\\ \boldsymbol{A}\times\boldsymbol{n}_\Omega=\boldsymbol{0}, \qquad & \text{on } \Gamma_{\mathrm{D}},\\ (\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega=\boldsymbol{0}, \quad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega=0 \qquad & \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

Weak formulation (consequence)

$\boldsymbol{A}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ *satisfies*

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Nédélec finite element discretization (consequence)

$\boldsymbol{V}_h:=\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, $p\geq 0$; $\boldsymbol{A}_h\in\boldsymbol{V}_h$ *satisfies*

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$\boldsymbol{V}_h:=\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, $p\ge 0$; $\boldsymbol{A}_h\in\boldsymbol{V}_h$ *satisfies*

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The curl-curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

The curl–curl problem

Find the magnetic vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ *such that*

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{A}) = \boldsymbol{j}, \quad \nabla \cdot \boldsymbol{A} = 0 \qquad & \text{in } \Omega, \\ \boldsymbol{A} \times \boldsymbol{n}_\Omega = \boldsymbol{0}, \qquad & \text{on } \Gamma_{\mathrm{D}}, \\ (\nabla \times \boldsymbol{A}) \times \boldsymbol{n}_\Omega = \boldsymbol{0}, \quad \boldsymbol{A} \cdot \boldsymbol{n}_\Omega = 0 \qquad & \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

Weak formulation (consequence)

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$(\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

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Discrete patchwise/broken patchwise equilibrated fluxes

Definition (Chaumont-Frelet, Vohralík (2021))

For each **vertex** $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

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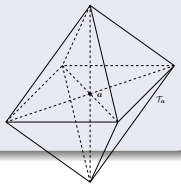

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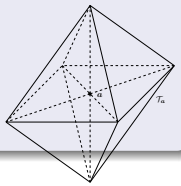

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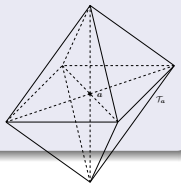

Key points

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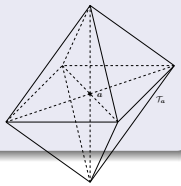

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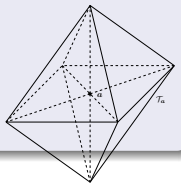

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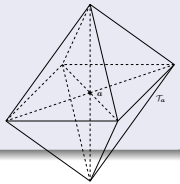

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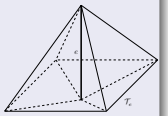

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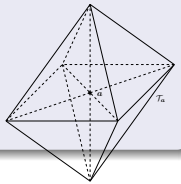

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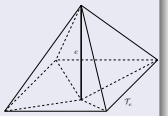

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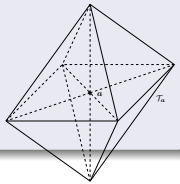

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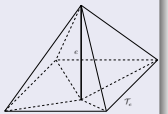

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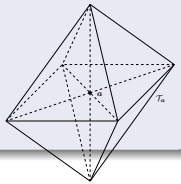

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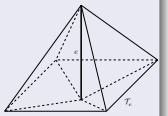

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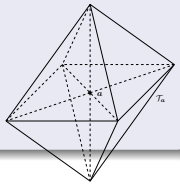

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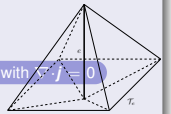
▸ **well-posed** for $\boldsymbol{j}|_{\omega_e} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_e) \cap \boldsymbol{H}(\mathrm{div}, \omega_e)$ with $\nabla \cdot \boldsymbol{j} = 0$



Key points

- **homogeneous tangential BC** on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$
- **global equilibrium** $\nabla \times \boldsymbol{h}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \nabla \times \boldsymbol{h}_h^{\boldsymbol{a}}$
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Key points

- **no BC** on $\partial\omega_e$: $\boldsymbol{h}_h = \sum_{e \in \mathcal{E}_h} \boldsymbol{h}_h^e \notin \boldsymbol{H}(\mathrm{curl}, \Omega)$
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Discrete patchwise/broken patchwise equilibrated fluxes

Definition (Chaumont-Frelet, Vohralík (2021))

For each **vertex** $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \boldsymbol{j} + \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)}} \|\psi^{\boldsymbol{a}} (\nabla \times \boldsymbol{A}_h) - \boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$

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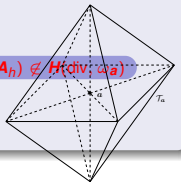

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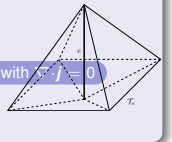

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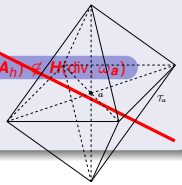

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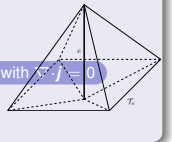

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Bottom line

Continuous case

- When there exist $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v} = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Discrete case

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The curl–curl case

Guaranteed upper bound

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}} \le \underbrace{\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h\|}_{\text{computable estimator}}$$

$$\le 2\left\{\sum_{\boldsymbol{a}\in\mathcal{V}_h}\|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)-\boldsymbol{h}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2\right\}^{1/2}$$

p-robust local efficiency (Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009, 2012); Chaumont-Frelet & V. (2021))

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Patchwise/broken patchwise flux equilibration

Patchwise flux equilibration

- **globally** *equilibrated flux*
- *Prager–Synge* **constant-free** *upper bound*
- *equilibration in several stages, more expensive*
- *additional layer for efficiency*
- *p*-robust

Broken patchwise flux equilibration

- **locally** *equilibrated flux*
- $6^{1/2}$, $C_{\mathrm{cont,PF}}$, *and* C_{L} *in the upper bound*
  $C_{\mathrm{L}} = 1$ *if* Ω *is convex and no mixed BCs*
- *equilibration in a single stage, cheaper, explicit for* $p = 0$
- *both estimator and efficiency on* $\omega_{\boldsymbol{a}}$
- *p*-robust

Lift constant C_{L} such that for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, $\nabla\times\boldsymbol{w} = \nabla\times\boldsymbol{v}$, and

$$\|\nabla\boldsymbol{w}\| \le C_{\mathrm{L}}\|\nabla\times\boldsymbol{v}\|$$

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- $6^{1/2}$, $C_{\mathrm{cont,PF}}$, *and* C_{L} *in the upper bound*
  $C_{\mathrm{L}} = 1$ *if* Ω *is convex and no mixed BCs*
- *equilibration in a* **single stage**, **cheaper**, **explicit** *for* $p = 0$
- *both estimator and efficiency on* $\mathcal{W}_h$
- *p-robust*

Lift constant C_{L} such that for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{v}$, and

$$\|\nabla \boldsymbol{w}\| \leq C_{\mathrm{L}} \|\nabla \times \boldsymbol{v}\|$$

Patchwise/broken patchwise flux equilibration

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Outline

1 Introduction

2 Reminder on the H^1-case

3 The $\boldsymbol{H}$(curl)-case

4 $\boldsymbol{H}$(curl) patchwise equilibration

5 Stable (broken) $\boldsymbol{H}$(curl) polynomial extensions

6 Numerical experiments

7 Conclusions

Stage 1: overconstrained Raviart–Thomas projection

Projection of $\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)$ to a Raviart–Thomas space

For all vertices $\boldsymbol{a}\in\mathcal{V}_h$, consider p' $:=\min\{p,1\}$-degree patchwise minimizations:

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}}} \|\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)-\boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}.$$

$$(\boldsymbol{v}_h,\boldsymbol{r}_h)_K=(\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h\in[\mathcal{P}_0(K)]^3,\forall K\in\mathcal{T}_{\boldsymbol{a}}$$

Comments

- $\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)\notin\mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$
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  - crucial for stage 2
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Stage 2: divergence-free decomposition of the given divergence-free Raviart-Thomas piecewise polynomial $\boldsymbol{\delta}_h$

Divergence-free decomposition of $\boldsymbol{\delta}_h$

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$-degree elementwise minimizations:

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Comments

- patchwise contributions

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{and} \quad \nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h$$

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Stage 2: divergence-free decomposition of the given divergence-free current density $\boldsymbol{j}$

Divergence-free decomposition of the current density $\boldsymbol{j}$

Set

$$\boldsymbol{j}_h^{\boldsymbol{a}} := \psi^{\boldsymbol{a}} \boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}.$$

Then

$$\begin{aligned} \boldsymbol{j}_h^{\boldsymbol{a}} &\in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \\ \nabla \cdot \boldsymbol{j}_h^{\boldsymbol{a}} &= 0, \\ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} &= \boldsymbol{j}. \end{aligned}$$

Stage 3: discrete patchwise equilibrated fluxes

Definition (Chaumont-Frelet, Vohralík (2021))

For each **vertex** $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0(\mathrm{curl},\omega_{\boldsymbol{a}}) \\ \nabla\times\boldsymbol{v}_h = f_h^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h) - \boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$

and combine

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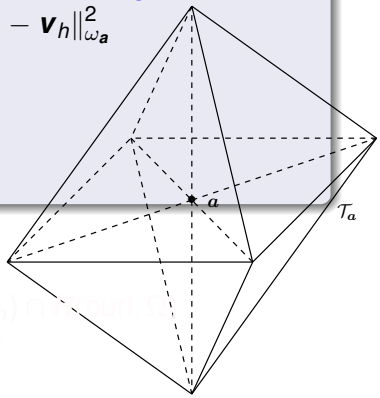

Key points

- **homogeneous tangential BC** on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap H_0(\mathrm{curl},\Omega)$
- **global equilibrium** $\nabla\times\boldsymbol{h}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\times\boldsymbol{h}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} f_h^{\boldsymbol{a}} = f$

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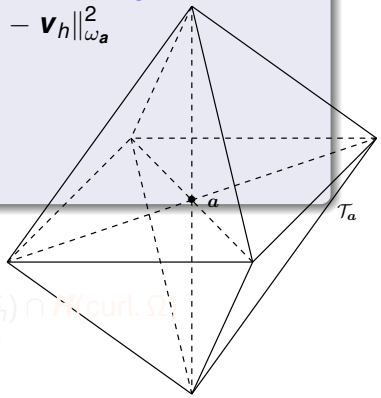

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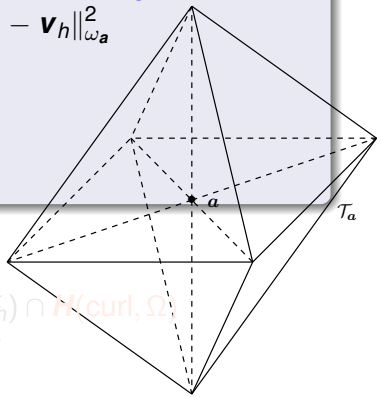

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- **global equilibrium** $\nabla\times\boldsymbol{h}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\times\boldsymbol{h}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{j}$

Stage 3: discrete patchwise equilibrated fluxes

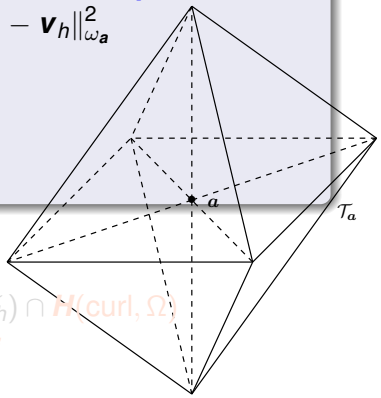
Definition (Chaumont-Frelet, Vohralík (2021))

For each **vertex** $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{j}_h^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}^2$$

and combine

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}.$$



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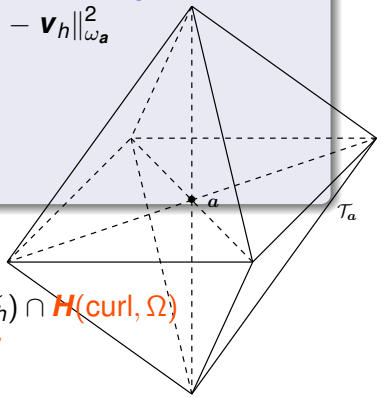

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$\boldsymbol{H}$(curl) polynomial extension on a tetrahedron

Theorem ($\boldsymbol{H}$(curl) polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009); Chaumont-Frelet, Ern, & V. (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $\boldsymbol{r}_K \in \mathcal{RT}_p(K)$ such that $\nabla\cdot\boldsymbol{r}_K = 0$, and for all $\boldsymbol{r}_{\mathcal{F}} \in \mathcal{N}_p^\tau(\Gamma_{\mathcal{F}})$ such that $\boldsymbol{r}_K\cdot\boldsymbol{n}_F = \mathrm{curl}_F(\boldsymbol{r}_F)$ for all $F \in \mathcal{F}$, there holds

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{\mathcal{N}}_p(K)\\ \nabla\times\boldsymbol{v}_p=\boldsymbol{r}_K\\ \boldsymbol{v}_p|_{\mathcal{F}}^{\tau}=\boldsymbol{r}_{\mathcal{F}}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},K)\\ \nabla\times\boldsymbol{v}=\boldsymbol{r}_K\\ \boldsymbol{v}|_{\mathcal{F}}^{\tau}=\boldsymbol{r}_{\mathcal{F}}}} \|\boldsymbol{v}\|_K.$$

Comments

- C_{st} only depends on the shape-regularity of K
- for (pw) p-polynomial data $\boldsymbol{r}_K$, $\boldsymbol{r}_{\mathcal{F}}$, minimization over $\boldsymbol{\mathcal{N}}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{curl}, K)$
- extension to an edge patch: Chaumont-Frelet, Ern, & V. (2021)
- extension to a vertex patch: conjecture

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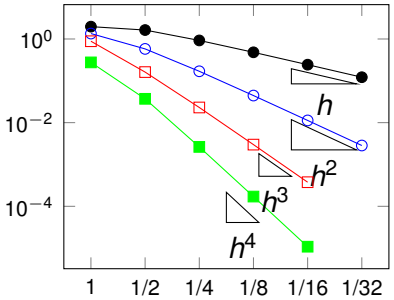
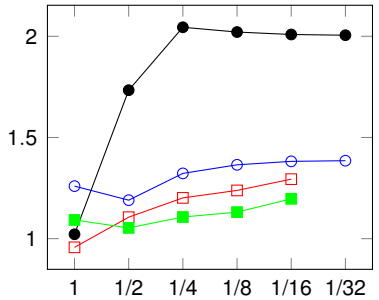
Broken patchwise equilibration, **smooth solution**, h-refinement

$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



$\max_{e\in\mathcal{E}_h}\eta_e/\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\omega_e}$



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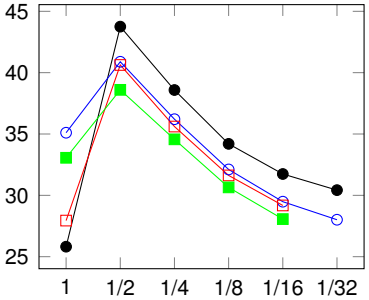
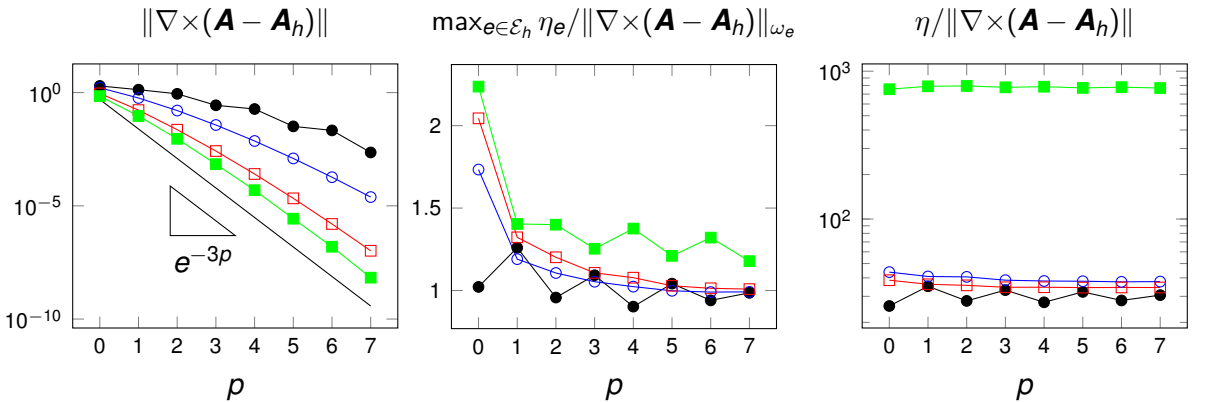
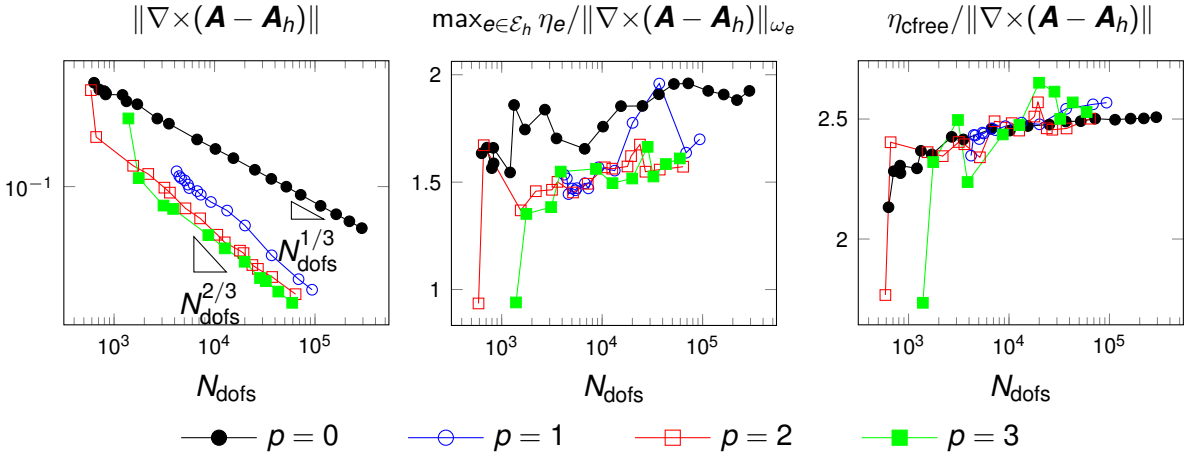


$p=0$ $p=1$ $p=2$ $p=3$

Broken patchwise equilibration, **smooth solution**, p-refinement



Broken patchwise equilibration, **singular solution**, **adap.** refinement



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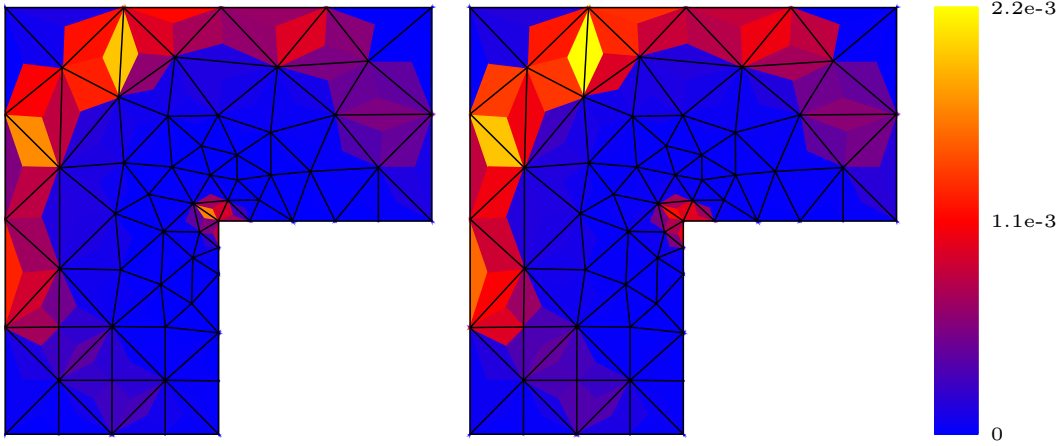
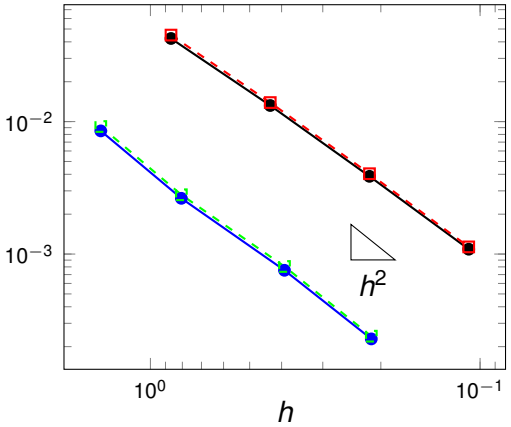


Estimated (left) and actual error (right), $p = 3$

Patchwise equilibration, $\boldsymbol{H}^3$ solution, h-refinement

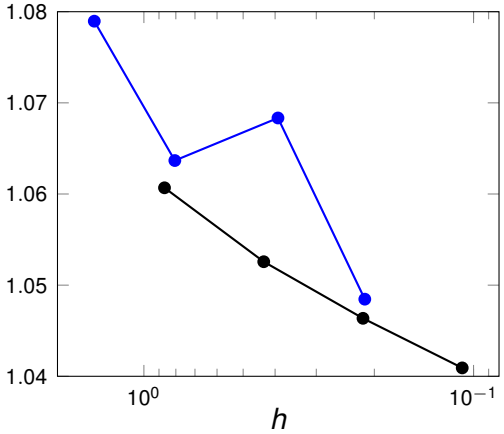
$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



error/eff. ind. estimate

$p=1$

Effectivity index $\eta/\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$

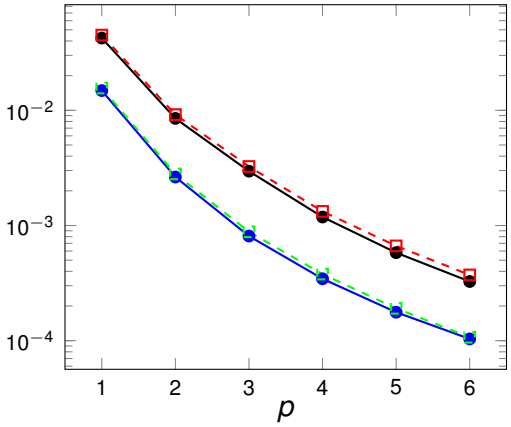


error/eff. ind. estimate

$p=2$

Patchwise equilibration, $\boldsymbol{H}^3$ solution, p-refinement

$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



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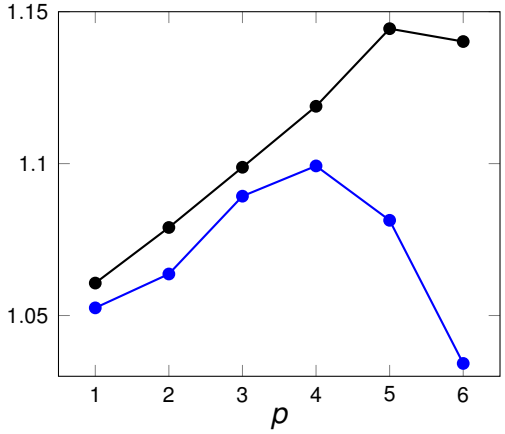
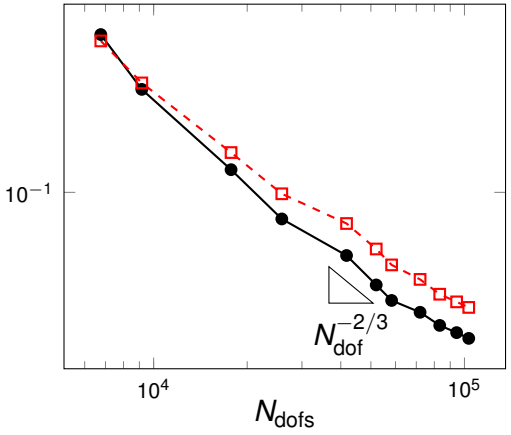


Patchwise equilibration, **singular solution**, **adap.** refinement ($p = 2$)

$\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|$



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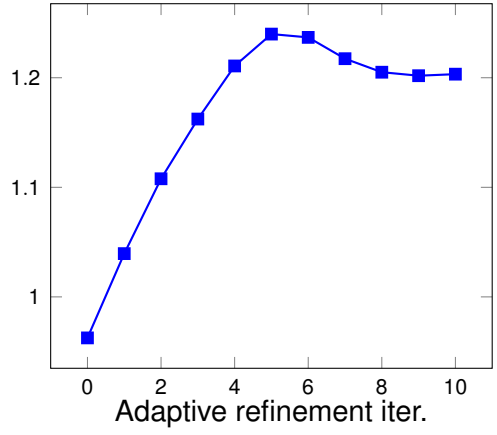

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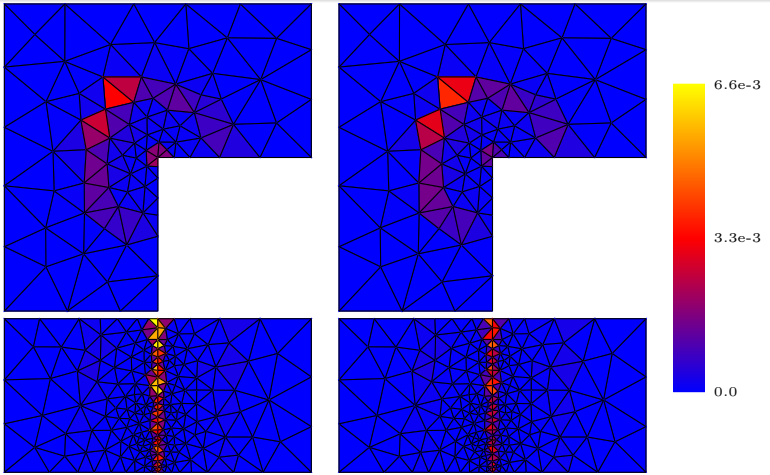

Estimated (left) and actual error (right), adaptive mesh refinement iteration #10.
Top view (top) and side view (bottom)

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Thank you for your attention!

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