A posteriori error estimates robust with respect to nonlinearities and orthogonal decomposition based on iterative linearization

Martin Vohralík

in collaboration with André Harnist, Koondanibha Mitra, and Ari Rappaport

Inria Paris & Ecole des Ponts

Tianyuan Mathematics Research Center, June 5, 2025

Outline

Goals

Error control

a posteriori error estimates

$$\underbrace{|||u - u_\ell|||}_{\text{unknown error}} \leq \underbrace{\eta(u_\ell)}_{\text{computable estimator}}$$

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Guaranteed a posteriori error estimates

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Guaranteed a posteriori error estimates **efficient**

$$\underbrace{|||u - u_\ell|||}_{\text{unknown error}} \leq \underbrace{\eta(u_\ell)}_{\text{computable estimator}} \leq C_{\text{eff}} |||u - u_\ell|||,$$

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Error control

Guaranteed a posteriori error estimates **efficient** and **robust** with respect to the **strength of nonlinearities**:

$$\underbrace{|||u - u_\ell|||}_{\text{unknown error}} \leq \underbrace{\eta(u_\ell)}_{\text{computable estimator}} \leq C_{\text{eff}} |||u - u_\ell|||, \quad C_{\text{eff}} \text{ independent of } \textit{data}$$

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Guaranteed a posteriori error estimates **efficient** and **robust** with respect to the **strength of nonlinearities**:

$$\underbrace{|||u - u_\ell|||}_{\text{unknown error}} \le \left\{ \sum_{K \in \mathcal{T}_\ell} \underbrace{\eta_K(u_\ell)^2}_{\text{element estimator}} \right\}^{1/2} \le C_{\text{eff}} |||u - u_\ell|||,$$

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$$\eta_K(u_\ell) \leq C_{\mathrm{eff}} |||u - u_\ell|||_{\omega_K}, \qquad \text{for all } K \in \mathcal{T}_\ell.$$

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Question

- **what to choose for $|||\cdot|||$?**

Previous results

Model nonlinear elliptic problem

Find $u : \Omega \to \mathbb{R}$ such that

$$-\nabla\cdot(a(|\nabla u|)\nabla u) = f \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

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- $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, open bounded polytope with Lipschitz boundary $\partial\Omega$
- a strongly monotone (a_{m}) and Lipschitz continuous (a_{c})
- f piecewise polynomial for simplicity
- numerical approximation u_ℓ

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- a strongly monotone (a_m) and Lipschitz continuous (a_c)
- f piecewise polynomial for simplicity
- numerical approximation u_ℓ
- strength of the nonlinearity (“nonlinear condition number”): $a_\mathrm{c}/a_\mathrm{m}$

Previous results

Sobolev norm

$$a_{\mathrm{m}} \|\nabla(u_\ell - u)\| \leq \eta(u_\ell) \leq C_{\mathrm{eff}} a_{\mathrm{c}} \|\nabla(u_\ell - u)\|$$

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Sobolev norm (**not robust** wrt $\frac{a_c}{a_m}$)

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Energy difference

$$\mathcal{J}(u_\ell) - \mathcal{J}(u) \le \eta(u_\ell)^2 \le C_{\text{eff}}^2 \frac{a_c^2}{a_m^2} (\mathcal{J}(u_\ell) - \mathcal{J}(u))$$

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- El Alaoui, Ern, & Vohralík (2011), Blechta, Málek, & Vohralík (2020), ...

Outline

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Find $u \in H_0^1(\Omega)$ such that

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Assumption (Gradient-dependent diffusivity)

Function $a : [0, \infty) \to (0, \infty)$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$|a(|\boldsymbol{x}|)\boldsymbol{x} - a(|\boldsymbol{y}|)\boldsymbol{y}| \leq a_{\rm c}|\boldsymbol{x} - \boldsymbol{y}| \qquad \text{(Lipschitz continuity)},$$

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Example of the nonlinear function a

Example (Mean curvature nonlinearity)

$$a(r) := a_{\mathrm{m}} + \frac{a_{\mathrm{c}} - a_{\mathrm{m}}}{\sqrt{1 + r^2}}.$$

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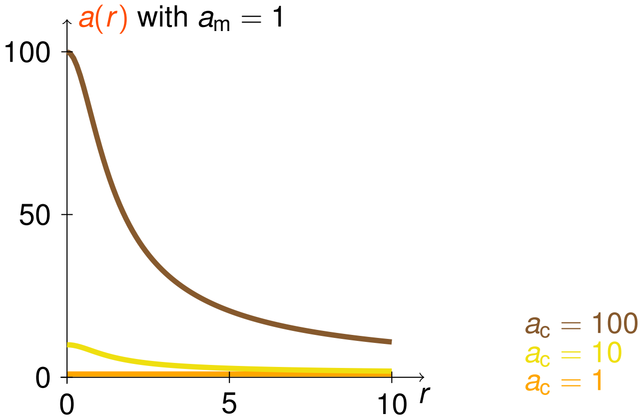

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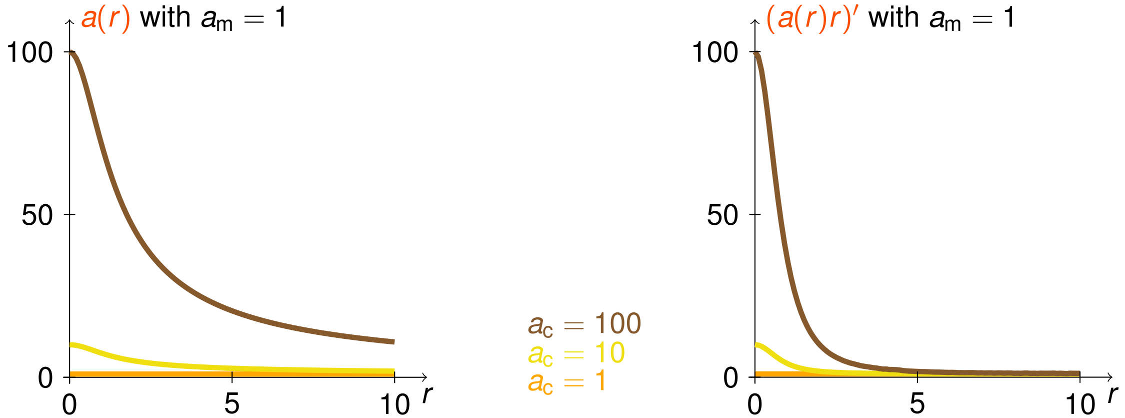

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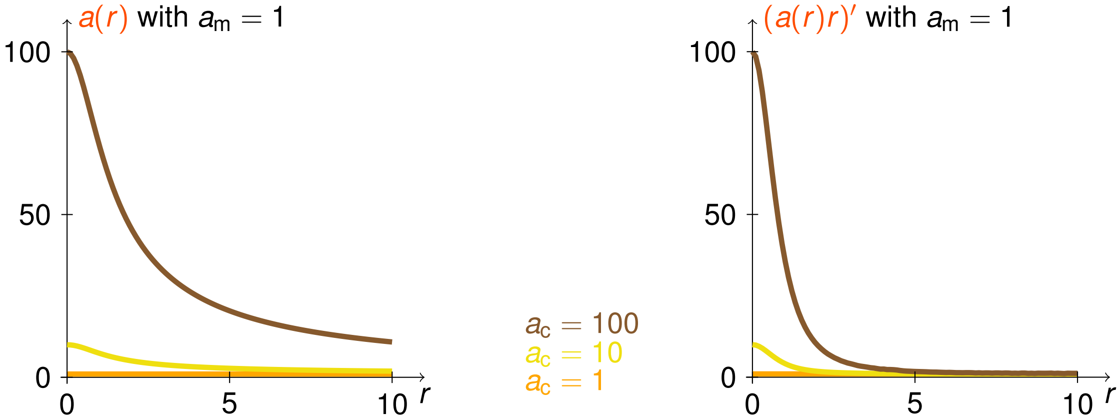
Strength of the nonlinearity

$$\frac{a_{\mathrm{c}}}{a_{\mathrm{m}}} = \frac{\text{Lipschitz continuity}}{\text{strong monotonicity}}$$



A model nonlinear problem II

Find $u \in H_0^1(\Omega)$ such that

$$\langle \mathcal{R}(u), v \rangle := (\tau \boldsymbol{K}(\underbrace{a(u)}_{\text{diffusion}} \nabla u + \underbrace{\boldsymbol{q}(u)}_{\text{advection}}), \nabla v) + (\underbrace{f(u)}_{\text{reaction}}, v) = 0 \qquad \forall v \in H_0^1(\Omega).$$

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- semilinear equations with $a = 1$, $\boldsymbol{K} = \boldsymbol{I}_d$, $\tau = 1$, $\boldsymbol{q} = \boldsymbol{0}$: ignition of gases, gravitational influences on stars, quantum field theory ...
- implicit time-discretization of nonlinear (degenerate) parabolic equations ($\tau > 0$ a time step size): Fischer–KPP, porous medium, Richards, biofilms ...

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Assumption (Gradient-independent diffusivity)

- $|a(\boldsymbol{x}_1, u_1) - a(\boldsymbol{x}_2, u_2)| \le L_a(|\boldsymbol{x}_1 - \boldsymbol{x}_2| + |u_1 - u_2|) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega$ *and* $u_1, u_2 \in \mathbb{R}$
- $0 \le f(\boldsymbol{x}, u_2) - f(\boldsymbol{x}, u_1) \le L_f(u_2 - u_1) \quad \forall \boldsymbol{x} \in \Omega$ *and* $u_1, u_2 \in \mathbb{R}, u_2 \ge u_1$
- $\boldsymbol{K}$ *is uniformly symmetric positive definite and bounded with eigenvalues* $a_{\rm m}$, $a_{\rm c}$
- $\boldsymbol{q}$ *is "small" wrt* a

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Linear case

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Classical choices discussed above

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Discretization and fixed-point iterative linearization

- **discretization**: finite element subspace $V_\ell \subset H_0^1(\Omega)$, find $u_\ell \in V_\ell$ s.t.

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- fixed-point **iterative linearization**: from $u_\ell^{k-1} \in V_\ell$, find $u_\ell^k \in V_\ell$ such that

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Trivial nonlinear case: main idea

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- fixed-point **iterative linearization**: from $u_\ell^{k-1} \in V_\ell$, find $u_\ell^k \in V_\ell$ such that

$$(\boldsymbol{K} a(u_\ell^{k-1}) \nabla u_\ell^k, \nabla v_\ell) + (f, v_\ell) = 0 \qquad \forall v_\ell \in V_\ell.$$

Iteration-dependent discrete energy norm

$$|||\cdot|||_{u_\ell^{k-1}} := \|\boldsymbol{K}^{1/2} a^{1/2}(u_\ell^{k-1}) \nabla(\cdot)\|$$

Trivial nonlinear case: main idea

Find $u \in H_0^1(\Omega)$ such that

$$\langle \mathcal{R}(u), v \rangle := (\boldsymbol{K} a(u) \nabla u, \nabla v) + (f, v) = 0 \qquad \forall v \in H_0^1(\Omega).$$

Ideal but impossible choice

$$|||\cdot||| := \|\boldsymbol{K}^{1/2} a^{1/2}(u) \nabla (\cdot)\|$$

Discretization and fixed-point iterative linearization

- **discretization**: finite element subspace $V_\ell \subset H_0^1(\Omega)$, find $u_\ell \in V_\ell$ s.t.

$$(\boldsymbol{K} a(u_\ell) \nabla u_\ell, \nabla v_\ell) + (f, v_\ell) = 0 \qquad \forall v_\ell \in V_\ell.$$

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Iteration-dependent discrete energy and dual norms

$$|||\cdot|||_{u_\ell^{k-1}} := \|\boldsymbol{K}^{1/2} a^{1/2}(u_\ell^{k-1}) \nabla (\cdot)\|, \qquad |||\mathcal{R}(u_\ell)|||_{-1, u_\ell^{k-1}} := \sup_{v \in H_0^1(\Omega)} \frac{\langle \mathcal{R}(u_\ell), v \rangle}{|||v|||_{u_\ell^{k-1}}}$$

Main idea

Main idea

Apply in the **a posteriori analysis** and in **adaptivity**, to define norms, the **iterative linearization** on the **discrete level**.

Outline

Weak solution

Definition (Weak solution)

Find $u \in H_0^1(\Omega)$ s.t.

$$\langle \mathcal{R}(u), v \rangle = 0 \qquad \forall v \in H_0^1(\Omega).$$

Finite element discretization

Definition (Finite element discretization)

Find $u_\ell \in V_\ell$ s.t.

$$\langle \mathcal{R}(u_\ell), v_\ell \rangle = 0 \qquad \forall v_\ell \in V_\ell.$$

Finite element discretization

Definition (Finite element discretization)

Find $u_\ell \in V_\ell$ s.t.

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- $\mathcal{T}_\ell$ simplicial mesh of Ω
- $p \geq 1$ polynomial degree
- $V_\ell := \mathcal{P}_p(\mathcal{T}_\ell) \cap H_0^1(\Omega)$
- conforming finite elements

Iterative linearization

Definition (Iterative linearization)

Find $u_\ell^k \in V_\ell$ s.t.

$$((\underbrace{u_\ell^k - u_\ell^{k-1}}_{\text{increment}}, v_\ell))_{u_\ell^{k-1}} = -\langle \underbrace{\mathcal{R}(u_\ell^{k-1})}_{\text{residual}}, v_\ell \rangle \qquad \forall v_\ell \in V_\ell.$$

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- iterative linearization index $k \geq 1$

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- $u_\ell^0 \in V_\ell$ a given initial guess

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- iterative linearization index $k \geq 1$
- $u_\ell^0 \in V_\ell$ a given initial guess
- **iteration-dependent** reaction–diffusion **scalar product**

$$((w, v))_{u_\ell^{k-1}} := (L_\ell^{k-1} w, v) + (\boldsymbol{A}_\ell^{k-1} \nabla w, \nabla v), \quad w, v \in H_0^1(\Omega)$$

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- $\boldsymbol{A}_\ell^{k-1} : \Omega \to \mathbb{R}^{d \times d}$ matrix-valued function constructed from u_ℓ^{k-1}, $L_\ell^{k-1} : \Omega \to \mathbb{R}$ scalar-valued function constructed from u_ℓ^{k-1}

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- $L_\ell^{k-1} = 0$ if $f = f(\boldsymbol{x})$ (linear, source term)

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- examples for the gradient-dependent diffusivity:
  - $\boldsymbol{A}_\ell^{k-1} = a(|\nabla u_\ell^{k-1}|)\boldsymbol{I}_d$, $L_\ell^{k-1} = \partial_u f(u_\ell^{k-1})$ (Kačanov)
  - $\boldsymbol{A}_\ell^{k-1} = a(|\nabla u_\ell^{k-1}|)\boldsymbol{I}_d + \frac{a'(|\nabla u_\ell^{k-1}|)}{|\nabla u_\ell^{k-1}|}\nabla u_\ell^{k-1} \otimes \nabla u_\ell^{k-1}$, $L_\ell^{k-1} = \partial_u f(u_\ell^{k-1})$ (Newton)
  - $\boldsymbol{A}_\ell^{k-1} = \gamma \boldsymbol{I}_d$ with $\gamma \geq \frac{a_c^2}{a_m}$ a **constant parameter**, $L_\ell^{k-1} = 0$ (Zarantonello)

Iterative linearization

Definition (Iterative linearization)

Find $u_\ell^k \in V_\ell$ s.t.

$$((\underbrace{u_\ell^k - u_\ell^{k-1}}_{\text{increment}}, v_\ell))_{u_\ell^{k-1}} = -\langle \underbrace{\mathcal{R}(u_\ell^{k-1})}_{\text{residual}}, v_\ell \rangle \qquad \forall v_\ell \in V_\ell.$$

- examples for the gradient-independent diffusivity:
  - $\boldsymbol{A}_\ell^{k-1} = \tau \boldsymbol{K} a(u_\ell^{k-1})$ (fixed point)

Iterative linearization

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- examples for the gradient-independent diffusivity:
  - $\boldsymbol{A}_\ell^{k-1} = \tau \boldsymbol{K} a(u_\ell^{k-1})$ (fixed point)
  - Picard: $L_\ell^{k-1} = \partial_u f(u_\ell^{k-1})$
  - Jäger–Kačur: $L_\ell^{k-1} = \max_{u \in \mathbb{R}} \left(\frac{f(u) - f(u_\ell^{k-1})}{u - u_\ell^{k-1}} \right)$
  - L-scheme: $L_\ell^{k-1} = \text{cnst} \geq \frac{1}{2} \sup \partial_u f$
  - M-scheme: $L_\ell^{k-1} = \partial_u f(u_\ell^{k-1}) + \tau \times \text{cnst}$

Iterative linearization

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Find $u_\ell^k \in V_\ell$ s.t.

$$\langle \mathcal{R}_{\mathrm{disc}}^{u_\ell^{k-1}}(u_\ell^k), v_\ell \rangle := ((u_\ell^k - u_\ell^{k-1}, v_\ell))_{u_\ell^{k-1}} + \langle \mathcal{R}(u_\ell^{k-1}), v_\ell \rangle = 0 \qquad \forall v_\ell \in V_\ell.$$

Outline

Iteration-dependent norm

Definition (Iteration-dependent norm)

$$|||v|||^2_{u_\ell^{k-1}} := ((v,\, v))_{u_\ell^{k-1}} = \|(L_\ell^{k-1})^{1/2} v\|^2 + \|(\boldsymbol{A}_\ell^{k-1})^{1/2} \nabla v\|^2, \quad v \in H_0^1(\Omega)$$

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- induced by the linearization scalar product

An orthogonal decomposition of the total residual/error

Theorem (Orthogonal decomposition of the total error into linearization and discretization components)

For all linearization steps $k \geq 1$, there holds

$$\underbrace{|||\mathcal{R}(u_\ell^{k-1})|||^2_{-1,u_\ell^{k-1}}}_{\textit{total residual/error}} = \underbrace{|||u_\ell^{k-1} - u_\ell^k|||^2_{u_\ell^{k-1}}}_{\textit{linearization error}} + \underbrace{|||\mathcal{R}_{\mathrm{disc}}^{u_\ell^{k-1}}(u_\ell^k)|||^2_{-1,u_\ell^{k-1}}}_{\textit{discretization residual/error}}.$$

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- $u^k_{\langle\ell\rangle} \in H_0^1(\Omega)$ such that $\langle\mathcal{R}_{\text{disc}}^{u_\ell^{k-1}}(u^k_{\langle\ell\rangle}), v\rangle = 0 \ \forall v \in H_0^1(\Omega)$

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- **orthogonal decomposition** into **error components**
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- $\langle \mathcal{R}_{\text{disc}}^{u_\ell^{k-1}}(u_\ell^k), v\rangle = ((u_\ell^k - u_\ell^{k-1}, v))_{u_\ell^{k-1}} + \langle \mathcal{R}(u_\ell^{k-1}), v\rangle$, $v \in H_0^1(\Omega)$ (0 if $v \in V_\ell$)
- $u_{\langle\ell\rangle}^k \in H_0^1(\Omega)$ such that $\langle \mathcal{R}_{\text{disc}}^{u_\ell^{k-1}}(u_{\langle\ell\rangle}^k), v\rangle = 0 \ \forall v \in H_0^1(\Omega)$
- **discretization error** is given by a **linear** reaction–diffusion problem

Outline

A posteriori error estimates in an iteration-dependent norm

Theorem (A posteriori estimate in iteration-dependent norm)

For all linearization steps $k \geq 1$,

$$|||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} \leq \eta(u_\ell^k).$$

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For all linearization steps $k \geq 1$,

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Moreover, for all linearization steps $k \geq 1$, there holds

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✓ $C_\ell^k = 1$ for Zarantonello $\Longrightarrow$ **robustness** wrt the **strength of nonlinearities**

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- ✓ $C_\ell^k = 1$ for Zarantonello $\Longrightarrow$ **robustness** wrt the **strength of nonlinearities**
- ✓ C_K^k given by **local conditioning** of the linearization matrix $\boldsymbol{A}_\ell^{k-1}$ (and scalar L_ℓ^{k-1}):

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- ✓ C_K^k given by **local conditioning** of the linearization matrix $\boldsymbol{A}_\ell^{k-1}$ (and scalar L_ℓ^{k-1}): typically **much better** than global conditioning (= worst-case scenario)
- ✓ C_K^k **computable**: we can affirm **robustness *a posteriori***, for the given case

A posteriori error estimates in an iteration-dependent norm

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- ✓ **local efficiency**

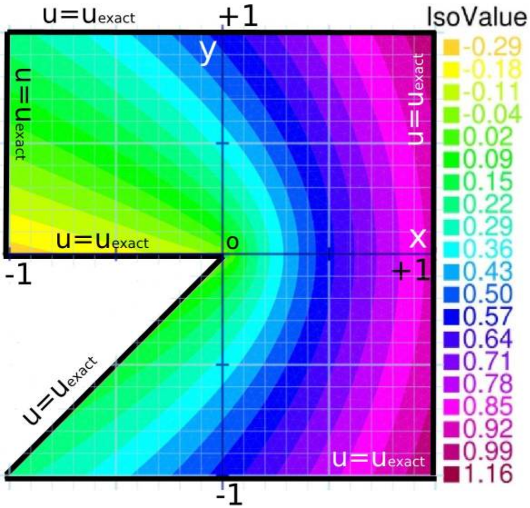
Outline

Gradient-dependent diffusivity

Setting

- mean curvature with $a(r) := a_{\rm m} + \frac{a_{\rm c} - a_{\rm m}}{\sqrt{1+r^2}}$
- $a_{\rm c} := a_{\rm m} + 1$, $a_{\rm m} := 2\tau$, $\frac{a_{\rm c}}{a_{\rm m}} = 1 + 0.5\frac{1}{\tau}$, $\frac{1}{\tau} \in [1, 10^3]$
- $f(\boldsymbol{x}, \xi) := \nu\,\xi - g(\boldsymbol{x})$, $\nu = 10^{-2}$
- $g(\boldsymbol{x}) := \left[r^{-1} \frac{(1-\lambda)(\lambda\, r^{\lambda-1})^3}{(1+(\lambda\, r^{\lambda-1})^2)^{\frac{3}{2}}} + \nu r^{\lambda} \right] \cos(\lambda\theta)$, $\lambda := 4/7$
- weak solution $u(\boldsymbol{x}) := r^{\lambda} \cos(\lambda\theta)$

Solution u


u=uexact
+1
IsoValue
-0.29
-0.18
-0.11
-0.04
0.02
0.09
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0.22
0.29
0.36
0.43
0.50
0.57
0.64
0.71
0.78
0.85
0.92
0.99
1.16
y
u=uexact
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-1
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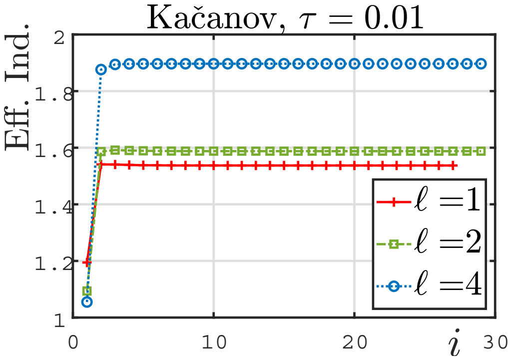
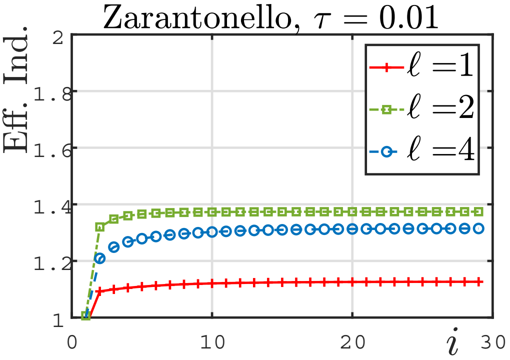
How **large** is the error?





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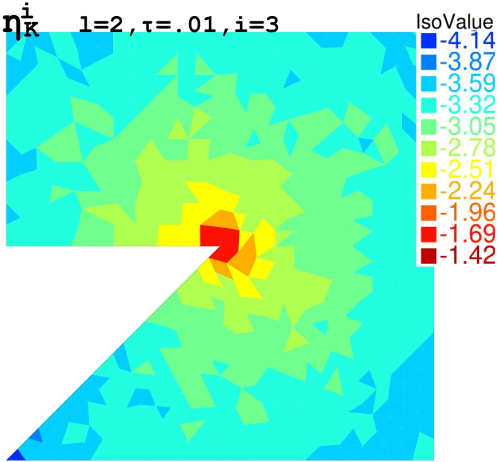
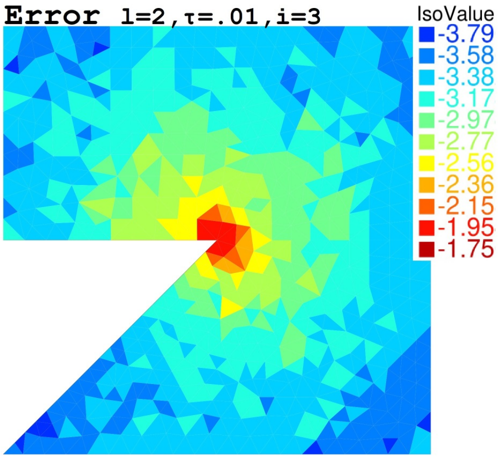
Where is the error **localized**?



Estimated error, $\tau = 0.01$, Kačanov



Exact error, $\tau = 1$, $\tau = 0.01$, Kačanov

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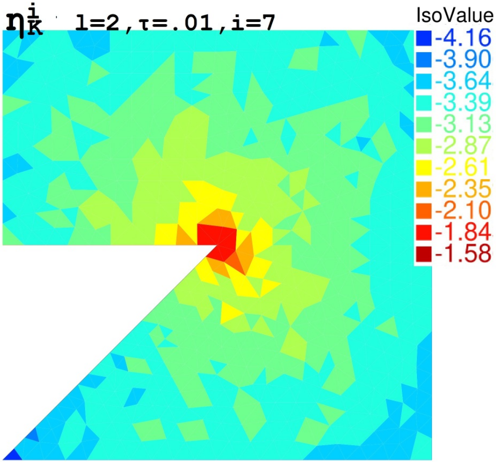
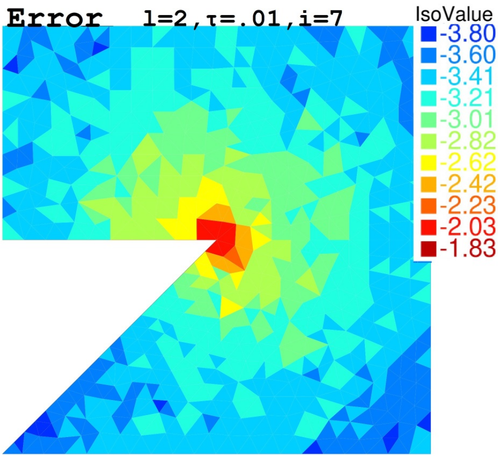


Estimated error, $\tau = 0.01$, Zarantonello



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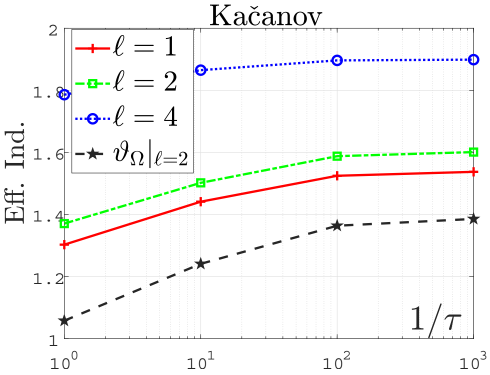
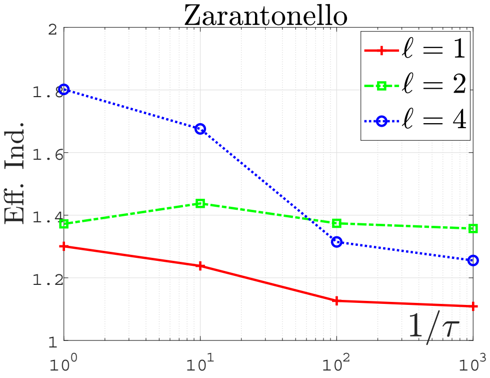
Robustness wrt the nonlinearities





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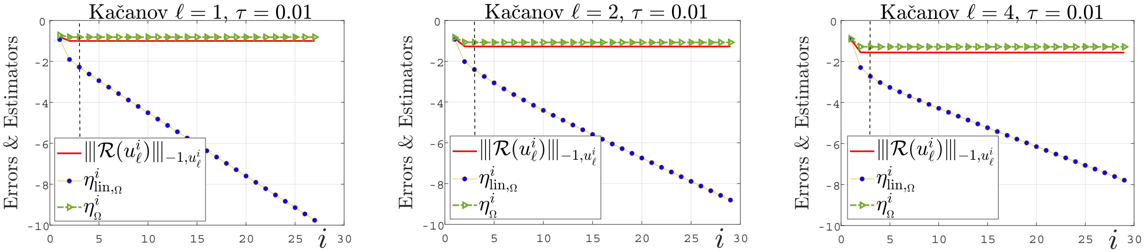
Error components and adaptivity via **stopping criteria**



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Gradient-independent diffusivity

Setting

- one time step of the Richards equation
- unit square $\Omega = (0,1)^2$
- realistic data

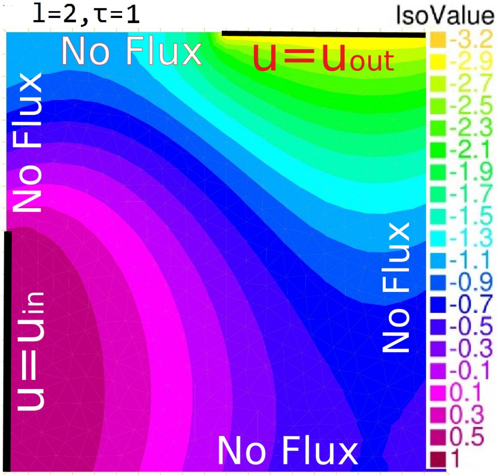
$$f(\boldsymbol{x},u) = S(u) - S(u_\ell^{n-1}(\boldsymbol{x})), \quad a(\boldsymbol{x},u) = \kappa(S(u)), \quad \boldsymbol{q}(\boldsymbol{x},u) = -\kappa(S(u))\,\boldsymbol{g},$$
$$\boldsymbol{K} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **van Genuchten saturation** and **permeability** laws

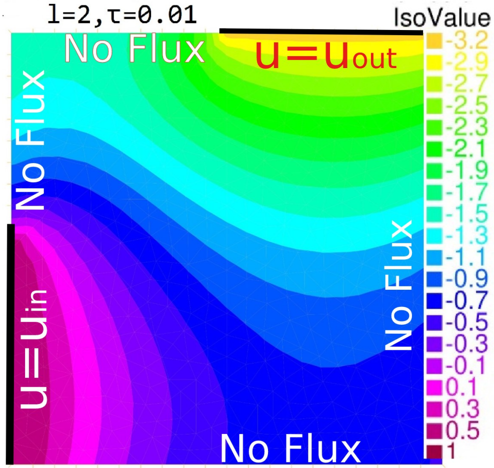
$$S(u) := \left(1 + (2-u)^{\frac{1}{1-\lambda}}\right)^{-\lambda}, \quad \kappa(s) := \sqrt{s}\left(1 - (1 - s^{\frac{1}{\lambda}})^{\lambda}\right)^2, \quad \lambda = 0.5$$

- time step length $\tau \in [10^{-3}, 1]$

Solution u

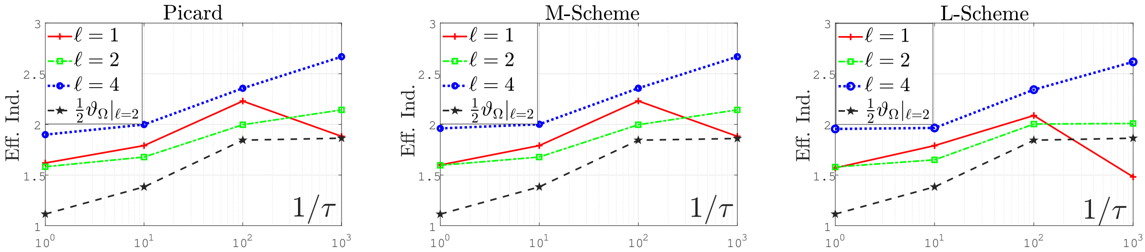


Time step length $\tau = 1$



Time step length $\tau = 0.01$

How **large** is the error? **Robustness** wrt the nonlinearities



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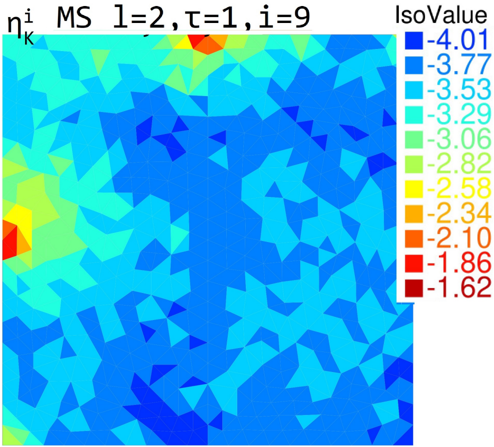
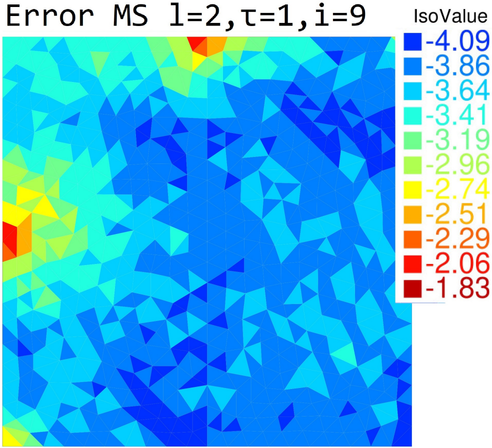


Estimated error, $\tau = 1$



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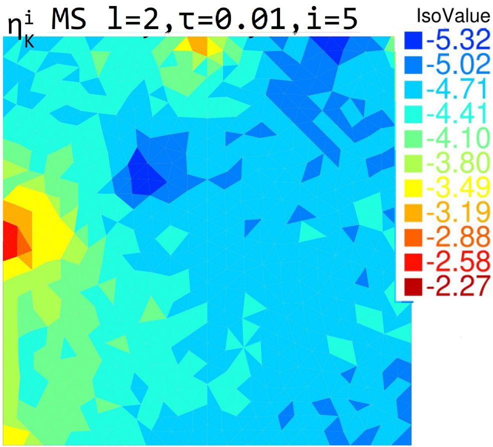

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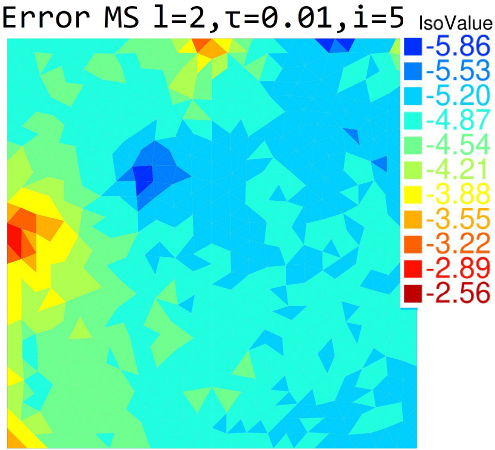

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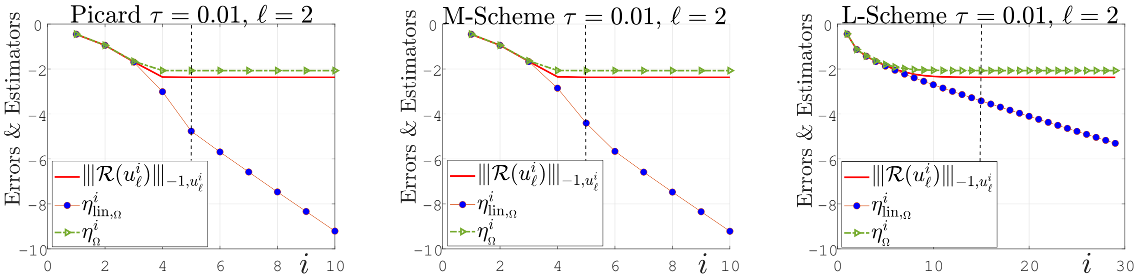

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Outline

A model nonlinear problem

Nonlinear elliptic problem

Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(a(|\nabla u|)\nabla u) &= f \quad \text{in} \quad \Omega,\\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

- $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, open bounded polytope with Lipschitz boundary $\partial\Omega$
- f piecewise polynomial for simplicity

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Assumption (Gradient-dependent diffusivity)

Function $a : [0, \infty) \to (0, \infty)$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$\begin{aligned} |a(|\boldsymbol{x}|)\boldsymbol{x} - a(|\boldsymbol{y}|)\boldsymbol{y}| &\leq a_{\mathrm{c}}|\boldsymbol{x} - \boldsymbol{y}| && \textit{(Lipschitz continuity)}, \\ (a(|\boldsymbol{x}|)\boldsymbol{x} - a(|\boldsymbol{y}|)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) &\geq a_{\mathrm{m}}|\boldsymbol{x} - \boldsymbol{y}|^2 && \textit{(strong monotonicity)}. \end{aligned}$$

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- $a_{\mathrm{m}} \le a(r) \le a_{\mathrm{c}}$, $a_{\mathrm{m}} \le (a(r)r)' \le a_{\mathrm{c}}$

Outline

Weak solution

Definition (Weak solution)

$u \in H_0^1(\Omega)$ such that

$$(a(|\nabla u|)\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega).$$

Energy and energy minimization

Definition (Energy functional)

$\mathcal{J} : H_0^1(\Omega) \to \mathbb{R}$

$$\mathcal{J}(v) := \int_\Omega \phi(|\nabla v|) - (f, v), \quad v \in H_0^1(\Omega),$$

with function $\phi : [0, \infty) \to [0, \infty)$ such that, for all $r \in [0, \infty)$,

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Equivalently

$$u = \arg \min_{v \in H_0^1(\Omega)} \mathcal{J}(v)$$

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$u_\ell \in V_\ell$ such that

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Energy difference

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$$\mathcal{J}(u_\ell) - \mathcal{J}(u)$$

- $\mathcal{J}(u_\ell) - \mathcal{J}(u) \geq 0$, $\mathcal{J}(u_\ell) - \mathcal{J}(u) = 0$ if and only if $u_\ell = u$
- physically-based error measure

Outline

Iterative linearization

Definition (Linearized finite element approximation)

$u_\ell^k \in V_\ell$ such that

$$(\boldsymbol{A}_\ell^{k-1} \nabla u_\ell^k, \nabla v_\ell) = (f, v_\ell) + (\boldsymbol{b}_\ell^{k-1}, \nabla v_\ell) \qquad \forall v_\ell \in V_\ell.$$

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- $u_\ell^0 \in V_\ell$ a given initial guess
- iterative linearization index $k \geq 1$
- $\boldsymbol{A}_\ell^{k-1} : \Omega \to \mathbb{R}^{d \times d}$ matrix-valued function constructed from u_ℓ^{k-1}, $\boldsymbol{b}_\ell^{k-1} : \Omega \to \mathbb{R}^d$ vector-valued function constructed from u_ℓ^{k-1}

Examples

Example (Picard (fixed-point))

$$\boldsymbol{A}_\ell^{k-1} = a(|\nabla u_\ell^{k-1}|)\boldsymbol{I}_d, \quad \boldsymbol{b}_\ell^{k-1} = \boldsymbol{0}.$$

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$$\boldsymbol{A}_\ell^{k-1} = \gamma \boldsymbol{I}_d, \quad \boldsymbol{b}_\ell^{k-1} = (\gamma - a(|\nabla u_\ell^{k-1}|))\nabla u_\ell^{k-1},$$

with $\gamma \geq \frac{a_c^2}{a_m}$ *a* **constant parameter**.

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Example (Newton)

$$\boldsymbol{A}_\ell^{k-1} = a(|\nabla u_\ell^{k-1}|)\boldsymbol{I}_d + \frac{a'(|\nabla u_\ell^{k-1}|)}{|\nabla u_\ell^{k-1}|}\nabla u_\ell^{k-1} \otimes \nabla u_\ell^{k-1},$$

$$\boldsymbol{b}_\ell^{k-1} = a'(|\nabla u_\ell^{k-1}|)|\nabla u_\ell^{k-1}|\nabla u_\ell^{k-1}.$$

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Apply in the **a posteriori analysis** and in **adaptivity**, to define the way how we measure the error, the **iterative linearization** on the **discrete level**.

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Continuous minimizer of the linearized energy functional

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Augmented energy difference

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Practically

$$\mathcal{E}_\ell^k = \mathcal{J}(u_\ell^k) - \mathcal{J}(u) \text{ at convergence}$$

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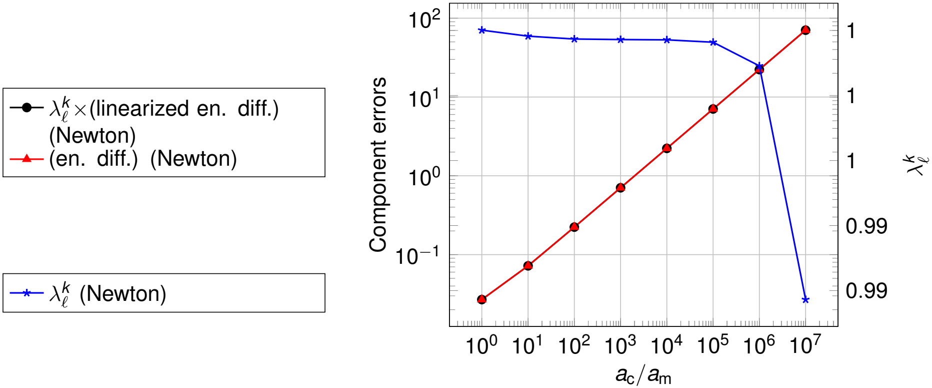

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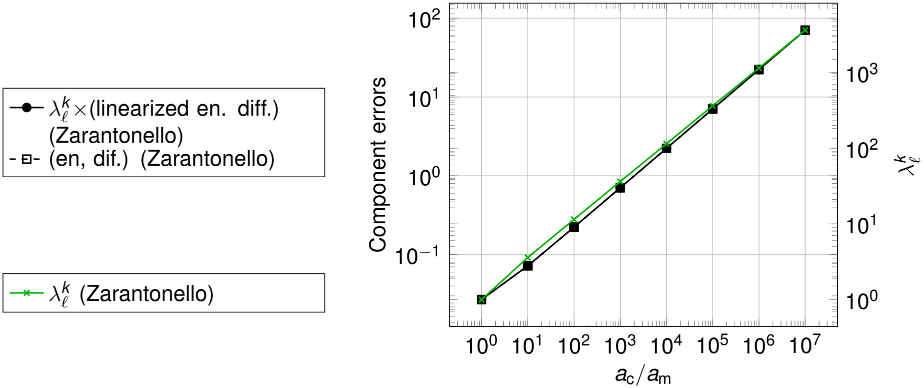

Outline

A posteriori error estimates for an augmented energy difference

Theorem (A posteriori estimate of augmented energy)

For all linearization steps $k \geq 1$,

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where

$$C_\ell^k \begin{cases} = 1 & \textit{Zarantonello} \end{cases}$$

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Moreover, for all $k \geq 1$, there holds

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where

$$C_\ell^k \quad \begin{cases} = 1 & \textit{Zarantonello} \\ & \end{cases}$$

✓ $C_\ell^k = 1$ for Zarantonello $\Longrightarrow$ **robustness** wrt the **strength of nonlinearities**

A posteriori error estimates for an augmented energy difference

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- ✓ C_ℓ^k **computable**: we can affirm **robustness *a posteriori***, for the given case

Outline

Fenchel conjugate, dual energy, estimator, flux equilibration

Definition (Fenchel conjugate)

$$\phi^*(\cdot, s) := \sup_{r \in [0,\infty)} (sr - \phi(\cdot, r)).$$

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Definition (Flux equilibration: $\boldsymbol{\sigma}_\ell^k = \sum_{\boldsymbol{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\boldsymbol{a},k}$)

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Outline

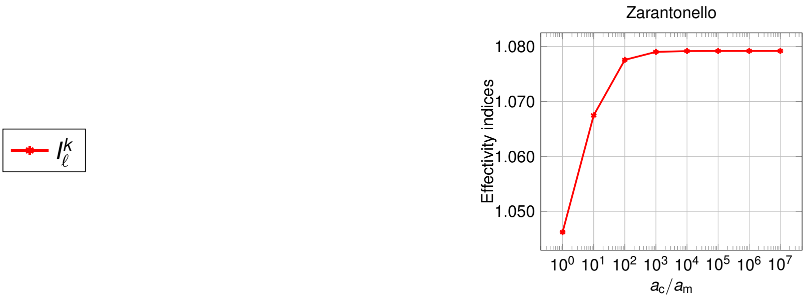
Smooth solution

Setting

- unit square $\Omega = (0,1)^2$
- known smooth solution $u(x,y) := 10\,x(x-1)y(y-1)$
- $p = 1$
- effectivity indices

$$\underbrace{I_\ell^k := \left(\frac{\eta_\ell^k}{\mathcal{E}_\ell^k}\right)^{\frac{1}{2}},}_{\text{total}} \quad \underbrace{I_{\mathrm{N},\ell}^k := \left(\frac{\mathcal{J}(u_\ell^k) - \mathcal{J}^*(\boldsymbol{\sigma}_\ell^k)}{\mathcal{J}(u_\ell^k) - \mathcal{J}(u)}\right)^{\frac{1}{2}}}_{\text{energy difference}}$$

How **large** is the error? **Robustness** wrt the nonlinearities $\left(a(r) = a_{\rm m} + \frac{a_{\rm c} - a_{\rm m}}{\sqrt{1+r^2}}\right)$



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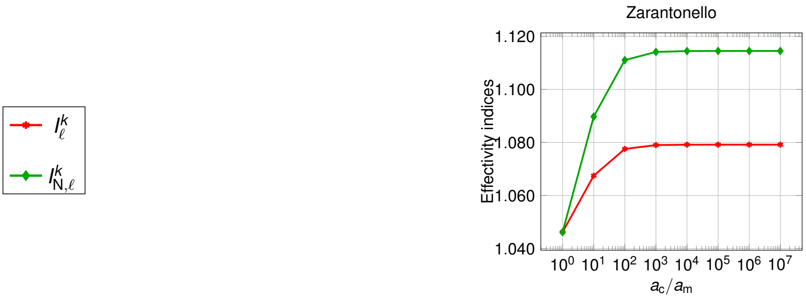

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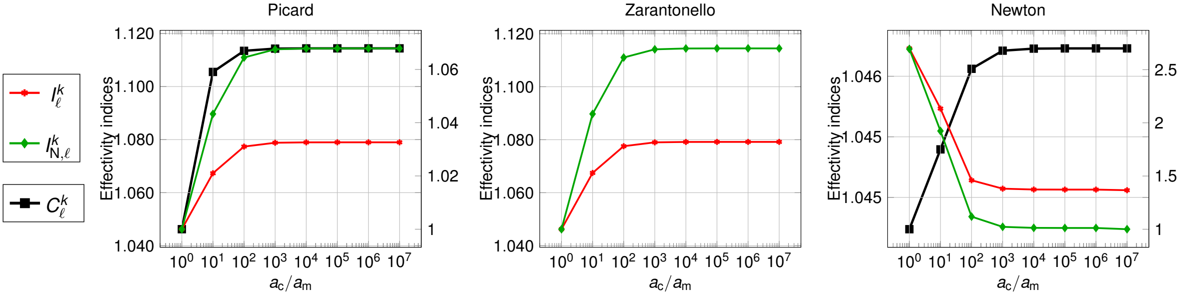
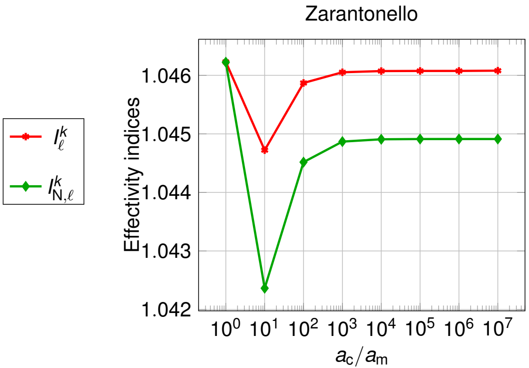


A. Harnist, K. Mitra, A. Rappaport, M. Vohralík, preprint (2023)

How **large** is the error? **Robustness** wrt the nonlinearities

$$\left(a(r) = a_{\mathrm{m}} + (a_{\mathrm{c}} - a_{\mathrm{m}})\frac{1-e^{-\frac{3}{2}r^2}}{1+2e^{-\frac{3}{2}}}\right)$$



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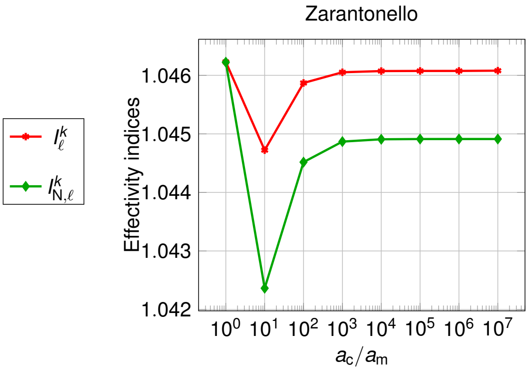
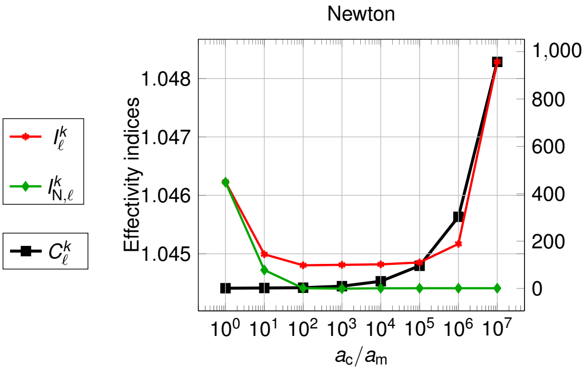
$(a(r) = a_{\mathrm{m}} + (a_{\mathrm{c}} - a_{\mathrm{m}})\frac{1-e^{-\frac{3}{2}r^2}}{1+2e^{-\frac{3}{2}}}$, robustness only for Zarantonello)





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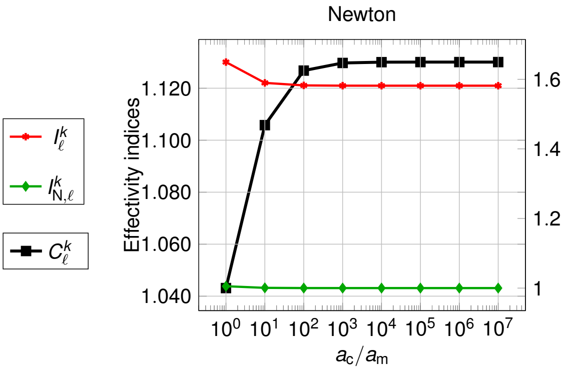
Singular solution

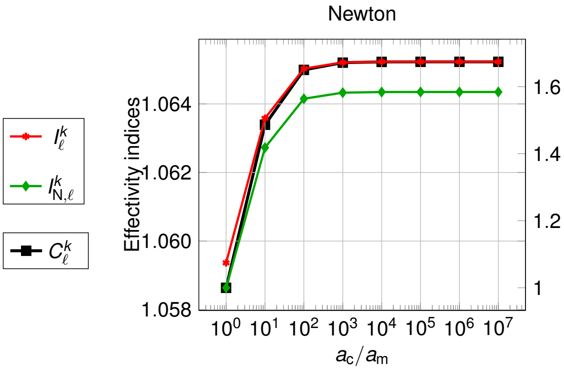
Setting

- L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$
- known singular solution $u(\rho,\theta) = \rho^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$
- $a(r) = a_\mathrm{m} + (a_\mathrm{c} - a_\mathrm{m}) \frac{1-e^{-\frac{3}{2}r^2}}{1+2e^{-\frac{3}{2}}}$
- $p = 1$
- uniform or adaptive mesh refinement

How **large** is the error? **Robustness** wrt the nonlinearities



Uniform mesh refinement



Adaptive mesh refinement

A. Harnist, K. Mitra, A. Rappaport, M. Vohralík, preprint (2023)

Outline

Conclusions

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- a posteriori **certification** of the **error** for nonlinear problems
- **robustness** with respect to the **strength of nonlinearities**
- employing iteration-dependent norms
- **orthogonal decomposition** based on iterative linearization
- augmenting the energy difference by the (discretization) error on the given linearization step

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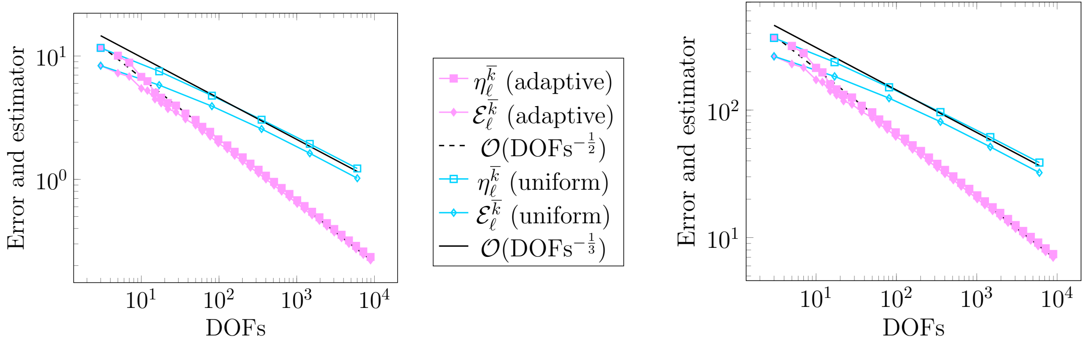
Thank you for your attention!

Outline

Decreasing the error efficiently: optimal decay rate wrt DoFs

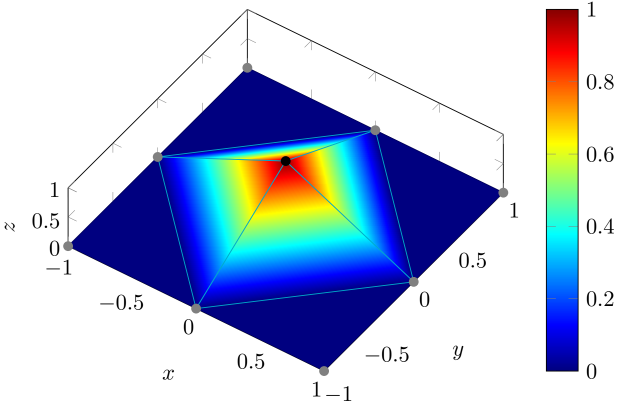


$\frac{a_c}{a_m} = 10^3$ $\frac{a_c}{a_m} = 10^6$

Outline

Partition of unity

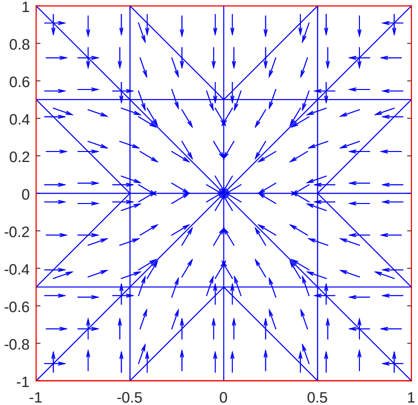
$$\sum_{\boldsymbol{a}\in\mathcal{V}_\ell} \psi^{\boldsymbol{a}} = 1$$



Hat basis function $\psi^{\boldsymbol{a}}$

Equilibrated flux reconstruction

Destuynder and Métivet (1998), Braess & Schöberl (2008), Ern & Vohralík (2013)



Flux $\boldsymbol{\iota}_\ell \notin \boldsymbol{H}(\mathrm{div})$ (e.g. FE flux $-\nabla u_\ell$)

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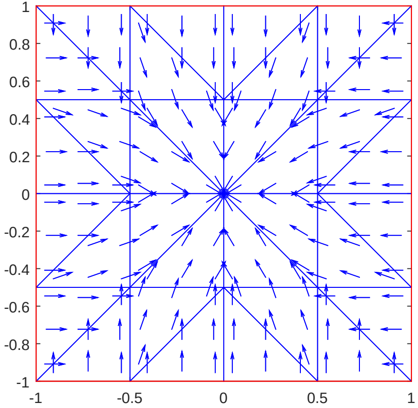

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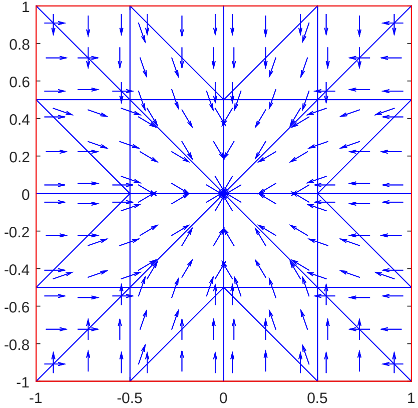

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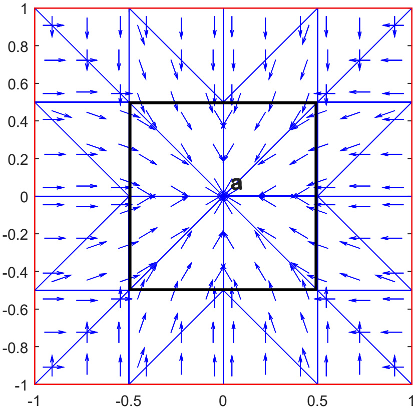

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$$\underbrace{\boldsymbol{\iota}_\ell \in \mathcal{RT}_p(\mathcal{T}_\ell),\, f \in \mathcal{P}_p(\mathcal{T}_\ell)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_\ell,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_\ell^{\mathrm{int}}}$$

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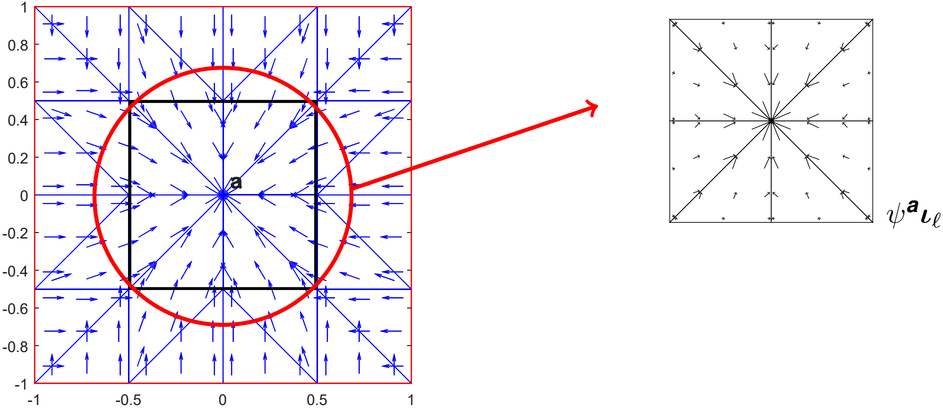

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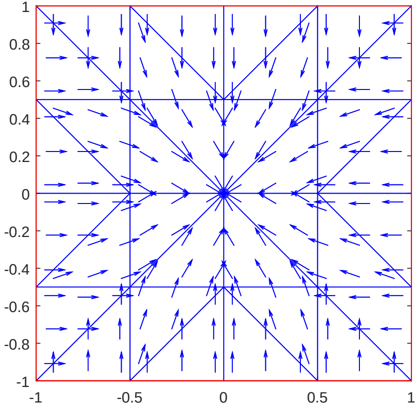

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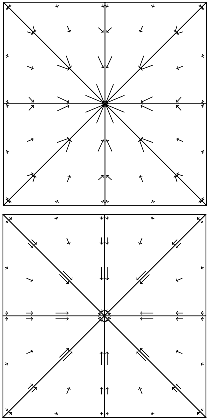

$\psi^{\boldsymbol{a}}\iota_\ell$

$\sigma_\ell^{\boldsymbol{a}}$

$$\underbrace{\iota_\ell \in \mathcal{RT}_p(\mathcal{T}_\ell),\, f \in \mathcal{P}_p(\mathcal{T}_\ell)} \qquad \sigma_\ell^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_\ell \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_\ell = f\psi^{\boldsymbol{a}} + \iota_\ell\cdot\nabla\psi^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\iota_\ell - \boldsymbol{v}_\ell\|^2_{\omega_{\boldsymbol{a}}}$$

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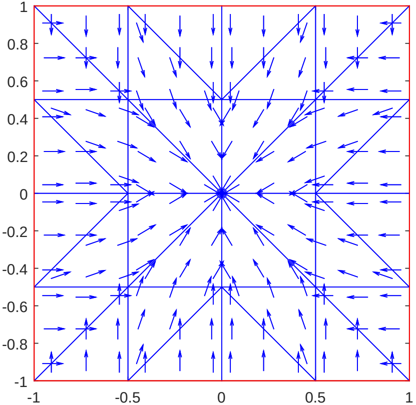

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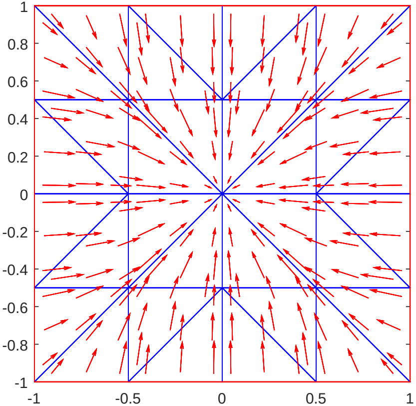

Equilibrated flux $\boldsymbol{\sigma}_\ell \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_\ell = f$

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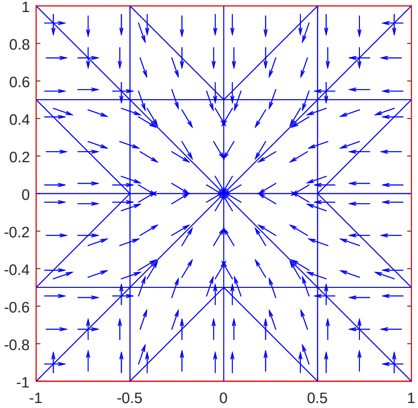

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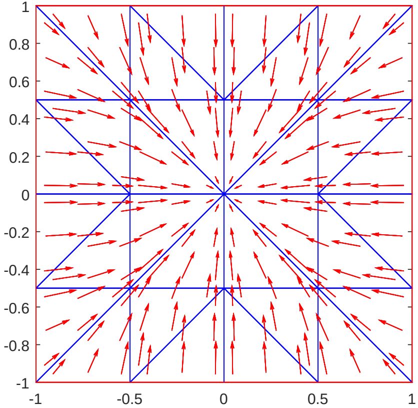

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