Polynomial-degree-robust multilevel and domain decomposition methods with optimal step-sizes for mixed finite element discretizations of elliptic problems

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Inria Paris & Ecole des Ponts

Bergen, June 19, 2023

Outline

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The model problem

The Dracy porous media flow problem

Find the pressure head $\gamma : \Omega \to \mathbb{R}$ *and the Darcy velocity* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *such that*

$$\begin{aligned} \boldsymbol{u} &= -\boldsymbol{K}\nabla\gamma && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= f && \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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Setting

- $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$: interval/polygon/polyhedron
- $\boldsymbol{K} \in [L^\infty(\Omega)]^{d\times d}$: symmetric and positive definite diffusion tensor
- $f \in L^2(\Omega)$ of mean value 0: source term

Mixed finite element approximation

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Find $\boldsymbol{u}_J \in \boldsymbol{V}_J$ *and* $\gamma_J \in W_J$ *such that*

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\begin{aligned}
(\boldsymbol{K}^{-1}\boldsymbol{u}_J, \boldsymbol{v}_J) - (\gamma_J, \nabla \cdot \boldsymbol{v}_J) &= 0 \qquad && \forall \boldsymbol{v}_J \in \boldsymbol{V}_J, \\
(\nabla \cdot \boldsymbol{u}_J, w_J) &= (f, w_J) \qquad && \forall w_J \in W_J.
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Setting

- $\mathcal{T}_J$: simplicial mesh of Ω
- $\boldsymbol{V}_J := \{\boldsymbol{v}_J \in \boldsymbol{H}_0(\mathrm{div},\Omega),\ \boldsymbol{v}_J|_K \in \mathbf{RT}_p(K)\ \forall K \in \mathcal{T}_J\}$: Raviart–Thomas space (piecewise vector-valued polynomials on $\mathcal{T}_J$) of degree p, normal trace continuous and 0 on $\partial\Omega$ ($\boldsymbol{H}_0(\mathrm{div},\Omega)$-conforming)
- W_J: piecewise polynomials on $\mathcal{T}_J$ of degree p and mean value 0 on Ω

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MG solvers for mixed finite elements

Saddle-point solvers

- after a choice of basis: find algebraic vectors U and Γ such that

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \mathsf{U} \\ \Gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{F} \end{pmatrix}$$

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SPD reformulations and solvers

- equivalent reformulation via hybridization: find algebraic vector Λ such that

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SPD reformulations and solvers

- equivalent reformulation via hybridization: find algebraic vector Λ such that

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- symmetric and positive definite system matrix
- preconditioned conjugate gradients possible, multigrid not straightforward: Λ (pressure heads on the mesh faces) belong to non-nested spaces (Brenner (1992), Chen (1996), Wheeler, Yotov (2000))

Flux-only reformulation and corresponding MG solvers

Equivalent reformulation

Find $\boldsymbol{u}_J \in \boldsymbol{V}_J^f$ *such that*

$$(\boldsymbol{K}^{-1}\boldsymbol{u}_J, \boldsymbol{v}_J) = 0 \qquad \forall \boldsymbol{v}_J \in \boldsymbol{V}_J^0.$$

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- $\boldsymbol{V}_J^{g} := \{\boldsymbol{v}_J \in \boldsymbol{V}_J : (\nabla\cdot\boldsymbol{v}_J, w_J) = (g, w_J)\ \forall w_J \in W_J\}$
- only flux unknowns
- multigrid becomes easily possible (Mathew (1993), Ewing, Wang (1994), Hiptmair, Hoppe (1999))

DD solvers for mixed finite elements

Domain decomposition solvers

- Glowinski, Wheeler (1988), Cowsar, Mandel, Wheeler (1995), ...
- Ewing, Wang (1992), ...

A few central reflections

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- **one post-smoothing step** enough
- **Pythagoras formula** for **error decrease**
- **built-in** a posteriori **estimate** on the **algebraic error** (**adaptive smoothing**)
- p-**robustness**
- **unified** treatment of **multigrid** and **DD**

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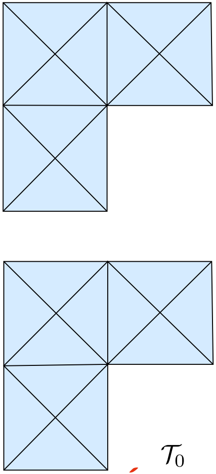
A hierarchy of meshes

Example: Two mesh hierarchies with $J = 3$ refinements.



Assumption: The meshes $\{\mathcal{T}_j\}_{0 \le j \le J}$ can be quasi-uniform or graded, satisfying:

- quasi-uniform $\mathcal{T}_0$,
- shape-regularity,
- maximum strength of refinement.

For given polynomial degree p and J, choose *increasing* level-wise polynomial degrees p_j, $j \in \{0, \dots, J\}$,

and define the spaces

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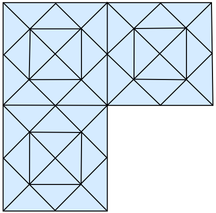

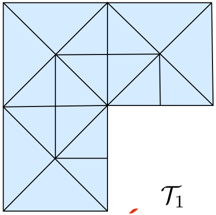

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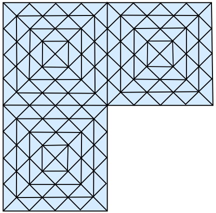

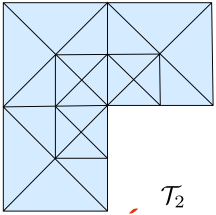

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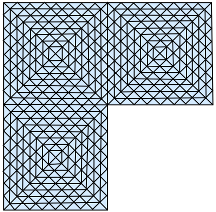

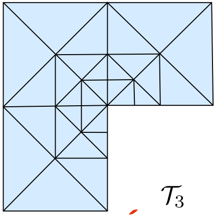

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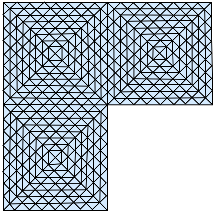

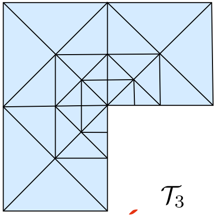

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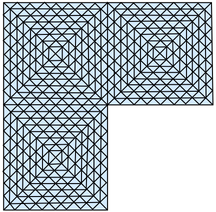

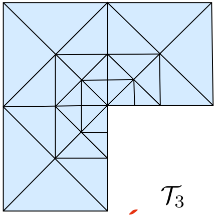

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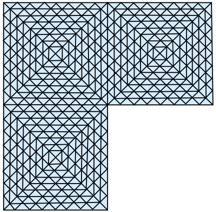

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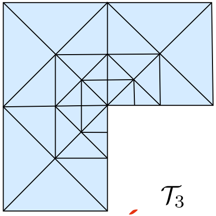

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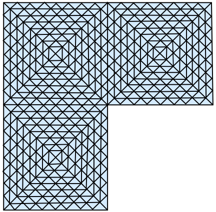

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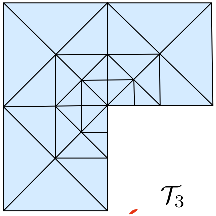

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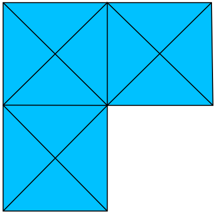

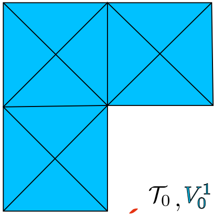

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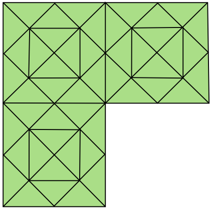

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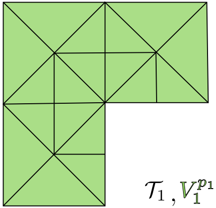
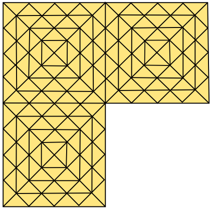


A hierarchy of meshes and spaces

Example: Two mesh hierarchies with $J = 3$ refinements.

Assumption: The meshes $\{\mathcal{T}_j\}_{0\le j\le J}$ can be quasi-uniform or graded, satisfying:

- quasi-uniform $\mathcal{T}_0$,
- shape-regularity,
- maximum strength of refinement.



For given polynomial degree p and J, choose *increasing* level-wise polynomial degrees p_j, $j \in \{0,\ldots,J\}$,

$$0 = p_0 \le p_1 \le p_2 \le \ldots \le p_J = p,$$

and define the spaces

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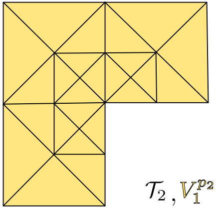

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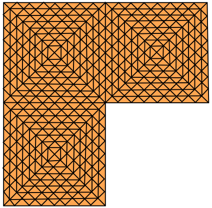

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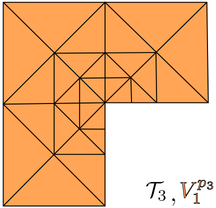
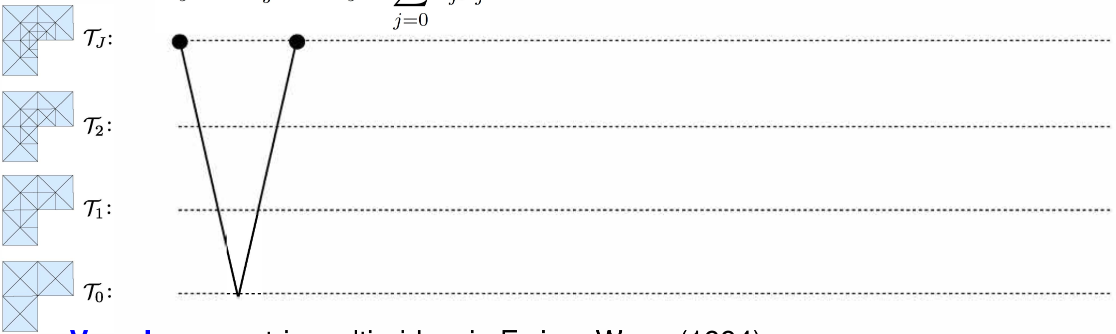


$\mathcal{T}_3, V_1^{p_3}$

Outline

V-cycle multigrid

$$\mathbf{u}_J^i \qquad \mathbf{u}_J^{i+1} = \mathbf{u}_J^i + \sum_{j=0}^{J} \lambda_j^i \boldsymbol{\rho}_j^i$$



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V(0,1)-cycle multigrid

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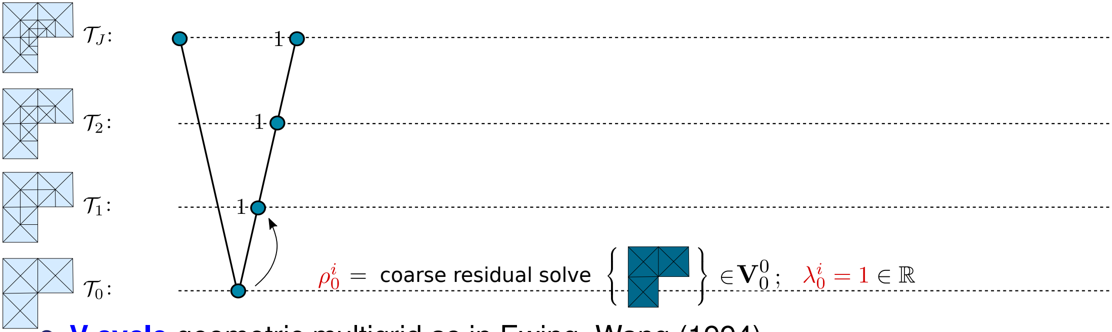

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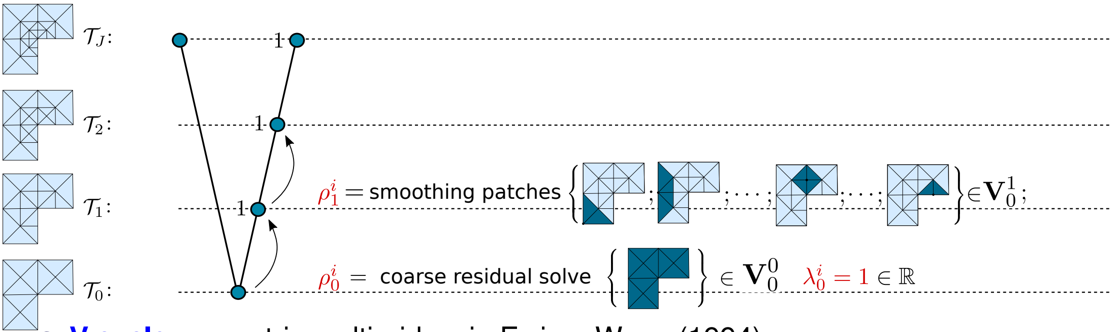
V(0,1)-cycle multigrid with block-Jacobi smoothing and line search



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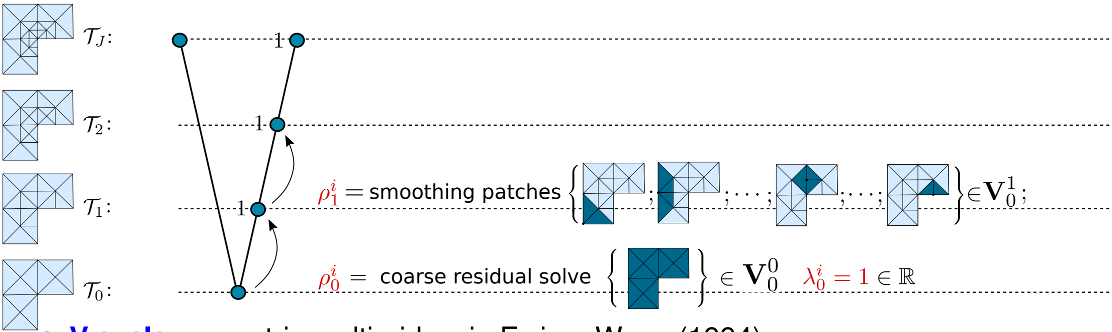

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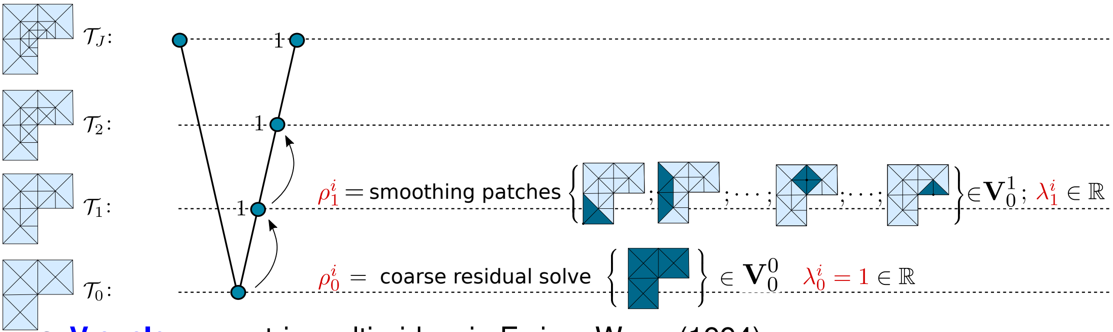

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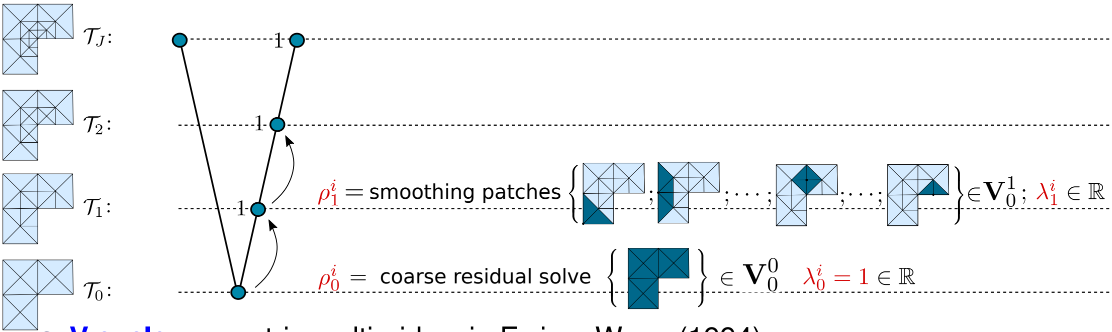

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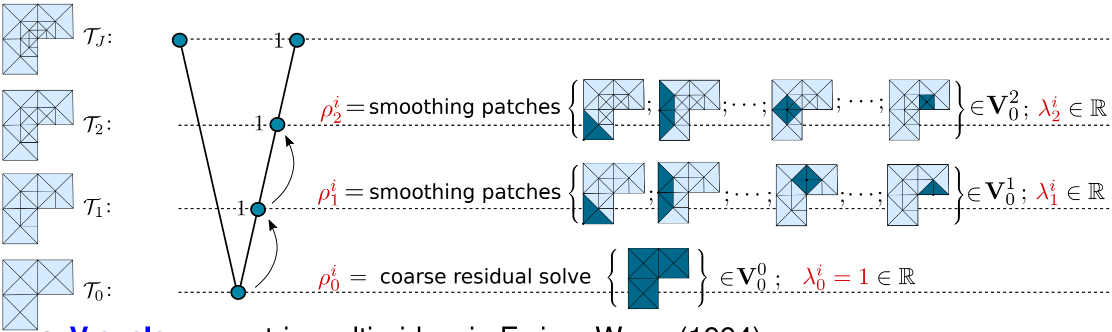

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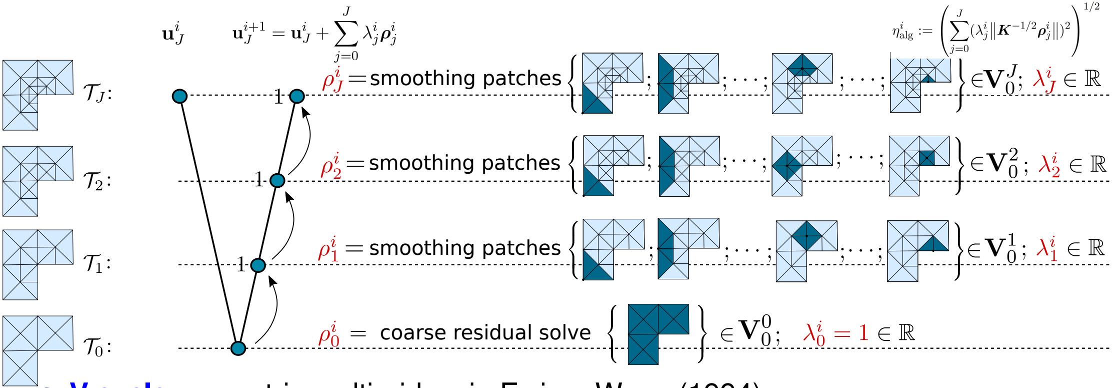

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Outline

Functional writing: for $\boldsymbol{u}_J^0 \in \boldsymbol{V}_J^f$ arbitrary, construct ρ_j^i, λ_j^i via $\boldsymbol{u}_j^i$

Coarse solve: $\rho_0^i \in \boldsymbol{V}_0^0$ s.t. $\underbrace{(\boldsymbol{K}^{-1}\rho_0^i, \boldsymbol{v}_0)}_{\text{global lifting}} = \underbrace{-(\boldsymbol{K}^{-1}\boldsymbol{u}_J^i, \boldsymbol{v}_0)}_{\text{global algebraic residual}} \quad \forall \boldsymbol{v}_0 \in \boldsymbol{V}_0^0$; $\boldsymbol{u}_0^i := \boldsymbol{u}_J^i + \rho_0^i$

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Outline

Pythagorean error formula and bound on the algebraic error

Theorem (Pythagorean error representation)

There holds

$$\underbrace{\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i+1})\right\|^2}_{\text{new error}} = \underbrace{\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\right\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_j^i \|\boldsymbol{K}^{-1/2}\boldsymbol{\rho}_j^i\|\right)^2}_{\left(\eta_{\mathrm{alg}}^i\right)^2}.$$

Corollary (Guaranteed lower bound on the algebraic error)

There holds:

$$\eta_{\mathrm{alg}}^i \leq \left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\right\|.$$

- similar situation to the conjugate gradients method, see Meurant (1997) and Strakoš and Tichý (2002)

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p-robust error contraction and algebraic estimator efficiency

Theorem (p-robust error contraction of the multilevel solver)

There holds

$$\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i+1})\| \leq \alpha \|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\|, \qquad 0 < \alpha(\kappa_{\mathcal{T}}, d, \mathbf{K}, J) < 1.$$

Theorem (p-robust reliable and efficient bound on the algebraic error)

There holds $\eta_{\mathrm{alg}}^i \leq \|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^i)\|$ *and, with* $\beta = \sqrt{1-\alpha^2}$,

$$\eta_{\mathrm{alg}}^i \geq \beta \|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^i)\|.$$

Corollary (Equivalence of the two main results)

The solver **contraction** *is* **equivalent** *to the* **efficiency** *of the estimator* η_{alg}^i.

- α is **independent** of the **polynomial degree** p

p-robust error contraction and algebraic estimator efficiency

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There holds $\eta_{\rm alg}^i \le \left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\right\|$ *and, with* $\beta = \sqrt{1-\alpha^2}$,

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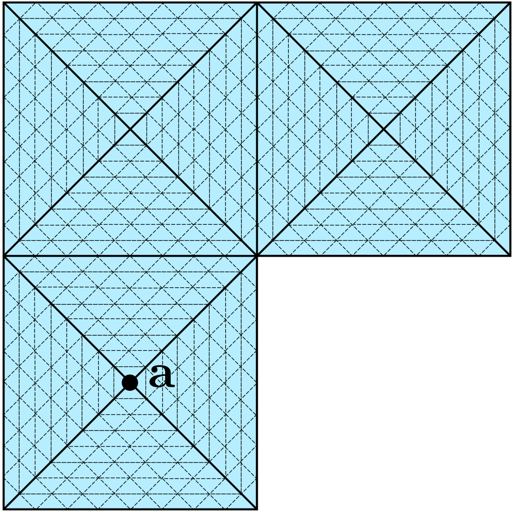
Domain decomposition for high-order mixed finite elements



Coarse grid $\mathcal{T}_H$ (solid line), fine grid $\mathcal{T}_J$ (dashed line), patch domain $\omega^{\mathbf{a}}$

Outline

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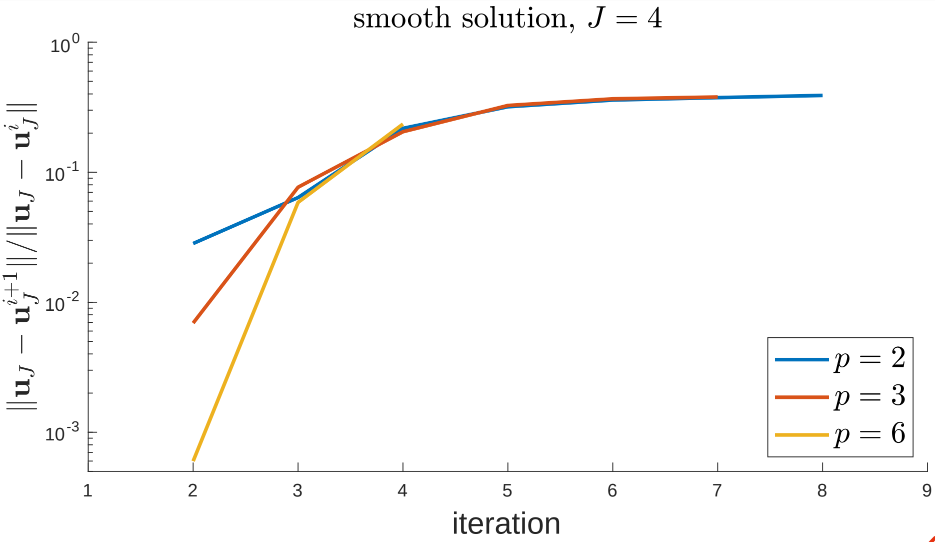
Smooth solution and uniform mesh refinement

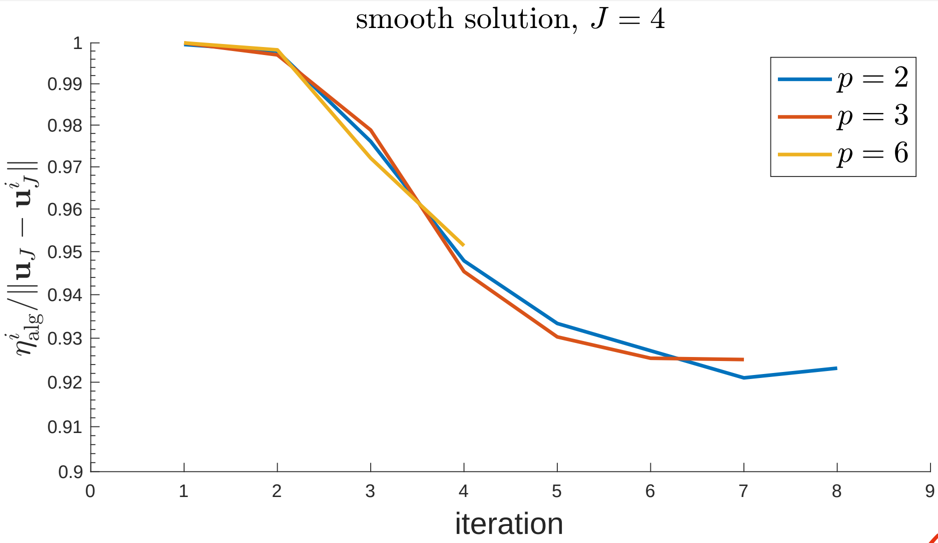
Setting

- Ω: unit square
- $\boldsymbol{K} = \mathrm{Id}$
- $\gamma(x, y) = \cos(\pi x)\cos(\pi y)$
- $\mathcal{T}_0$ with mesh size $h_0 = 0.3$, uniform mesh refinement

Contraction factors



Effectivity indices of the guaranteed lower bound η^i_{alg}



Number of iterations to decrease η^i_{alg} by 10^5

p	$J = 2$	$J = 3$	$J = 4$
2	9	9	8
3	9	8	7
6	6	5	4

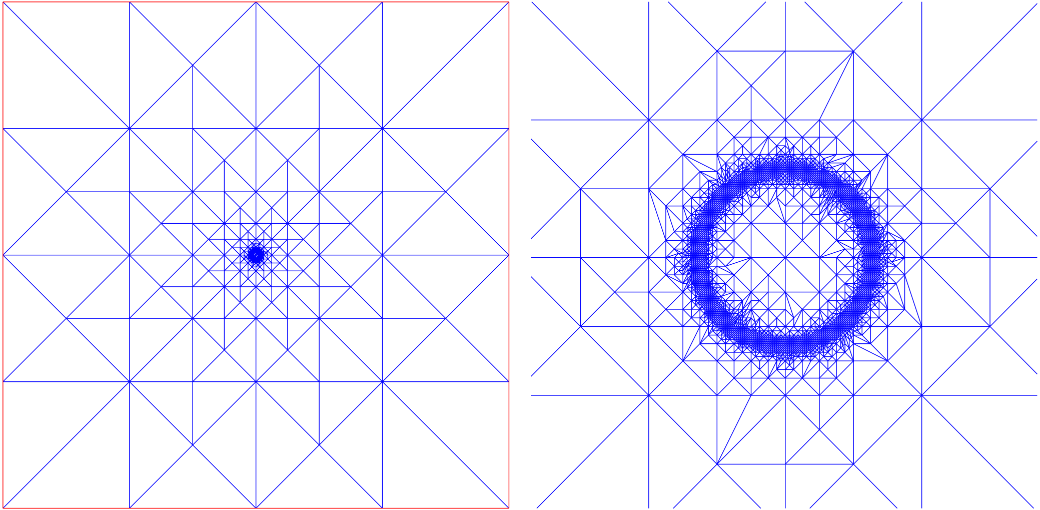
Outline

Rough solution and adaptive mesh refinement

Setting

- Ω: unit square
- $\boldsymbol{K} = \mathrm{Id}$
- $\gamma(x, y) = \tan^{-1}(\alpha(r - r_0))$, where $r = \sqrt{(x - x_\mathrm{c})^2 + (y - y_\mathrm{c})^2}$
- $\alpha = 1000$, $x_\mathrm{c} = 0.5$, $y_\mathrm{c} = 0.5$, $r_0 = 0.01$
- $\mathcal{T}_0$ with mesh size $h_0 = 0.3$, adaptive mesh refinement

Adaptive mesh: $J = 12$, $p = 6$, & Dörfler marking parameter $\theta = 0.8$



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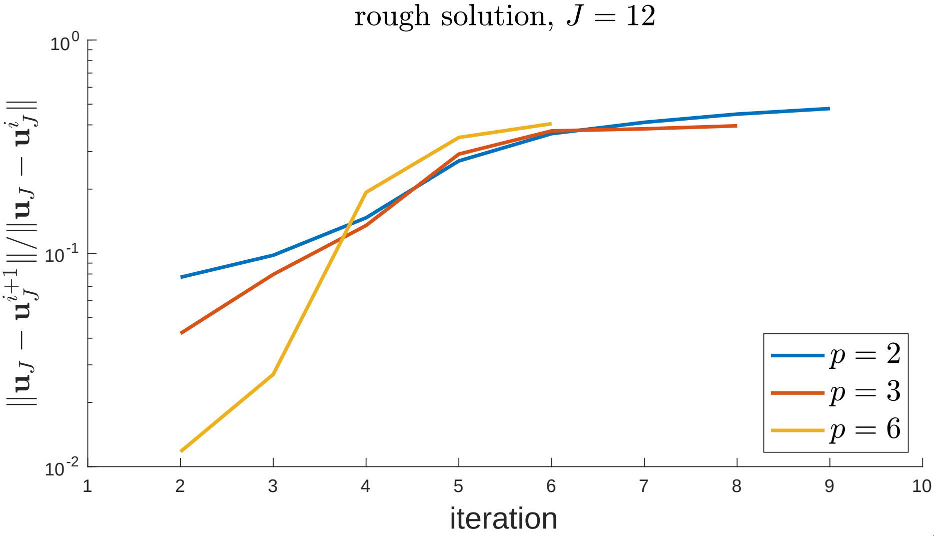

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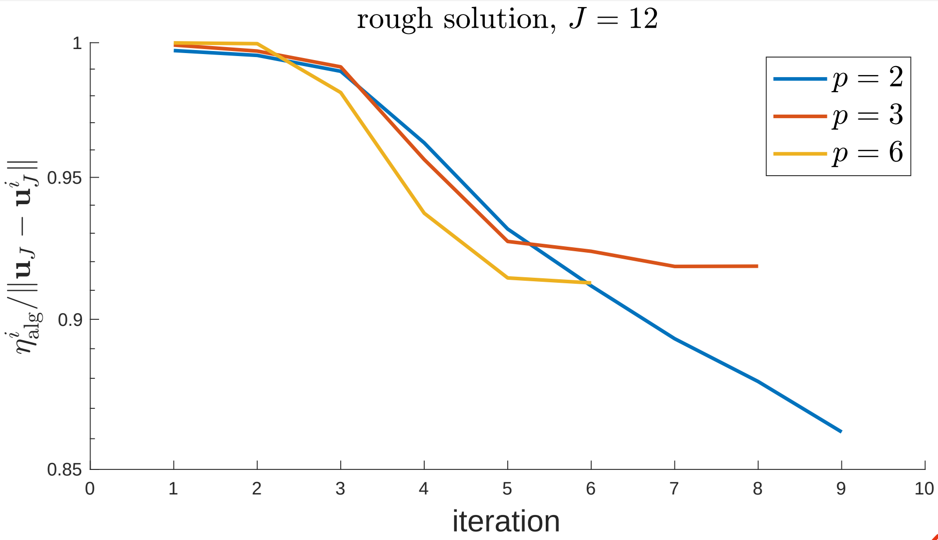

Number of iterations to decrease η^i_{alg} by 10^5

p	$J = 3$	$J = 6$	$J = 12$
2	13	12	9
3	12	11	8
6	13	10	6

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Conclusions

p-robust MG and DD methods for mixed finite elements

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Thank you for your attention!

Outline

Localized algebraic error estimate

Theorem (Localized algebraic error estimate)

There holds

$$\left(\eta_{\mathrm{alg}}^i\right)^2 = \left\|\boldsymbol{K}^{-1/2}\boldsymbol{\rho}_0^i\right\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\boldsymbol{a}\in\mathcal{V}_j} \left\|\boldsymbol{K}^{-1/2}\boldsymbol{\rho}_{j,\boldsymbol{a}}^i\right\|_{\omega_j^{\boldsymbol{a}}}^2.$$

Outline

Solvers for high-order finite elements

Algebraic problem

Find $\mathsf{U}_J \in \mathbb{R}^{|V_J^p|}$ such that

$$\mathbb{A}_J \mathsf{U}_J = \mathsf{F}_J$$

- $\mathbb{A}_J$ less and less sparse for big p
- $\mathbb{A}_J$ worse and worse conditioned for big p
- $\mathbb{A}_J$ looses structure on graded meshes $\mathcal{T}_J$
- $\mathbb{A}_J$ is dependent on the basis of V_J^p

Black-box iterative solvers

do not work well for high p & on highly graded meshes $\mathcal{T}_J$

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