Guaranteed a posteriori bounds for eigenvalues and eigenvectors: multiplicities and clusters

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Outline

1. Introduction
2. Orthogonal projectors
3. Eigenvalue–eigenvector–residual equivalences
4. Applications to finite elements and planewaves
   - Finite elements for the Laplace operator
   - Planewaves for the Schrödinger operator
5. Numerical experimens
   - Finite elements for the Laplace operator
   - Planewaves for the Schrödinger operator
6. Conclusions and outlook

Eigenvalue problem

Setting

- $\mathcal{H}$: real separable Hilbert space, inner product $(\cdot, \cdot)$, norm $\|\cdot\|$
- A: linear self-adjoint operator on $\mathcal{H}$ with domain $D(A)$, bounded-below, with compact resolvent
- **eigenvalues** λ_k and **eigenvectors** $\varphi_k^0 \in D(A)$, $k \geq 1$, s.t.

$$A\varphi_k^0 = \lambda_k \varphi_k^0 \qquad \forall k \geq 1$$

- $\lambda_k \in \mathbb{R}_+$, $\lambda_k \to +\infty$, φ_k^0 form an orthonormal basis of $\mathcal{H}$

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Laplace operator on a polytope $\Omega \subset \mathbb{R}^d$ with hom. Dirichlet BCs

- $\mathcal{H} = L^2(\Omega)$, $A = -\Delta$, $D(A) = H_0^1(\Omega) \cap \{v | \, \Delta v \in L^2(\Omega)\}$

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Schrödinger operator on a cubic box $\Omega \subset \mathbb{R}^d$ with periodic BCs

- $\mathcal{H} = L^2_\#(\Omega)$, $A = -\Delta + V$, $D(A) = H^2_\#(\Omega)$

Weak form, numerical approximation, examples

Weak form

- find $(\varphi_k^0, \lambda_k) \in D(A^{1/2}) \times \mathbb{R}_+$, $(\varphi_k^0, \varphi_j^0) = \delta_{kj}$, $1 \leq k, j$, s.t.

$$(A^{1/2}\varphi_k^0, A^{1/2}v) = \lambda_k(\varphi_k^0, v) \qquad \forall v \in D(A^{1/2}), \forall k \geq 1$$

Conforming numerical approximation

- find $(\varphi_{kh}, \lambda_{kh}) \in V_h \subset D(A^{1/2}) \times \mathbb{R}_+$, $(\varphi_{kh}, \varphi_{jh}) = \delta_{kj}$, $1 \leq k, j \leq \dim V_h$, s.t.

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Laplace operator on a polytope $\Omega \subset \mathbb{R}^d$ with hom. Dirichlet BCs

- $D(A^{1/2}) = H_0^1(\Omega)$, $\|A^{1/2}v\| = \left(\int_\Omega |\nabla v|^2\right)^{1/2}$

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Schrödinger operator on a cubic box $\Omega \subset \mathbb{R}^d$ with periodic BCs

- $D(A^{1/2}) = H^1_\#(\Omega)$, $\|A^{1/2}v\| = \left(\int_\Omega (|\nabla v|^2 + V|v|^2)\right)^{1/2}$

Previous results, eigenvalue bounds

- Armentano and Durán (2004), Plum (1997), Goerisch and He (1989), Still (1988), Kuttler and Sigillito (1978), Moler and Payne (1968), Fox and Rheinboldt (1966), Bazley and Fox (1961), Weinberger (1956), Forsythe (1955), Kato (1949)
- ...

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- Carstensen and Gedicke (2014) & Liu (2015): ⊕ guaranteed bound, arbitrarily coarse mesh; ⊖ a priori arguments (largest mesh element diameter), only lowest-order nonconforming FEs
- Šebestová and Vejchodský (2014), Kuznetsov and Repin (2013): ⊕ general guaranteed bounds for any conforming discretization; ⊖ suboptimal convergence speed
- Liu and Oishi (2013): ⊕ guaranteed bound; ⊖ only lowest-order conforming FEs, auxiliary eigenvalue problem on nonconvex domains
- Wang, Chamoin, Ladevèze, Zhong (2016): ⊕ general bounds for any conforming discretization; ⊖ infinite-dimensional local problem (loss of the guaranteed bound)
- Cancès, Dusson, Maday, Stamm, Vohralík (2017, 2018): ⊕ general framework (planewaves, conforming FEs, nonconforming FEs, mixed FEs, DGs; any order; optimal convergence); ⊖ gap condition, simple eigenvalues

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Previous results, eigenvector bounds

- Boffi, Gallistl, Gardini, Gastaldi (2017), Boffi, Durán, Gardini, Gastaldi (2017), Bonito and Demlow (2016), Dai, He, Zhou (2015), Gallistl (2014), Carstensen and Gedicke (2011), Bank, Grubišić, Ovall (2013), Rannacher, Westenberger, Wollner (2010), Grubišić and Ovall (2009), Durán, Padra, Rodríguez (2003), Heuveline and Rannacher (2002), Larson (2000), Maday and Patera (2000), Verfürth (1994) . . .
- . . . typically contain uncomputable terms, higher-order on fine enough meshes

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Setting

Eigenvalue cluster

- J eigenvalues $(\lambda_m, \ldots, \lambda_M)$ (allowing for **multiplicities**)
- corresponding J eigenvectors $\Phi^0 := (\varphi_m^0, \ldots, \varphi_M^0)$

Approximate eigenvalue cluster

- $(\lambda_{mh}, \ldots, \lambda_{Mh})$ with $\dim V_h \geq M$, $\Phi_h := (\varphi_{mh}, \ldots, \varphi_{Mh})$

Assumption A (Continuous–discrete gap conditions)

$$\lambda_{m-1} \leq \lambda_{(m-1)h} < \underline{\lambda}_m \leq \lambda_m \leq \lambda_{mh}, \quad \textit{when } m > 1,$$

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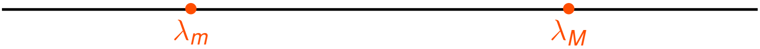

exact eigenvalues

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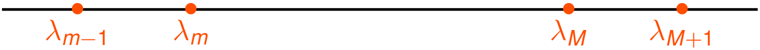

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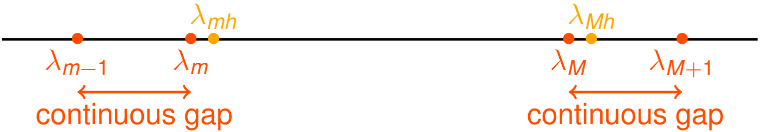

exact eigenvalues, appr. eigenvalues

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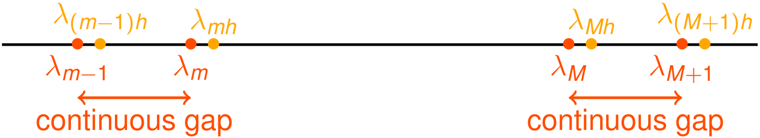

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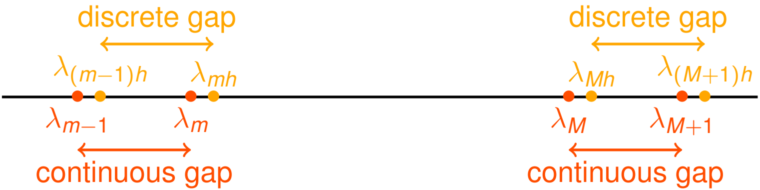

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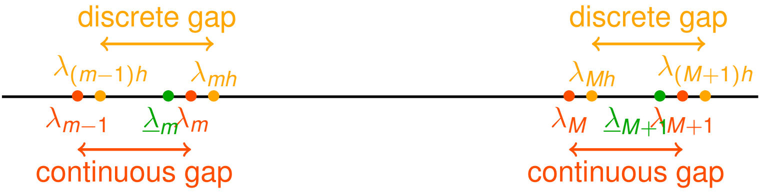

exact eigenvalues, appr. eigenvalues, guaranteed lower bounds

Main results

We bound

1. cluster **eigenvalue error**

$$0 \leq \sum_{i=m}^{M} (\lambda_{ih} - \lambda_i) \leq \eta^2$$

2. cluster **eigenvector energy error**

$$\|A^{1/2}(\gamma^0 - \gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})} \leq \eta \leq C_{\mathrm{eff}} \left(\|A^{1/2}(\gamma^0 - \gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})} + \text{h.o.t.}\right)$$

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- ✓ guaranteed and optimally converging bounds
- ✗ $C_{\rm eff}$ depends on the continuous–discrete relative gaps and on $\bar{c}_h := \max\left\{\left(\frac{\lambda_{Mh}}{\lambda_1} - 1\right)^2, 1\right\}$
- ✗ $C_{\rm eff}$ depends on the mesh shape regularity (FEs Laplace)
- ✓ $C_{\rm eff}$ independent of polynomial degree in V_h (FEs Laplace)

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✗ C_{eff} depends on the continuous–discrete relative gaps and on $\bar{c}_h := \max\left\{\left(\frac{\lambda_{Mh}}{\lambda_1} - 1\right)^2, 1\right\}$

✗ C_{eff} depends on the mesh shape regularity (FEs Laplace)

✓ C_{eff} independent of polynomial degree in V_h (FEs Laplace)

Main results

We bound

1. cluster **eigenvalue error**

$$0 \leq \sum_{i=m}^{M} (\lambda_{ih} - \lambda_i) \leq \eta^2$$

2. cluster **eigenvector energy error**

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Outline

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Continuous and discrete orthogonal projectors

Non-uniqueness issue

- multiple eigenvalues $\lambda_m = \ldots = \lambda_M$: for any orthogonal matrix $\mathbf{U} \in O(J) = \{\mathbf{U} \in \mathbb{R}^{J\times J};\ \mathbf{U}^T\mathbf{U} = \mathbf{1}_J\}$, $\Phi^0\mathbf{U}$ is **also** orthonormal set of eigenvectors for $(\lambda_m, \ldots, \lambda_M)$
- measure the errors in the **spaces** spanned by **eigenvectors**, uniquely determined even for multiple eigenvalues (under Assumption A)

Continuous orthogonal projector onto Span Φ^0

$$\forall v \in \mathcal{H}, \quad \gamma^0 v := \sum_{i=m}^{M} (v, \varphi_i^0)\varphi_i^0 \in D(A^{1/2})$$

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Unitary transformed approximate eigenvectors

Assumption B (Non-orthogonality of exact and approximate eigenspaces)

There holds

$$\forall v \in Span\{\varphi_m^0, \ldots, \varphi_M^0\} \backslash \{0\}, \quad \|\gamma_h v\| \neq 0.$$

Abstract unitary transformed approximate eigenvectors

- closest set of discrete eigenvectors

$$\Phi_h^0 := (\varphi_{mh}^0, \ldots, \varphi_{Mh}^0) := \mathrm{argmin}_{\mathbf{U} \in O(J)} \|\mathbf{U}\Phi_h - \Phi^0\|$$

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Equivalence between projection & eigenvector errors

Hilbert–Schmidt norm

$$\|B\|_{\mathfrak{S}_2(\mathcal{H})} := \left\{ \sum_{k\geq 1} \|Be_k\|^2 \right\}^{1/2}, \quad e_k \text{ arbitrary orthonormal basis of } \mathcal{H}$$

Lemma (Equivalence between projection & $\mathcal{H}$ / energy errors)

There holds

$$\frac{1}{\sqrt{2}} \|\gamma^0 - \gamma_h\|_{\mathfrak{S}_2(\mathcal{H})} \leq \|\Phi^0 - \Phi_h^0\| \leq \|\gamma^0 - \gamma_h\|_{\mathfrak{S}_2(\mathcal{H})}.$$

Moreover,

$$\begin{aligned} &\frac{1}{\sqrt{2}} \|A^{1/2}(\gamma^0 - \gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})} \\ &\leq \|A^{1/2}(\Phi^0 - \Phi_h^0)\| \\ &\leq \Big(1 + \frac{\lambda_M}{4\lambda_m} \underbrace{\|\gamma^0 - \gamma_h\|^2_{\mathfrak{S}_2(\mathcal{H})}}_{\leq 4J}\Big)^{1/2} \|A^{1/2}(\gamma^0 - \gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})}. \end{aligned}$$

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Outline

Eigenvalue–eigenvector equivalence

Theorem (Eigenvalue–eigenvector equivalence)

There holds

$$\begin{aligned}
&\|A^{1/2}(\gamma^0 - \gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})} - \lambda_M \|\gamma^0 - \gamma_h\|^2_{\mathfrak{S}_2(\mathcal{H})} \\
\leq &\sum_{i=m}^{M} (\lambda_{ih} - \lambda_i) \\
\leq &\|A^{1/2}(\gamma^0 - \gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})}.
\end{aligned}$$

Single eigenpair and cluster residuals

Single eigenpair residual $\mathrm{Res}(\varphi_{ih}, \lambda_{ih}) \in D(A^{1/2})'$

$$\langle \mathrm{Res}(\varphi_{ih}, \lambda_{ih}), v \rangle_{D(A^{1/2})', D(A^{1/2})} := \lambda_{ih}(\varphi_{ih}, v) - \left(A^{1/2}\varphi_{ih}, A^{1/2}v\right), \; v \in D(A^{1/2})$$

$$\|\mathrm{Res}(\varphi_{ih}, \lambda_{ih})\|_{D(A^{1/2})'} := \sup_{v \in D(A^{1/2}),\, \|A^{1/2}v\|=1} \langle \mathrm{Res}(\varphi_{ih}, \lambda_{ih}), v \rangle_{D(A^{1/2})', D(A^{1/2})}$$

Cluster residual $\mathrm{Res}(\gamma_h) \in \mathfrak{S}_2(\mathcal{H})$

$$\mathrm{Res}(\gamma_h) := A^{1/2}\gamma_h - A^{-1/2}\left(A^{1/2}\gamma_h\right)^{\dagger} A^{1/2}\gamma_h$$

Lemma (Equivalence of cluster and single eigenpair residuals)

There holds

$$\|\mathrm{Res}(\gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})} = \sum_{i=m}^{M} \|\mathrm{Res}(\varphi_{ih}, \lambda_{ih})\|^2_{D(A^{1/2})'}.$$

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Eigenvector–residual equivalence I

Theorem (Upper bound of the energy error by the residual)

There holds

$$\|A^{1/2}(\gamma^0-\gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})} \leq \|\mathrm{Res}(\gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})}+(\lambda_M+\lambda_{Mh})\|\gamma^0-\gamma_h\|^2_{\mathfrak{S}_2(\mathcal{H})}.$$

Set

$$c_h := \max\left[\left(\frac{\lambda_{mh}}{\lambda_{m-1}}-1\right)^{-1}, \left(1-\frac{\lambda_{Mh}}{\lambda_{M+1}}\right)^{-1}\right].$$

Then

$$\|A^{1/2}(\gamma^0-\gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})} \leq 2c_h^2\|\mathrm{Res}(\gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})} + \frac{\lambda_M}{2}\|\gamma^0-\gamma_h\|^4_{\mathfrak{S}_2(\mathcal{H})}.$$

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Eigenvector–residual equivalence II

Theorem (Lower bound of the energy error by the residual)

Set

$$\bar{c}_h := \max\left\{\left(\frac{\lambda_{Mh}}{\lambda_1} - 1\right)^2, 1\right\}.$$

Then

$$\begin{aligned}
&\|\mathrm{Res}(\gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})}\\
\leq\ &\bar{c}_h\|A^{1/2}(\gamma^0-\gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})} + \frac{3(\lambda_M)^2}{4\lambda_m}\|\gamma^0-\gamma_h\|^4_{\mathfrak{S}_2(\mathcal{H})}\\
&+\frac{3}{\lambda_m}\left(1+\frac{1}{4}\|\gamma^0-\gamma_h\|^4_{\mathfrak{S}_2(\mathcal{H})}\right)\times\\
&\left[2\left(1+\frac{\lambda_M}{4\lambda_m}\|\gamma^0-\gamma_h\|^2_{\mathfrak{S}_2(\mathcal{H})}\right)^2\|A^{1/2}(\gamma^0-\gamma_h)\|^4_{\mathfrak{S}_2(\mathcal{H})}\right.\\
&\left.+2(\lambda_M)^2\|\gamma^0-\gamma_h\|^4_{\mathfrak{S}_2(\mathcal{H})}\right].
\end{aligned}$$

Upper bounds for the projection $\mathcal{H}$ error

Lemma (Upper bounds for the $\mathcal{H}$ error)

Set

$$\tilde{c}_h := \max\left[(\underline{\lambda}_{m-1})^{-1/2}\left(\frac{\lambda_{mh}}{\underline{\lambda}_{m-1}}-1\right)^{-1}, (\overline{\lambda}_{M+1})^{-1/2}\left(1-\frac{\lambda_{Mh}}{\underline{\lambda}_{M+1}}\right)^{-1}\right].$$

Then

$$\|\gamma^0-\gamma_h\|_{\mathfrak{S}_2(\mathcal{H})} \leq \sqrt{2}c_h\|A^{-1/2}\mathrm{Res}(\gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})}$$

and

$$\|\gamma^0-\gamma_h\|_{\mathfrak{S}_2(\mathcal{H})} \leq \sqrt{2}\tilde{c}_h\|\mathrm{Res}(\gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})}.$$

Outline

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Eigenvalues error

Theorem (Guaranteed bounds for the sum of eigenvalues)

Let $\eta_{\rm res}^2 := \sum_{i=m}^{M} \|\nabla \varphi_{ih} + \boldsymbol{\sigma}_{ih}\|^2$, $\boldsymbol{\sigma}_{ih}$ = *equilibrated fluxes. Then*

$$\boxed{0 \leq \sum_{i=m}^{M} (\lambda_{ih} - \lambda_i) \leq \eta^2.}$$

Case I *There always holds*

$$\eta^2 := (2c_h^2 + 2\lambda_{Mh}\tilde{c}_h^4 \eta_{\rm res}^2)\eta_{\rm res}^2.$$

Case II *Assume that for* $i = m, \ldots, M$, *the solutions* $\zeta_{(ih)}$ *of the residual source problems belong to* $H^{1+\delta}(\Omega)$, $0 < \delta \leq 1$, *so that*

$$\min_{v_h \in V_h} \|\nabla(\zeta_{(ih)} - v_h)\| \leq C_{\rm I} h^\delta |\zeta_{(ih)}|_{H^{1+\delta}(\Omega)},$$

$$|\zeta_{(ih)}|_{H^{1+\delta}(\Omega)} \leq C_{\rm S} \|r_{(ih)}\|.$$

Then

$$\eta^2 := (1 + 4\lambda_{Mh} c_h^2 C_{\rm I}^2 C_{\rm S}^2 h^{2\delta})\eta_{\rm res}^2.$$

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Eigenvectors error, efficiency, and robustness

Theorem (Guaranteed bounds for the projection energy error)

Let the assumptions of the previous theorem be verified. Then the projection energy error can be bounded via

$$\boxed{\||\nabla|(\gamma^0 - \gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})} \leq \eta.}$$

There also holds

$$\eta_{\mathrm{res}}^2 \leq (d+1)^2 C_{\mathrm{st}}^2 C_{\mathrm{cont,PF}}^2 \bar{c}_h \||\nabla|(\gamma^0 - \gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})}^2 + \textit{ h.o.t.}$$

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$$\boxed{\| |\nabla|(\gamma^0 - \gamma_h) \|_{\mathfrak{S}_2(\mathcal{H})} \leq \eta.}$$

There also holds

$$\eta_{\mathrm{res}}^2 \leq (d+1)^2 C_{\mathrm{st}}^2 C_{\mathrm{cont,PF}}^2 \bar{c}_h \| |\nabla|(\gamma^0 - \gamma_h) \|_{\mathfrak{S}_2(\mathcal{H})}^2 + \ h.o.t.$$

Outline

Eigenvalues error

Theorem (Guaranteed bounds, efficiency, and robustness)

Define

$$\eta_{\mathrm{res}}^2 := \sum_{i=m}^{M} \|\mathrm{Res}(\varphi_{iN}, \lambda_{iN})\|^2_{H_\#^{-1}(\Omega)} \qquad \textit{(computable)}.$$

Then

$$0 \leq \sum_{i=m}^{M} (\lambda_{ih} - \lambda_i) \leq \eta^2,$$

where

$$\eta^2 := \left(1 + \frac{1}{N^2} \frac{L^2 \lambda_{MN}}{\pi^2} c_N^2\right) \eta_{\mathrm{res}}^2.$$

Moreover,

$$\|(-\Delta + V)^{1/2}(\gamma^0 - \gamma_h)\|_{\mathfrak{S}_2(\mathcal{H})} \leq \eta.$$

Finally,

$$\eta_{\mathrm{res}}^2 \leq (\sup_{\Omega} V)\bar{c}_N \|(-\Delta + V)^{1/2}(\gamma^0 - \gamma_h)\|^2_{\mathfrak{S}_2(\mathcal{H})} + \textit{h.o.t.}$$

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Outline

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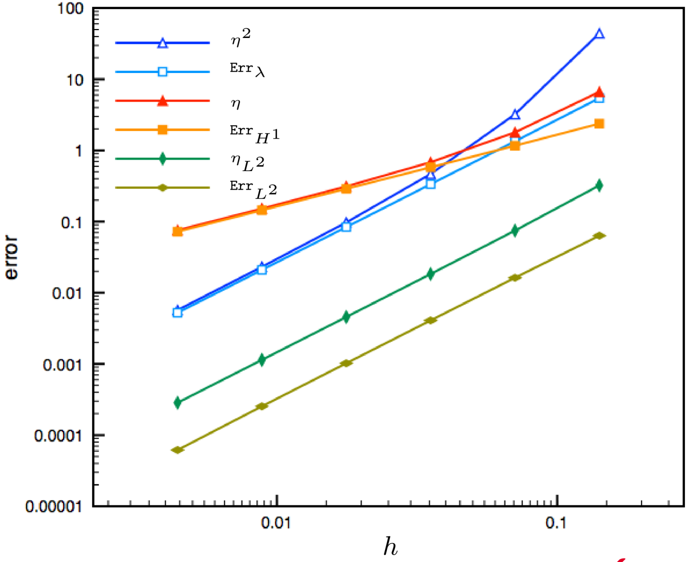
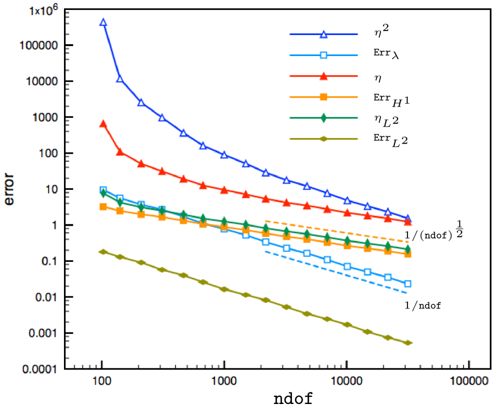
Finite elements, unit square, Case II, $m = 2$, $M = 3$

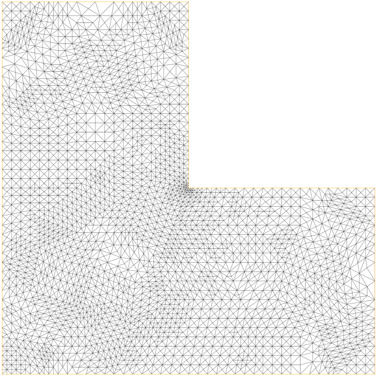


Finite elements, unit square, Case II, $m = 2$, $M = 4, 8$

	N	h	ndof	$\mathtt{Err}_\lambda$	η^2	I_λ^{eff}	$\mathtt{Err}_{H^1}$	η	$I_{H^1}^{\mathrm{eff}}$	$\mathtt{Err}_{L^2}$	η_{L^2}	$I_{L^2}^{\mathrm{eff}}$
$m = 1$	10	0.1414	121	13.5049	21673.5051	1604.86	4.1325	147.2192	35.63	0.2141	1.7415	8.13
$M = 4$	20	0.0707	441	3.4018	98.8430	29.06	1.9076	9.9420	5.21	0.0554	0.2274	4.10
$\mathcal{T}_{H,1}$	40	0.0354	1681	0.8519	5.0687	5.95	0.9297	2.2514	2.42	0.0139	0.0521	3.75
	80	0.0177	6561	0.2131	0.4708	2.21	0.4619	0.6862	1.49	0.0035	0.0128	3.67
	160	0.0088	25921	0.0533	0.0728	1.37	0.2306	0.2698	1.17	0.0009	0.0032	3.67
	320	0.0044	103041	0.0133	0.0155	1.16	0.1152	0.1243	1.08	0.0002	0.0008	3.71
$m = 1$	10	0.1414	121	72.9222	82403.2050	1130.02	9.3347	287.0596	30.75	0.3359	3.2521	9.68
$M = 8$	20	0.0707	441	18.0492	281.4040	15.59	4.3588	16.7751	3.85	0.0874	0.3923	4.49
$\mathcal{T}_{H,2}$	40	0.0354	1681	4.4994	15.9735	3.55	2.1323	3.9967	1.87	0.0221	0.0893	4.04
	80	0.0177	6561	1.1240	1.8566	1.65	1.0603	1.3626	1.29	0.0055	0.0219	3.94
	160	0.0088	25921	0.2810	0.3445	1.23	0.5294	0.5869	1.11	0.0014	0.0054	3.94
	320	0.0044	103041	0.0702	0.0788	1.12	0.2646	0.2808	1.06	0.0003	0.0014	4.00

Finite elements, L-shaped domain, Case I, $m = 3$, $M = 5$





Outline

Planewaves, $\Omega = (0, 2\pi) \times (0, 2\pi)$, various clusters

	N	ndof	Err_λ	η^2	I_λ^{eff}	Err_{H^1}	η	$I_{H^1}^{\mathrm{eff}}$	Err_{L^2}	η_{L^2}	$I_{L^2}^{\mathrm{eff}}$
$m = 1$	5	121	2.62e-05	2.02e-04	7.70	5.32e-03	1.42e-02	2.67	9.94e-04	6.18e-03	6.22
$M = 5$	15	961	4.12e-07	7.31e-07	1.77	6.45e-04	8.55e-04	1.32	4.47e-05	2.62e-04	5.85
	25	2601	5.32e-08	7.22e-08	1.36	2.31e-04	2.69e-04	1.16	9.99e-06	5.80e-05	5.81
$m = 6$	5	121	5.12e-05	2.80e-04	5.47	7.60e-03	1.67e-02	2.20	1.41e-03	5.90e-03	4.17
$M = 9$	15	961	7.51e-07	1.15e-06	1.53	8.73e-04	1.07e-03	1.23	6.05e-05	2.43e-04	4.02
	25	2601	9.63e-08	1.22e-07	1.26	3.11e-04	3.49e-04	1.12	1.35e-05	5.38e-05	4.00
$m = 10$	5	121	3.81e-05	1.79e-03	46.9	6.83e-03	4.23e-02	6.19	1.28e-03	1.30e-02	10.1
$M = 13$	15	961	4.47e-07	2.93e-06	6.55	6.77e-04	1.71e-03	2.53	4.69e-05	4.87e-04	10.4
	25	2601	5.64e-08	1.80e-07	3.18	2.39e-04	4.24e-04	1.78	1.03e-05	1.07e-04	10.4

Outline

Conclusions and outlook

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- general framework based on projection operators
- allows to deal with possible degeneracies or near-degeneracies
- gap between the considered eigenvalues and the rest of the spectrum needed

Outlook

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References

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Thank you for your attention!

Operators A^s, $s \geq 0$

- domain & norm

$$D(A^s) := \left\{ v \in \mathcal{H}; \quad \|A^s v\|^2 := \sum_{k \geq 1} \lambda_k^{2s} |(v, \varphi_k^0)|^2 < +\infty \right\}$$

- expression

$$A^s : v \in D(A^s) \mapsto \sum_{k \geq 1} \lambda_k^s (v, \varphi_k^0) \varphi_k^0 \in \mathcal{H}$$

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