p-robust global–local equivalence, p-stable local (commuting) projectors, and optimal elementwise hp approximation estimates in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$

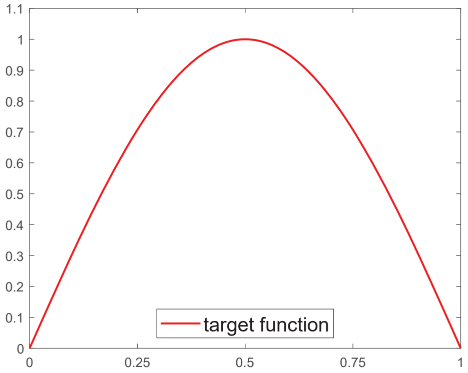
Théophile Chaumont-Frelet, Leszek Demkowicz, and **Martin Vohralík**

Inria Paris & Ecole des Ponts

Vienna, February 25, 2025

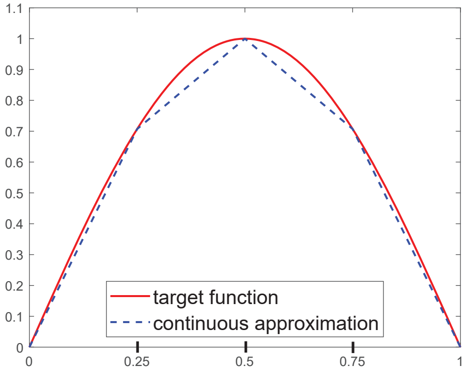
Outline

Equivalence of **global-** and **local-best** approximations in $H_0^1(\Omega)$: 1D



Target function in $H_0^1(\Omega)$

Equivalence of **global-** and **local-best** approximations in $H_0^1(\Omega)$: 1D



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

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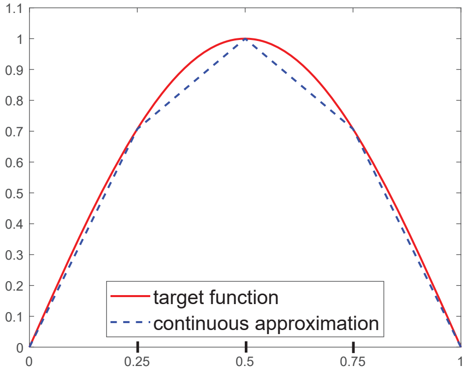

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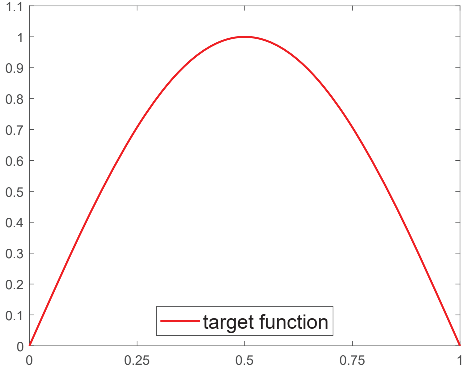

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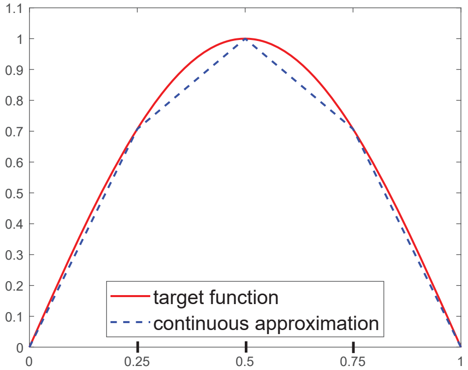
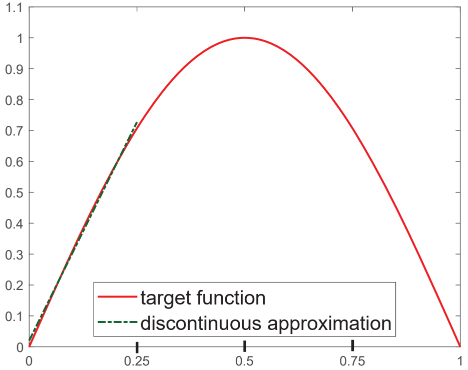
Equivalence of **global-** and **local-best** approximations in $H^1_0(\Omega)$: 1D



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H^1_0(\Omega)$, **global** problem



Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

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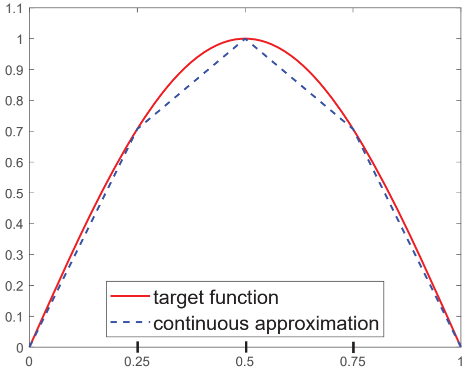

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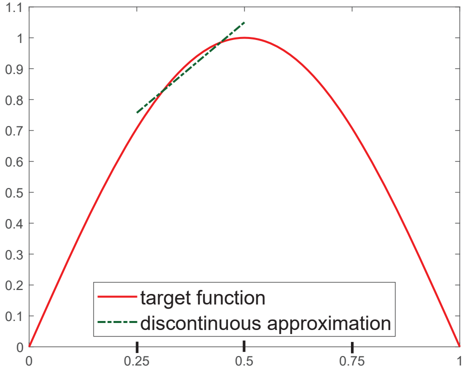

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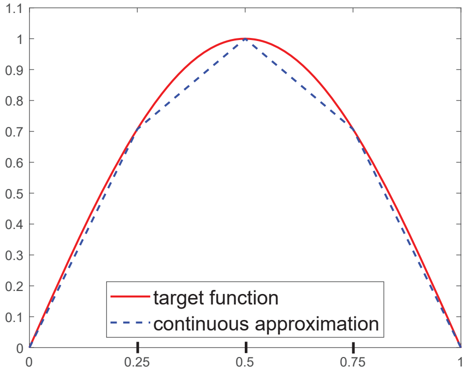
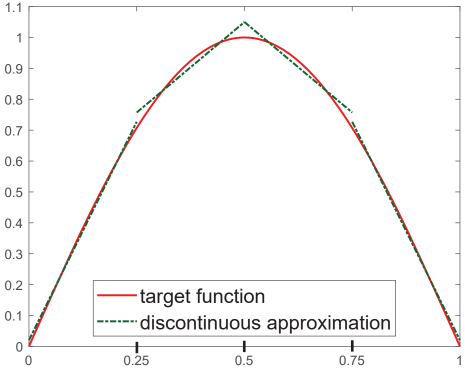
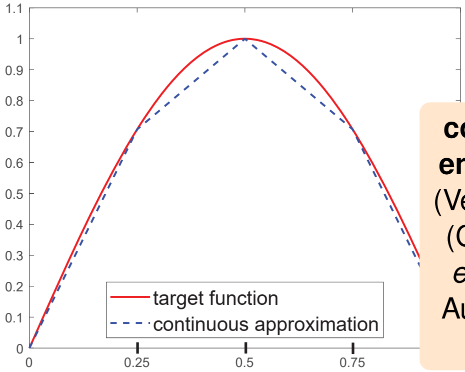


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

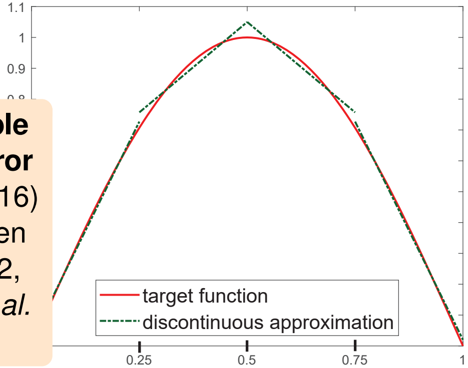


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of **global-** and **local-best** approximations in $H_0^1(\Omega)$: 1D



comparable energy error
(Veeser 2016)
(Carstensen *et al.* 2012, Aurada *et al.* 2013)



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Outline

1. Introduction
2. *p*-stable local (commuting) projectors
   - *p*-stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
   - *p*-stable local projector in $H^1(\Omega)$
3. *p*-robust global-best–local-best equivalence
   - *p*-robust global-best–local-best equivalence in $H^1(\Omega)$
   - *p*-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
4. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div}, \Omega)$
5. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - *p*-stable (broken) polynomial extensions
   - *p*-stable decompositions
6. Conclusions

Outline

Two key Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $\boldsymbol{L}^2(\Omega)$,
$H^1(\Omega) := \{v \in L^2(\Omega);\ \nabla v \in \boldsymbol{L}^2(\Omega)\}$

$\boldsymbol{H}(\mathrm{div},\Omega)$

vector-valued $\boldsymbol{L}^2(\Omega)$ functions with weak divergences in $L^2(\Omega)$,
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Two key Sobolev spaces with BCs

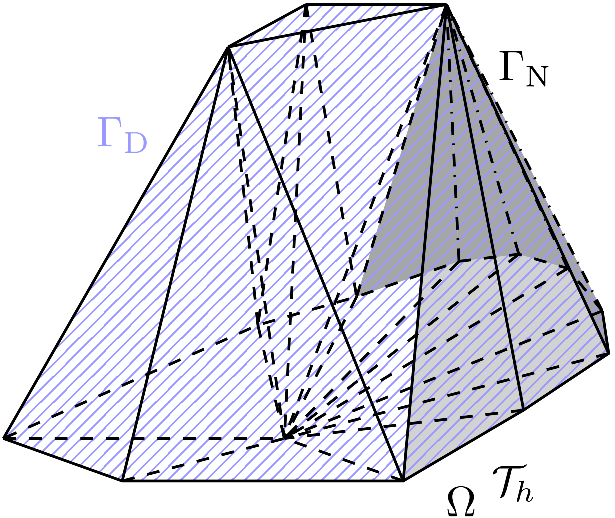
$H^1_{0,\mathrm{D}}(\Omega)$

$H^1_{0,\mathrm{D}}(\Omega) := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_\mathrm{D}\}$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{N}$ in appropriate sense$\}$

Meshes, elements, and patches



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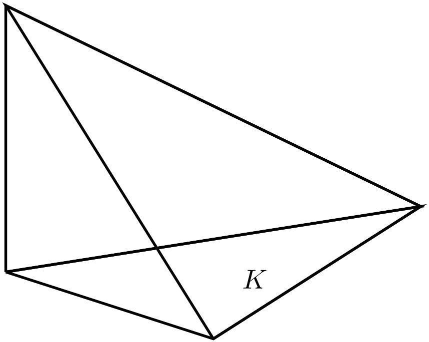

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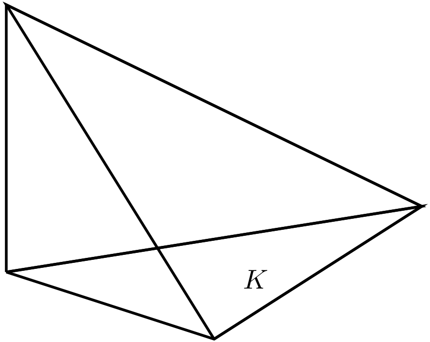

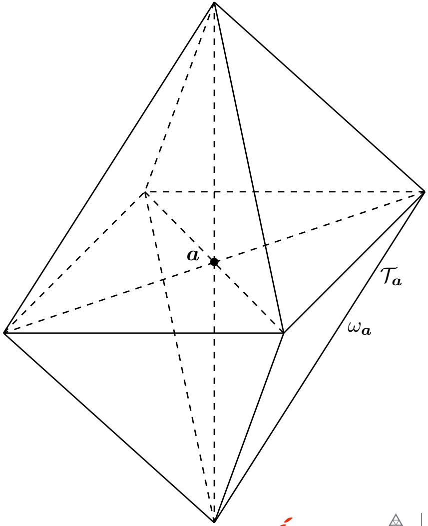

Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$

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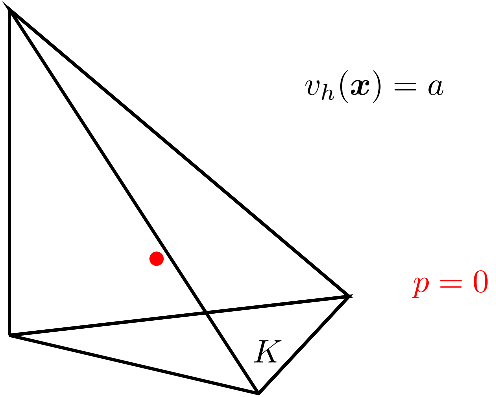

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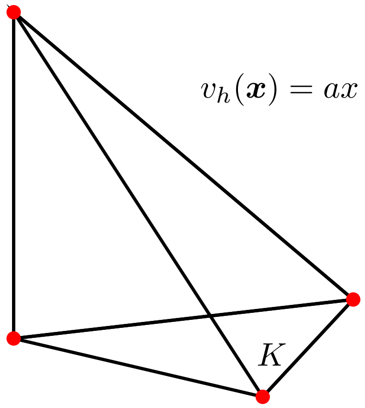
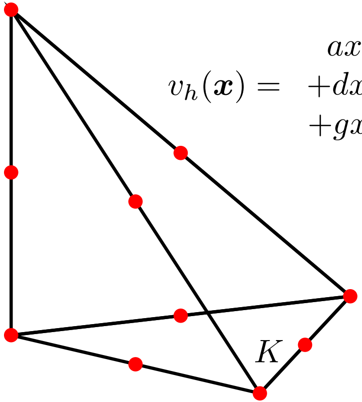


$$v_h(\boldsymbol{x}) = ax + by + cz + d$$

$p = 1$

Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$



$$v_h(\boldsymbol{x}) = \begin{array}{l} ax^2 + by^2 + cz^2 \\ +dxy + eyz + fxz \\ +gx + hy + iz + j \end{array}$$

$p = 2$

Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$

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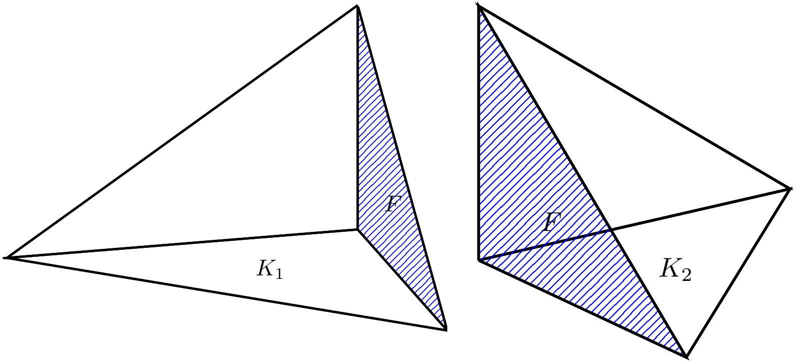

- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- $\Rightarrow$ ensure this by putting sufficient DoFs at the face F

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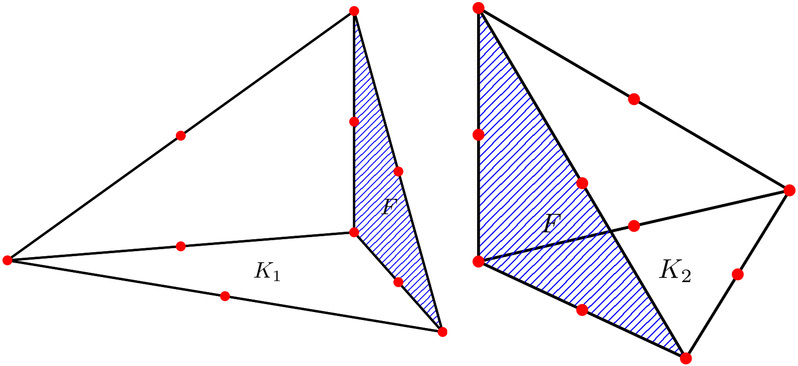

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Raviart–Thomas space $\mathcal{RT}_p(K) := [\mathcal{P}_p(K)]^3 + \mathcal{P}_p(K)\boldsymbol{x}$, $p \geq 0$

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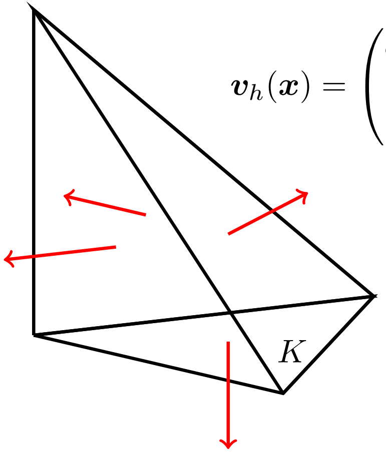
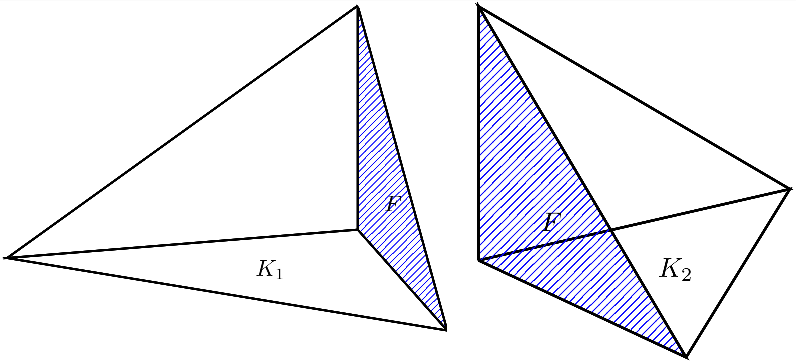


$$\boldsymbol{v}_h(\boldsymbol{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$p = 0$$

Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $p \geq 0$



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_2)$, and $(\boldsymbol{v}|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \cdot \boldsymbol{n}_F)|_F$ in appropriate sense
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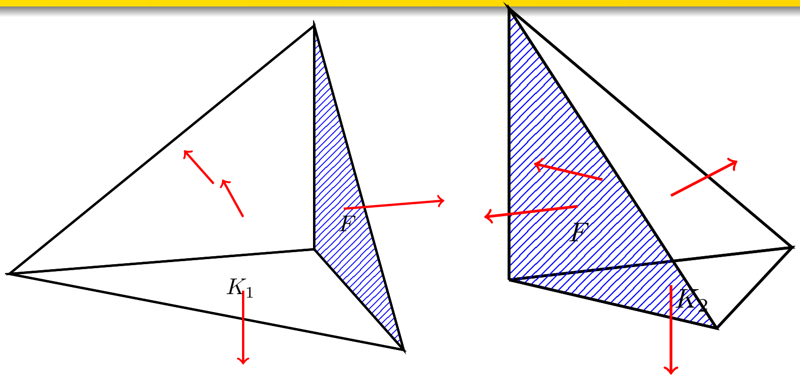

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- $\Rightarrow$ ensure this by putting sufficient DoFs at the face F

Outline

Commuting de Rham diagram

Commuting de Rham diagram

$$H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

Commuting de Rham diagram

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$$\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)$$

Commuting de Rham diagram

Commuting de Rham diagram

$$
\begin{tikzcd}
H^1_{0,\mathrm{N}}(\Omega) \arrow[r, "\nabla"] \arrow[d, "P^{\mathrm{grad}}_{h(p+1)}"] & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\boldsymbol{P}^{\mathrm{curl}}_{hp}"] & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, "\boldsymbol{P}^{\mathrm{div}}_{hp}"] & L^2_*(\Omega) \arrow[d, "\Pi^p_h"] \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) \arrow[r, "\nabla"] & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \arrow[r, "\nabla\cdot"] & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{tikzcd}
$$

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$
\begin{array}{ccc}
\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\Big\downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \Big\downarrow \Pi_h^p \\
\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
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Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

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Properties of $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ (**minimal regularity**, partial BCs)

Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

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Commuting de Rham diagram: operator $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_{hp}^{\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

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5. is **p-stable in $L^2(\Omega)$** (up to data oscillation)
6. satisfies the **commuting property** expressed by the arrows
7. is **projector**, i.e., leaves intact piecewise polynomials

Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$ regularity but not commuting
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- Ern, Gudi, Smears, and Vohralík (2022) & Chaumont-Frelet and Vohralík (2024): all the above properties
- none is p-robust

Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
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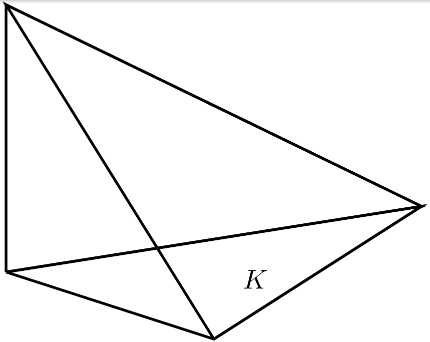
Classical elementwise interpolation



- $\|\boldsymbol{v}-\boldsymbol{v}_h\|^2 = \sum_{K\in\mathcal{T}_h}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2$
- $\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{div},K) \Rightarrow$ so interpolate $\boldsymbol{v}|_K$, elementwise interpolation

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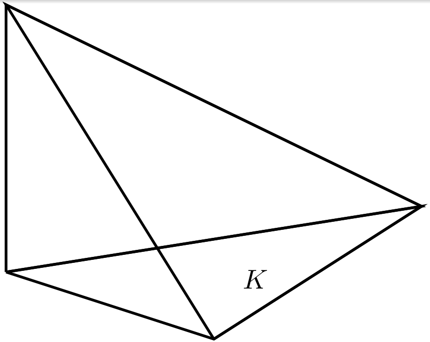

- $\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$
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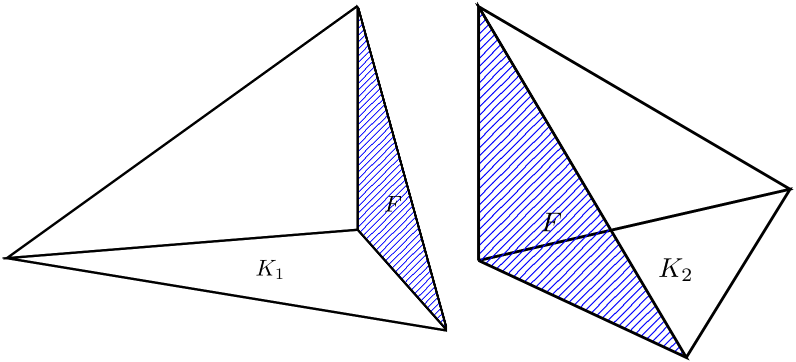
Classical elementwise interpolation: conformity enforcement



- $\boldsymbol{v}_h \in \mathcal{RT}_p(K_1, K_2) \cap \boldsymbol{H}(\mathrm{div}, K_1 \cup K_2)$ iff $\boldsymbol{v}_h|_{K_1} \in \mathcal{RT}_p(K_1)$, $\boldsymbol{v}_h|_{K_2} \in \mathcal{RT}_p(K_2)$, and $(\boldsymbol{v}_h|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}_h|_{K_2} \cdot \boldsymbol{n}_F)|_F$

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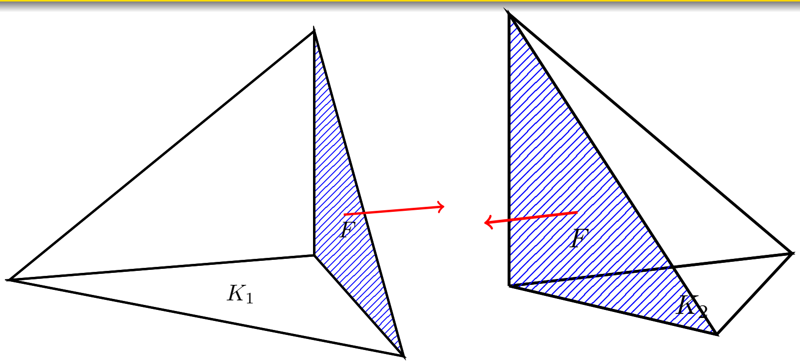

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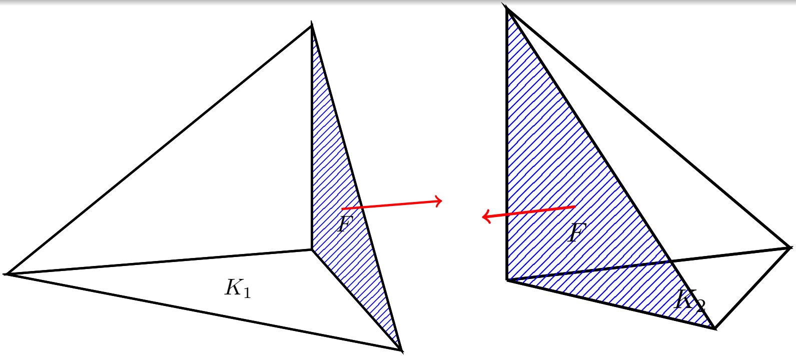

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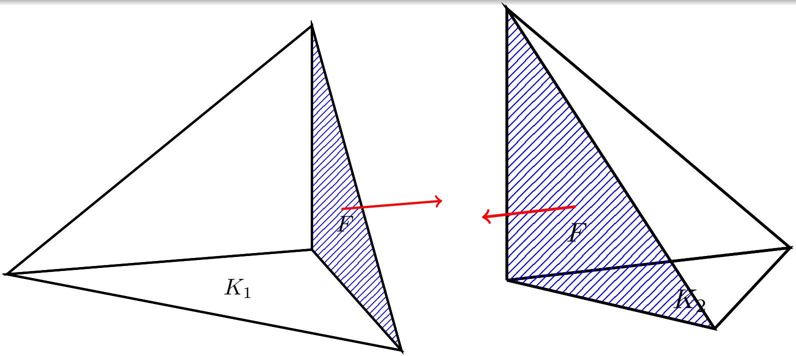
Clash

Face normal trace integrals $\langle \boldsymbol{v}{\cdot}\boldsymbol{n}_F, 1\rangle_F/|F|$ not available in $\boldsymbol{H}(\mathrm{div})$.

Classical elementwise interpolation: conformity enforcement



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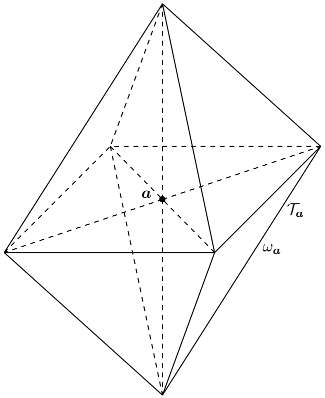
Conclusion

Not a single tetrahedron $K \in \mathcal{T}_h$ if the minimal regularity $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ requested.

Classical patchwise interpolation (Clément)



- some local-best polynomial approximation on $\omega_{\boldsymbol{a}}$
- values on $\omega_{\boldsymbol{a}}$ as coefficients for basis functions supported on $\omega_{\boldsymbol{a}}$

Conclusion

Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

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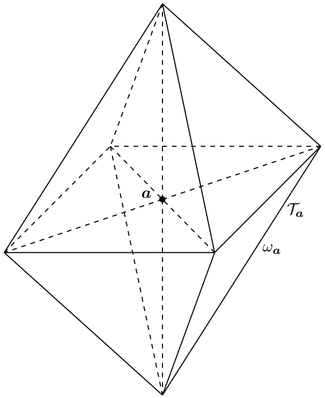

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A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ be given.

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1. For each $K \in \mathcal{T}_h$, define the **elementwise L^2-orthogonal projection**

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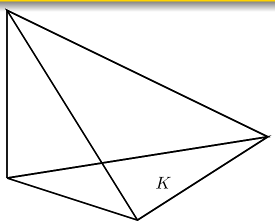

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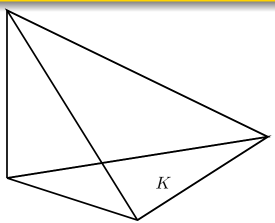

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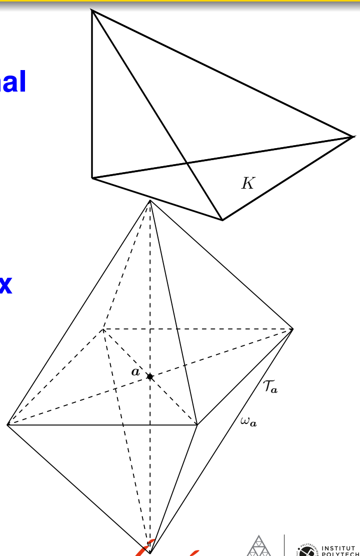

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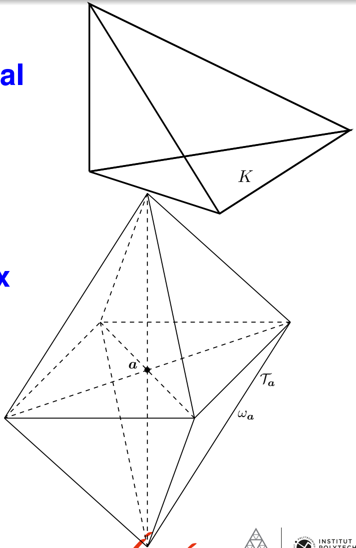

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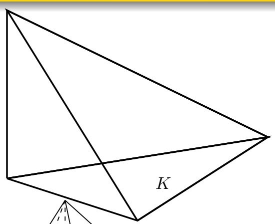
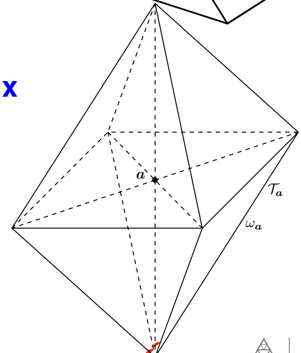
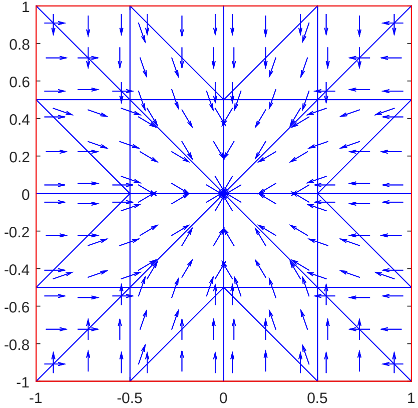
3. Apply a **p-stable decomposition** on extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$ to conforming projections of the reminder $\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h \Rightarrow$ correction $\boldsymbol{\zeta}_h$; $\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) := \boldsymbol{\sigma}_h + \boldsymbol{\zeta}_h$.





Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$

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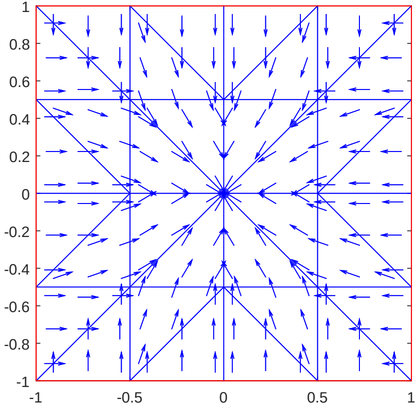

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Equilibration in $\boldsymbol{H}$(div)

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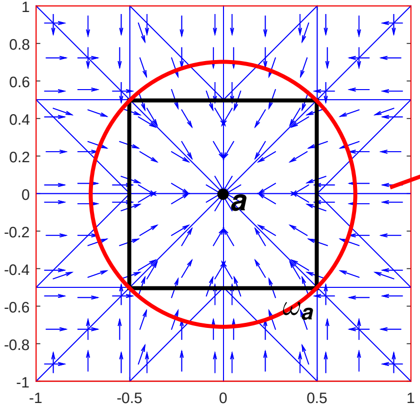

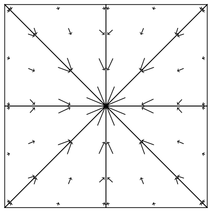

$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$

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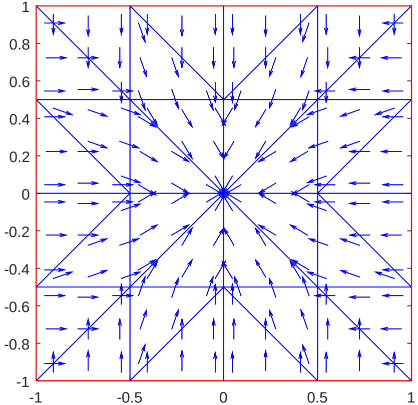

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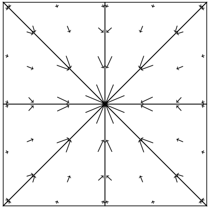

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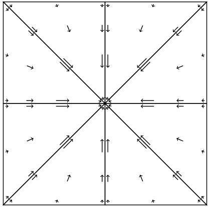

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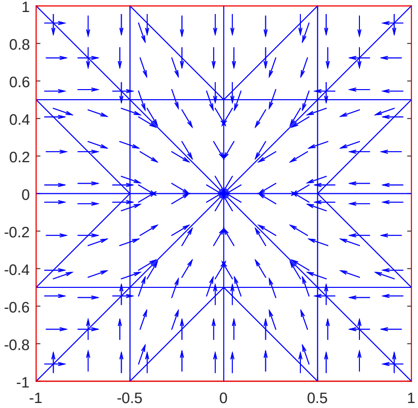

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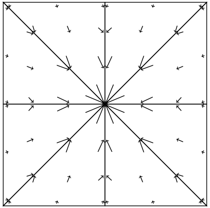

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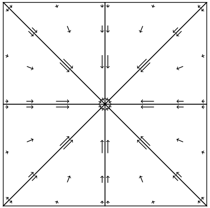

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Equilibration in $\boldsymbol{H}(\mathrm{div})$

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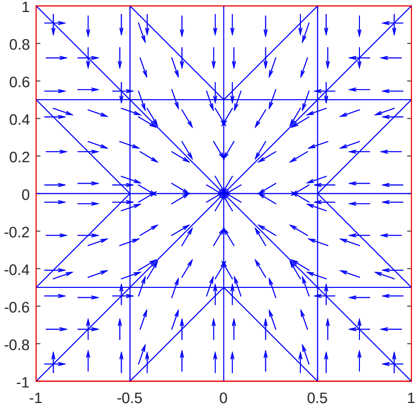

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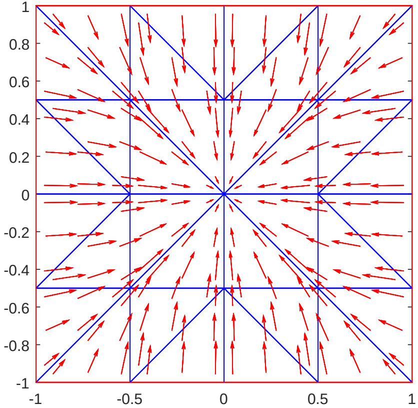

Equilibrated flux rec. $\boldsymbol{\sigma}_h$

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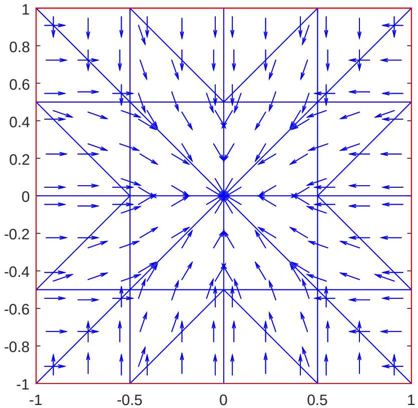

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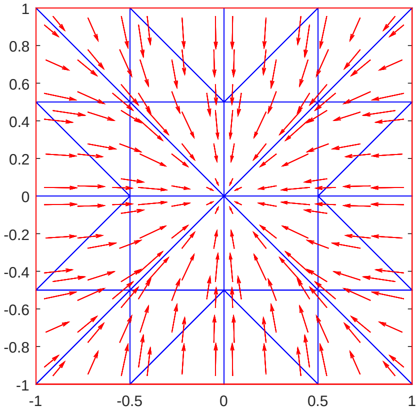

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   (patchwise **flux equilibration**)

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A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

2. $\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v})}} \left\| \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{v}_h \right\|_{\omega_{\boldsymbol{a}}}$

(patchwise **flux equilibration**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

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   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$ (**elementwise L^2-orthogonal projection**)

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   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h, \boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

   (patchwise divergence-free remainder equilibration with an additional constraint)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}} \quad \text{if } p \geq 1}} \left\|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

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   (patchwise divergence-free remainder equilibration with an additional constraint)

   $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{b} \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{b}}$$ with in particular $\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, $\nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$

   (patchwise **p-stable** remainder **decomposition**)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

1. $$\boldsymbol{\tau}_h|_K := \arg\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K$$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=\Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v}+\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v})\\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K=(\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h\in[\mathcal{P}_0(K)]^d,\forall K\in\mathcal{T}_{\boldsymbol{a}} \quad \text{if } p\geq 1}} \|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h)-\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

(patchwise **flux equilibration** with an **additional orthogonality constraint**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}})\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=0\\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K=(\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K=0 \quad \forall \boldsymbol{r}_h\in[\mathcal{P}_0(K)]^d,\forall K\in\widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h-\boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

(patchwise divergence-free remainder equilibration with an additional constraint)

$$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{b}\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{b}}$$ with in particular $\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, $\nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}=0$

(patchwise ***p*-stable** remainder **decomposition**)

$$\boldsymbol{\zeta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}$$ (gluing patchwise correction contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}} := \boldsymbol{\sigma}_h + \boldsymbol{\zeta}_h$

1. $$\boldsymbol{\tau}_h|_K := \arg\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K$$ (**elementwise L^2-orthogonal projection**)

2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=\Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v}+\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v})\\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K=(\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{r}_h)_K\quad\forall\boldsymbol{r}_h\in[\mathcal{P}_0(K)]^d,\forall K\in\mathcal{T}_{\boldsymbol{a}}\quad\text{if } p\geq 1}} \left\|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{h,p}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h)-\boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}}$$

   (patchwise **flux equilibration** with an **additional orthogonality constraint**)

   $$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}})\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=0\\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K=(\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K=0\quad\forall\boldsymbol{r}_h\in[\mathcal{P}_0(K)]^d,\forall K\in\widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h-\boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

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   (patchwise ***p*-stable** remainder **decomposition**)

   $$\boldsymbol{\zeta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}$$ (gluing patchwise correction contributions)

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_{hp}^{\mathrm{div}}$ *is* ***commuting*** *since*

$$\nabla{\cdot}\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla{\cdot}\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}}: \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_{hp}^{\mathrm{div}}$ *is* ***commuting*** *and* ***projector*** *since*

$$\begin{aligned}
\nabla\cdot\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) &= \Pi_h^p(\nabla\cdot\boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),\\
\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) &= \boldsymbol{v} && \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega).
\end{aligned}$$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}}: \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_{hp}^{\mathrm{div}}$ *is* ***commuting*** *and* ***projector*** *since*

$$\begin{aligned} \nabla\cdot\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) &= \Pi_h^p(\nabla\cdot\boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega), \\ \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) &= \boldsymbol{v} && \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega). \end{aligned}$$

It has ***p-robust local-best approximation property***

since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $K \in \mathcal{T}_h$,

$$\|\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\cdot(\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}))\|_K\right)^2$$

$$\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K}\left\{\min_{\boldsymbol{v}_h\in\mathcal{RT}_p(L)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2\right\},$$

A p-stable local commuting projector $\boldsymbol{P}_{hp}^{\mathrm{div}}$

Theorem ($\boldsymbol{P}_{hp}^{\mathrm{div}}: \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

*$\boldsymbol{P}_{hp}^{\mathrm{div}}$ is **commuting** and **projector** since*

$$\begin{aligned} \nabla\cdot\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) &= \Pi_h^p(\nabla\cdot\boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega), \\ \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}) &= \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega). \end{aligned}$$

*It has **p-robust local-best approximation property** and is **p-robustly L^2 stable** up to data oscillation, since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $K \in \mathcal{T}_h$,*

$$\begin{aligned} &\|\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\cdot(\boldsymbol{v} - \boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v}))\|_K\right)^2 \\ &\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \left\{ \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(L)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2 \right\}, \end{aligned}$$

$$\|\boldsymbol{P}_{hp}^{\mathrm{div}}(\boldsymbol{v})\|_K^2 \lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \left\{ \|\boldsymbol{v}\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2 \right\}.$$

Outline

A p-stable local projector $P^{\mathrm{grad}}_{h(p+1)}$

Theorem ($P^{\mathrm{grad}}_{h(p+1)} : H^1_{0,\mathrm{N}}(\Omega) \to \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega)$)

$P^{\mathrm{grad}}_{h(p+1)}$ *is a* ***projector*** *since*

$$P^{\mathrm{grad}}_{h(p+1)}(v) = v \qquad \forall v \in \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega).$$

A p-stable local projector $P^{\mathrm{grad}}_{h(p+1)}$

Theorem ($P^{\mathrm{grad}}_{h(p+1)} : H^1_{0,\mathrm{N}}(\Omega) \to \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega)$)

$P^{\mathrm{grad}}_{h(p+1)}$ *is a* ***projector*** *since*

$$P^{\mathrm{grad}}_{h(p+1)}(v) = v \qquad \forall v \in \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega).$$

It has ***p-robust local-best approximation property*** *since, for all $v \in H^1_{0,\mathrm{N}}(\Omega)$ and $K \in \mathcal{T}_h$,*

$$\|\nabla(v - P^{\mathrm{grad}}_{h(p+1)}(v))\|^2_K \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \min_{v_h \in \mathcal{P}_{p+1}(L)} \|\nabla(v - v_h)\|^2_L,$$

A p-stable local projector $P^{\mathrm{grad}}_{h(p+1)}$

Theorem ($P^{\mathrm{grad}}_{h(p+1)} : H^1_{0,\mathrm{N}}(\Omega) \to \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega)$)

$P^{\mathrm{grad}}_{h(p+1)}$ *is a* ***projector*** *since*

$$P^{\mathrm{grad}}_{h(p+1)}(v) = v \qquad \forall v \in \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega).$$

It has ***p-robust local-best approximation property*** *and is* ***p-robustly H^1 stable*** *since, for all $v \in H^1_{0,\mathrm{N}}(\Omega)$ and $K \in \mathcal{T}_h$,*

$$\|\nabla(v - P^{\mathrm{grad}}_{h(p+1)}(v))\|^2_K \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \min_{v_h \in \mathcal{P}_{p+1}(L)} \|\nabla(v - v_h)\|^2_L,$$

$$\|\nabla P^{\mathrm{grad}}_{h(p+1)}(v)\|^2_K \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \|\nabla v\|^2_L.$$

A p-stable local projector $P^{\mathrm{grad}}_{h(p+1)}$

Theorem ($P^{\mathrm{grad}}_{h(p+1)}: H^1_{0,\mathrm{N}}(\Omega) \to \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{N}}(\Omega)$)

$P^{\mathrm{grad}}_{h(p+1)}$ *is a* ***projector*** *since*

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It has ***p-robust local-best approximation property*** *and is* ***p-robustly H^1 stable*** *since, for all $v \in H^1_{0,\mathrm{N}}(\Omega)$ and $K \in \mathcal{T}_h$,*

$$\|\nabla(v - P^{\mathrm{grad}}_{h(p+1)}(v))\|^2_K \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \min_{v_h \in \mathcal{P}_{p+1}(L)} \|\nabla(v - v_h)\|^2_L,$$

$$\|\nabla P^{\mathrm{grad}}_{h(p+1)}(v)\|^2_K \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \|\nabla v\|^2_L.$$

- $\lesssim$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$

Outline

Global-best approximation ≈ local-best approximation

Previous contributions

- Carstensen, Peterseim, and Schedensack (2012): H^1 (lowest-order case $p = 1$)
- Aurada, Feischl, Kemetmüller, Page, and Praetorius (2013): H^1 (boundary approximation context)
- Veeser (2016): H^1 (any p, p-dependent constant)
- Canuto, Nochetto, Stevenson, and Verani (2017): H^1 (improvement of the p dependence of the equivalence constant in 2D)
- Ern, Gudi, Smears, and Vohralík (2022): $\boldsymbol{H}(\mathrm{div})$ (any p, p-dependent constant)
- Chaumont-Frelet and Vohralík (2021, 2022): $\boldsymbol{H}(\mathrm{curl})$ (any p, p-dependent constant)
- Gawlik, Holst, and Licht (2021): finite element exterior calculus context (any p, p-dependent constant)

Outline

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Global–local equivalence in H_0^1, Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Global–local equivalence in H_0^1, Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

$$\min_{\textit{smaller space}} \approx_p \min_{\textit{bigger space}}$$

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (Global–local equivalence in H^1_0, Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

$$\min_{\text{CG space}} \approx_p \min_{\text{DG space}}$$

- $\approx_p$: up to a generic constant that only depends on space dimension d, shape-regularity of the mesh $\mathcal{T}_h$, and polynomial degree p

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

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Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

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- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

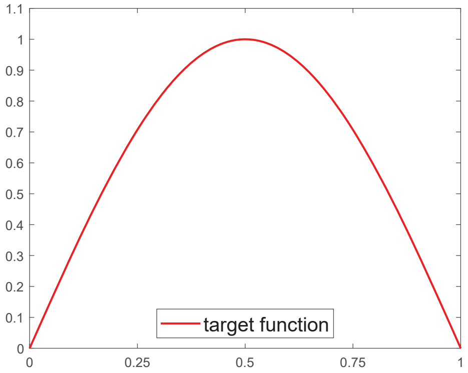
Theorem (p-robust global–local equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

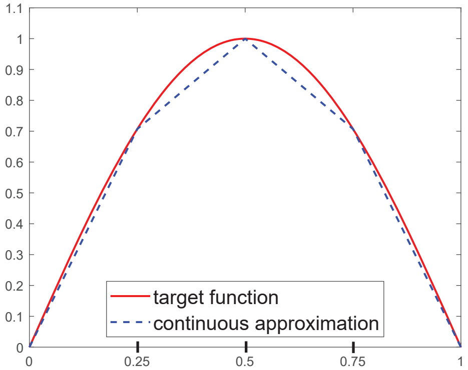
- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



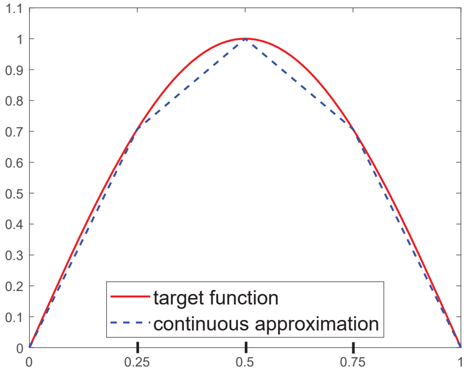
Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

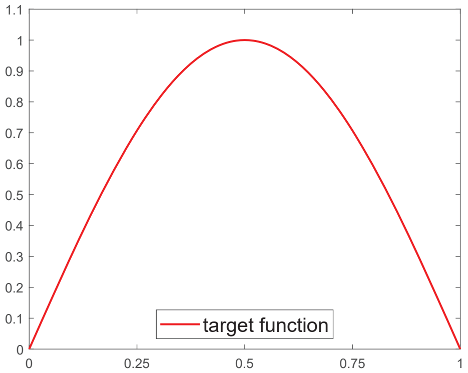


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

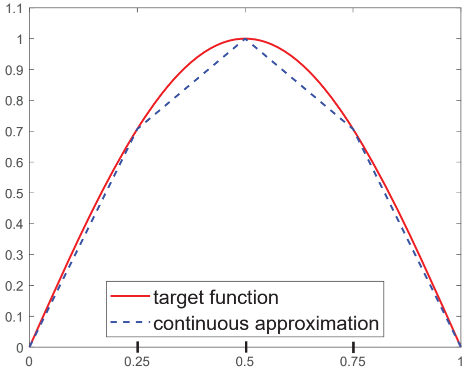


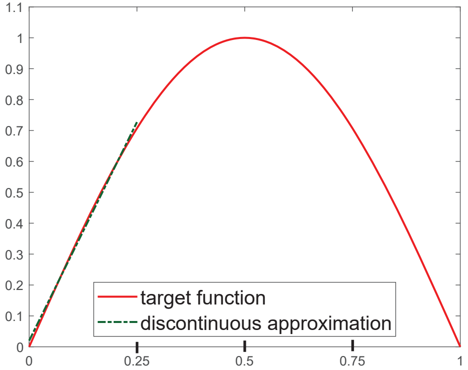
Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem



Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

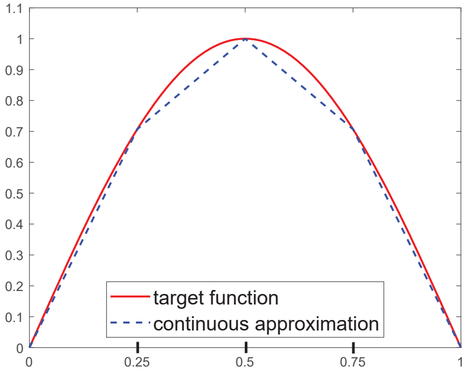




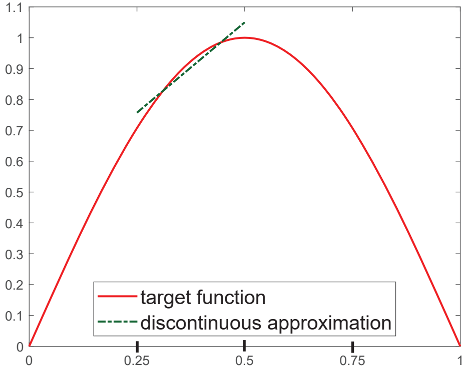
Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

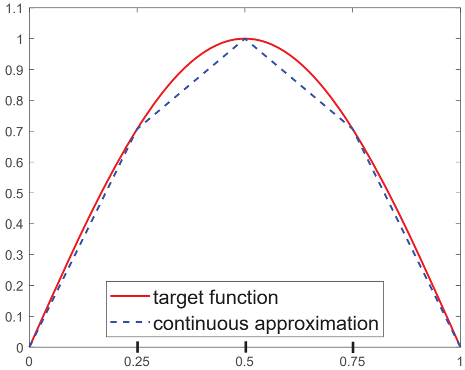


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

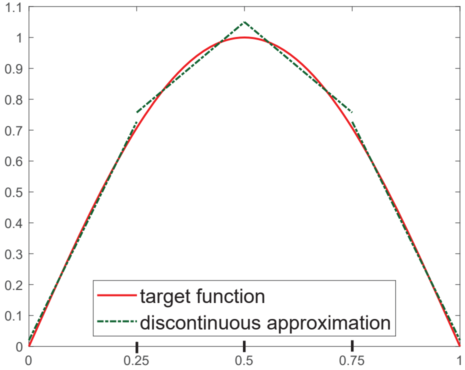


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

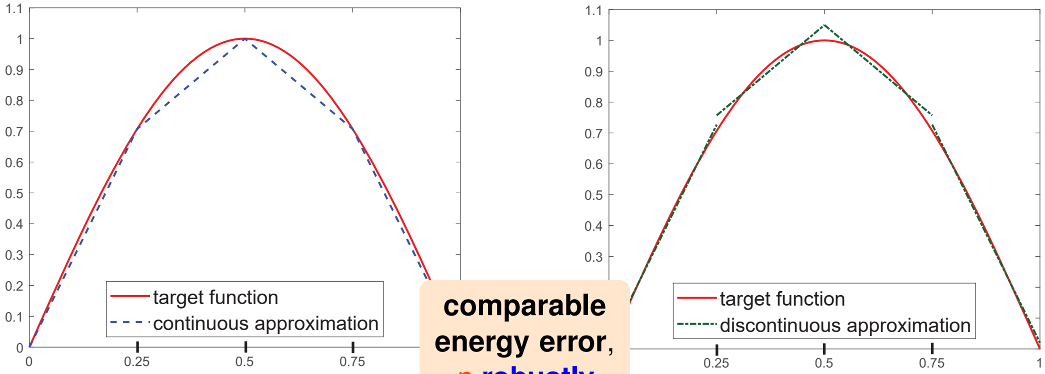


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem



Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



comparable energy error, *p*-robustly

Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Outline

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Global–local equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2022))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Global–local equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2022))

$$\min_{\textit{smaller space with constraints}} \approx_p \min_{\textit{bigger space without constraints}}$$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Global–local equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2022))

$$\min_{\text{MFE space with constraints}} \approx_p \min_{\text{broken MFE space without constraints}}$$

- $\approx_p$: only depends on space dimension d, shape-regularity of $\mathcal{T}_h$, and polynomial degree p

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div},\Omega)$

Theorem (p-robust global–local equivalence in $\boldsymbol{H}(\mathrm{div})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RTN}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on }\Omega\\ \text{normal trace-continuity constraint}\\ \text{divergence constraint}\\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h\in\mathcal{RTN}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2\right]}_{\substack{\text{local-best on each }K\\ \text{no normal trace-continuity constraint}\\ \text{no divergence constraint}\\ \text{broken MFE space (much bigger)}}}.$$

- $\approx$: only depends on d and shape-regularity of $\mathcal{T}_h$

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Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,

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Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (p-robust global–local equivalence in $\boldsymbol{H}(\mathrm{div})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RTN}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\textit{global-best on } \Omega \\ \textit{normal trace-continuity constraint} \\ \textit{divergence constraint} \\ \textit{MFE space (much smaller)}}}$$

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Outline

1. Introduction
2. *p*-stable local (commuting) projectors
   - *p*-stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
   - *p*-stable local projector in $H^1(\Omega)$
3. *p*-robust global-best–local-best equivalence
   - *p*-robust global-best–local-best equivalence in $H^1(\Omega)$
   - *p*-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
4. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div}, \Omega)$
5. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - *p*-stable (broken) polynomial extensions
   - *p*-stable decompositions
6. Conclusions

The Laplace equation (source term $f \in L^2(\Omega)$)

The Laplace equation

Find $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ -\nabla u \cdot \boldsymbol{n}_\Omega &= 0 && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \, \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

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Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Dual weak formulation

$\boldsymbol{\sigma} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with* $\nabla{\cdot}\boldsymbol{\sigma} = f$ *such that*

$$(\boldsymbol{\sigma}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \text{ with } \nabla{\cdot}\boldsymbol{v} = 0.$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \, \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

The Laplace equation (source term $f \in L^2(\Omega)$)

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$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

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$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

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Dual finite element approximation

$\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with* $\nabla \cdot \boldsymbol{\sigma}_h = \Pi^p_h f$ *such that*

$$(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \text{ with } \nabla \cdot \boldsymbol{v}_h = 0.$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

Dual weak formulation

$\boldsymbol{\sigma} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with* $\nabla{\cdot}\boldsymbol{\sigma} = f$ *such that*

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Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

$$(\nabla u_h, \nabla v_h) = (f, v_h) \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega).$$

Dual finite element approximation

$\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with* $\nabla{\cdot}\boldsymbol{\sigma}_h = \Pi_h^p f$ *such that*

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Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

Error characterisation

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla{\cdot}\boldsymbol{v}_h = \Pi_h^p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|$$

The Laplace equation (source term $f \in L^2(\Omega)$): **global-best approximations**

Error characterisation

$$\|\nabla(u-u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h)\cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u-v_h)\|$$

Error characterisation

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p f}} \|\boldsymbol{\sigma}-\boldsymbol{v}_h\|$$

The Laplace equation (source term $f \in L^2(\Omega)$): **global-best approximations (constrained)**

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

Error characterisation

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_h^p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|$$

Outline

Approximation error estimates in the $H^1(\Omega)$ context

h approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p, s-1\}} |v|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

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Approximation error estimates in the $H^1(\Omega)$ context

hp approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. Then

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Approximation error estimates in the $H^1(\Omega)$ context

hp approximation estimate

Let $v \in H^s(\cancel{\Omega})$, $s \geq 1$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, \cancel{p}) \cancel{\ln(p)} \frac{h^{\min\{p, s-1\}}}{p^{s-1}} \|v\|_{H^s(\cancel{\Omega})}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ with

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$.

- varying polynomial degree: $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$ is the smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

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$$\|\nabla(v - P_h^{p,\mathrm{grad}} v)\|_K^2 \leq C(\kappa_{\mathcal{T}_h}, \kappa_p, d, s)^2 \sum_{L \in \widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(\underline{p}_K, s_L - 1)}}{\underline{p}_K^{s_L - 1}} \|v\|_{H^{s_L}(L)} \right)^2 \qquad \forall K \in \mathcal{T}_h.$$

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Outline

Approximation error estimates in the $\boldsymbol{H}(\mathrm{div},\Omega)$ context

h approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) \quad h^{\min\{p+1,s\}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
- Monk (1994), rectangular meshes
- Demkowicz and Buffa (2005)(under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)
- Ern, Gudi, Smears, and Vohralík (2022), s uniform

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Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla \cdot \boldsymbol{v})|_K \in [\mathcal{P}_p(K)]^3 \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$.

- varying polynomial degree can be added

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for $s_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h=\nabla\cdot\boldsymbol{v}}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s)^2 \sum_{K\in\mathcal{T}_h} \left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}} \|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)} \right)^2.$$

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for $s_K \geq 0$ *and* $t_K \geq 0$.

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for $s_K \geq 0$ *and* $t_K \geq 0$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla{\cdot}\boldsymbol{v}_h = \Pi_h^p(\nabla{\cdot}\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla{\cdot}\boldsymbol{v}-\Pi_h^p(\nabla{\cdot}\boldsymbol{v})\|_K\right)^2\Bigg]$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s, t)^2 \sum_{K\in\mathcal{T}_h}\left[\left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2 + \left(\frac{h_K}{p+1}\frac{h_K^{\min\{p+1,t_K\}}}{(p+1)^{t_K}}\|\nabla{\cdot}\boldsymbol{v}\|_{\boldsymbol{H}^{t_K}(K)}\right)^2\right].$$

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$$\leq C(\kappa_{\mathcal{T}_h}, d, s, t)^2 \sum_{K\in\mathcal{T}_h}\Bigg[\left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2 + \left(\frac{h_K}{p+1}\frac{h_K^{\min\{p+1,t_K\}}}{(p+1)^{t_K}}\|\nabla\cdot\boldsymbol{v}\|_{\boldsymbol{H}^{t_K}(K)}\right)^2\Bigg].$$

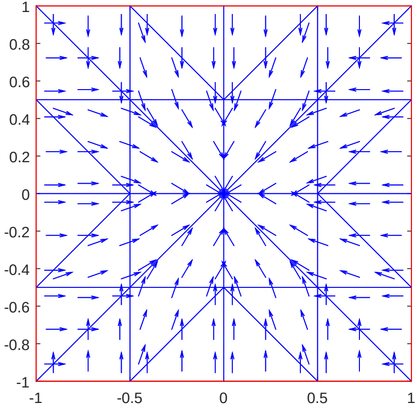
- varying polynomial degree can be added

Outline

1. Introduction
2. *p*-stable local (commuting) projectors
   - *p*-stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
   - *p*-stable local projector in $H^1(\Omega)$
3. *p*-robust global-best–local-best equivalence
   - *p*-robust global-best–local-best equivalence in $H^1(\Omega)$
   - *p*-robust global-best–local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
4. Optimal elementwise *hp* approximation error estimates
   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
   - Optimal elementwise *hp* approximation error estimates in $\boldsymbol{H}(\mathrm{div}, \Omega)$
5. Tools
   - Equilibration in $\boldsymbol{H}(\mathrm{div})$
   - *p*-stable (broken) polynomial extensions
   - *p*-stable decompositions
6. Conclusions

Outline

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$

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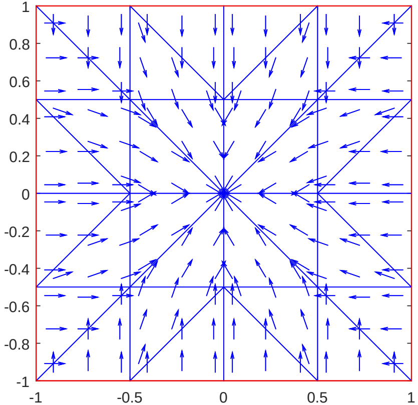

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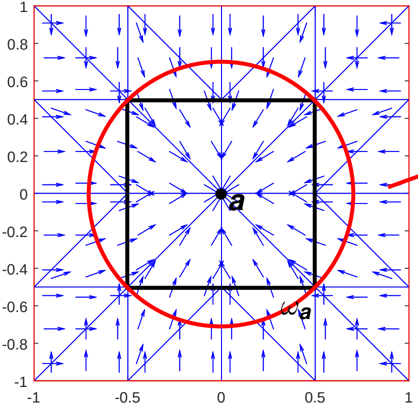
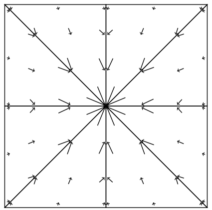
Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)





$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$

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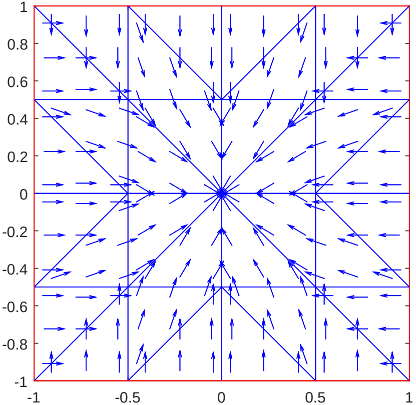
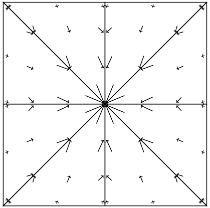
Equilibration in $\boldsymbol{H}(\mathrm{div})$

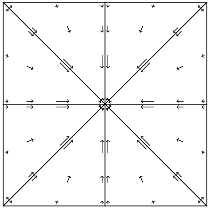
Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



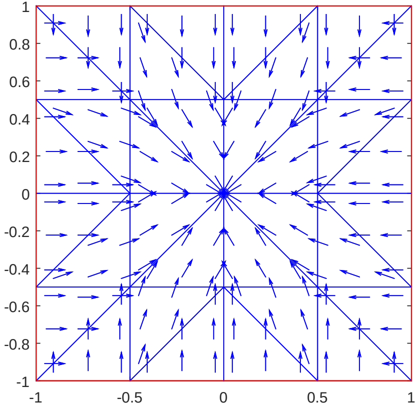
$\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$



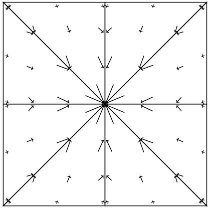
$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h),\ f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



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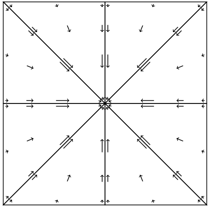
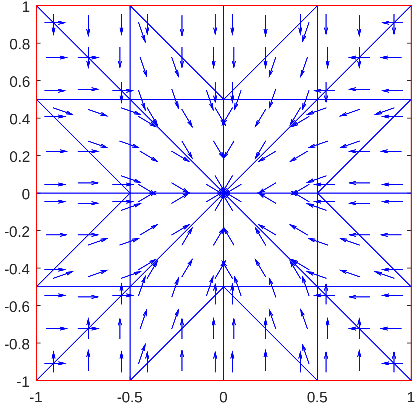


$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

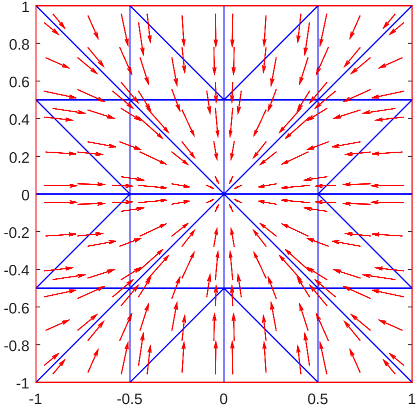
$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \qquad \boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=f\psi^{\boldsymbol{a}}+\boldsymbol{\tau}_h\cdot\nabla\psi^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}^2$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$

Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



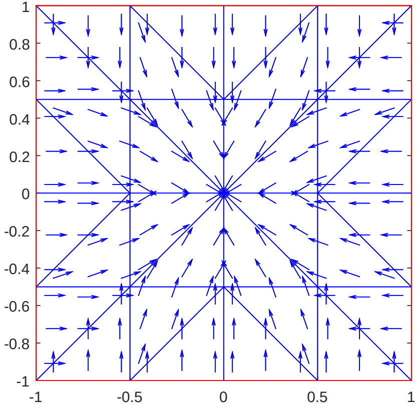
Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



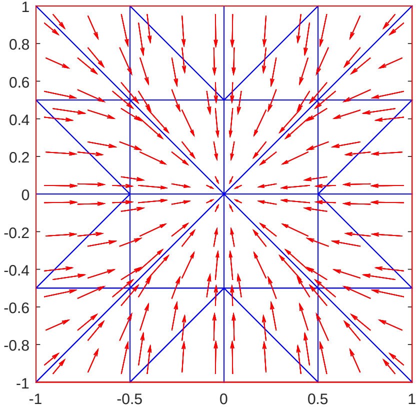
Equilibrated flux rec. $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\tau}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\tau}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}), \nabla\cdot\boldsymbol{\sigma}_h = f$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Bossavit (1998), Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\boldsymbol{\tau}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\tau}_h \neq f$



Equilibrated flux rec. $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_h = f$

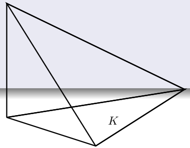
Outline

$\boldsymbol{H}(\mathrm{div})$ polynomial extensions on a tetrahedron

Theorem ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_\mathcal{F} \in \mathcal{P}_p(\Gamma_\mathcal{F})$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{RT}_p(K)\\ \nabla\cdot\boldsymbol{v}_p = r_K\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K)\\ \nabla\cdot\boldsymbol{v} = r_K\\ \boldsymbol{v}\cdot\boldsymbol{n}_\mathcal{F} = r_\mathcal{F}}} \|\boldsymbol{v}\|_K.$$



Comments

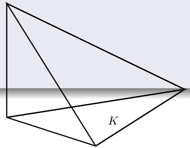
- C_{st} only depends on the shape-regularity of K
- p-robustness: for (pw) p-polynomial data r_K, $r_\mathcal{F}$, minimization over $\mathcal{RT}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{div}, K)$
- extension to a vertex patch: Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)

$\boldsymbol{H}(\mathrm{div})$ polynomial extensions on a tetrahedron

Theorem ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_{\mathcal{F}} \in \mathcal{P}_p(\Gamma_{\mathcal{F}})$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{RT}_p(K) \\ \nabla\cdot\boldsymbol{v}_p = r_K \\ \boldsymbol{v}_p\cdot\boldsymbol{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K) \\ \nabla\cdot\boldsymbol{v} = r_K \\ \boldsymbol{v}\cdot\boldsymbol{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\boldsymbol{v}\|_K.$$



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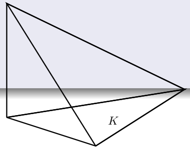

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$\boldsymbol{H}$(div) polynomial extensions on a tetrahedron and on patches

Theorem ($\boldsymbol{H}$(div) polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020)**)**

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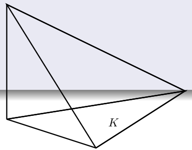

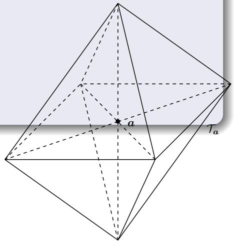

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Outline

$\boldsymbol{H}$(div) stable decomposition

Theorem ($\boldsymbol{H}$(div) stable decomposition in 2D; in extension of Schöberl, Melenk, Pechstein, & Zaglmayr (2008))

Let $d = 2$ and let $\overline{\Omega}$ be contractible. Let

$$\delta_p \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \quad \text{with} \quad \nabla{\cdot}\delta_p = 0, \qquad \textit{div-free}$$

$$(\delta_p, \boldsymbol{r}_h)_K = 0 \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d,\ \forall K \in \mathcal{T}_h. \qquad \textit{vanishing means}$$

Then there exists a decomposition of δ_p as

$$\delta_p = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \delta_p^{\boldsymbol{a}}, \qquad \textit{decomposition}$$

where

$\delta_p^{\boldsymbol{a}}$ are supported on the vertex patch subdomains $\omega_{\boldsymbol{a}}$, linearly depend on δ_p on the extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$,

and satisfy

$$\delta_p^{\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{with} \quad \nabla{\cdot}\delta_p^{\boldsymbol{a}} = 0, \qquad \textit{local div-free}$$

$$\|\delta_p^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\delta_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \qquad \textit{p-stable}$$

Outline

Conclusions

Piecewise polynomial approximation in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$:

- a p-stable **local (commuting) projector**
- p-robust **global**-best–**local**-best **equivalence**
- elementwise *hp* **approximation** error **estimates** under **minimal elementwise Sobolev regularity**

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Thank you for your attention!