Equivalence between mixed finite element and multi-point finite volume methods ⋆

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Abstract

We consider the lowest-order Raviart–Thomas mixed finite element method for elliptic problems on simplicial meshes in two or three space dimensions. This method produces saddle-point type problems for scalar and flux unknowns. We show how to easily eliminate the flux unknowns, which implies an equivalence between this method and a particular multi-point finite volume scheme, without any approximate numerical integration. We describe the stencil of the final matrix and give sufficient conditions for its symmetry and positive definiteness. We present a numerical example illustrating the performance of the proposed method. *To cite this article: M. Vohralík, C. R. Math. Acad. Sci. Paris (2004).*

Résumé

Equivalence entre les méthodes des éléments finis mixtes et des volumes finis à plusieurs points
Nous considérons la méthode des éléments finis mixtes de Raviart–Thomas de plus bas degré pour des problèmes elliptiques sur les maillages composés de triangles en dimension deux d'espace et de tétraèdres en dimension trois d'espace. Cette méthode aboutit à des problèmes de type point-selle pour les inconnues scalaires et les flux. Nous montrons comment facilement éliminer les flux, ce qui implique l'équivalence entre cette méthode et une méthode de type volumes finis à plusieurs points et ceci sans aucune intégration numérique approchée. Nous décrivons le nombre maximal des éléments non nuls sur chaque ligne de la matrice finale et présentons les conditions suffisantes pour qu'elle soit symétrique et définie positive. Nous présentons un essai numérique montrant la performance de la méthode proposée. *Pour citer cet article : M. Vohralík, C. R. Math. Acad. Sci. Paris (2004).*

Let us consider the elliptic problem

$$\mathbf{u} = -\mathbf{D}\nabla p \quad \text{in } \Omega\,, \tag{1a}$$

$$\nabla \cdot \mathbf{u} = q \quad \text{in } \Omega\,, \tag{1b}$$

$$p = p_D \quad \text{on } \partial\Omega\,, \tag{1c}$$

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where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain (open, bounded, and connected set), $\mathbf{D}$ is a bounded and uniformly positive definite tensor, $p_D \in H^{\frac{1}{2}}(\partial\Omega)$, and the source term q fulfills $q \in L_2(\Omega)$. Inhomogeneous Neumann or Robin boundary conditions can also be considered.

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω (consisting of triangles if $d = 2$ and of tetrahedra if $d = 3$). The approximation of the problem (1a)–(1c) by means of the mixed finite element method consists in finding $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in \Phi_h$ such that (see [3])

$$(\mathbf{D}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega = -\langle \mathbf{v}_h \cdot \mathbf{n}, p_D \rangle_{\partial\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h\,, \tag{2a}$$

$$-(\nabla \cdot \mathbf{u}_h, \phi_h)_\Omega = -(q, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi_h\,. \tag{2b}$$

Here, $\mathbf{V}_h$ and Φ_h are suitable finite-dimensional spaces defined on $\mathcal{T}_h$. The associated matrix problem is saddle-point when $\mathbf{D}$ is symmetric. It can be written in the form

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{3}$$

In the lowest-order Raviart–Thomas method [5] and its three-dimensional Nédélec variant [4], the scalar unknowns P are associated with the elements of $\mathcal{T}_h$ and U are the fluxes through the sides (edges if $d = 2$, faces if $d = 3$) of $\mathcal{T}_h$. Using the hybridization technique, one can decrease the number of unknowns to Lagrange multipliers associated with the sides and obtain a symmetric (when $\mathbf{D}$ is) and positive definite matrix, cf. [1]. The use of approximate numerical integration allows for the elimination of the fluxes, cf. e.g. [2]. Finally, the lowest-order Raviart–Thomas method can be in two space dimensions rewritten with one (new) unknown per element, see [7].

We show in this paper a new method which permits to efficiently reduce the system (3) onto a system for the (original) scalar unknowns P only. It shows that in the lowest-order Raviart–Thomas mixed finite element method, one can express, solving only local problems, the flux through each side using the scalar unknowns, sources, and possibly boundary conditions associated with the elements in a neighborhood of this side. This method is thus equivalent to a particular multi-point finite volume scheme, and this without any numerical integration. We describe the stencil of the final matrix and give sufficient conditions for its symmetry and positive definiteness. The numerical example at the end of this paper confirms considerable computational savings while using the proposed method. Finally, this approach seems to easily extend to nonlinear parabolic convection–reaction–diffusion problems and to higher-order schemes.

1. The elimination process

Let us denote the set of sides by $\mathcal{E}_h$. Let us consider simplices $K, L \in \mathcal{T}_h$ sharing an interior side σ. Let V_K be the vertex of K opposite to σ and V_L the vertex of L opposite to σ. A basis function $\mathbf{v}_\sigma \in \mathbf{V}_h$ associated with the side σ can be written in the form $\mathbf{v}_\sigma(\mathbf{x}) = \frac{1}{d|K|}(\mathbf{x} - V_K)$, $\mathbf{x} \in K$, $\mathbf{v}_\sigma(\mathbf{x}) = \frac{1}{d|L|}(V_L - \mathbf{x})$, $\mathbf{x} \in L$, $\mathbf{v}_\sigma(\mathbf{x}) = [0]^d$ otherwise. Here $|K|$ is the volume of the element K. We fix its orientation, i.e. the order of K and L. For a boundary side σ, the support of $\mathbf{v}_\sigma$ only consists of $K \in \mathcal{T}_h$ such that $\sigma \subset \partial K$. A basis function $\phi_K \in \Phi_h$ associated with an element $K \in \mathcal{T}_h$ is equal to 1 on K and to 0 otherwise.

Let us denote by $\mathcal{V}_h$ the set of all vertices and consider $V \in \mathcal{V}_h$. We call the set of all elements of $\mathcal{T}_h$ sharing this vertex a *cluster* associated with V and denote it by $\mathcal{C}_V$. Let us denote by $\mathcal{E}_V$ the set of all sides of $\mathcal{C}_V$, by $\mathcal{F}_V$ the set of all the sides sharing V, and by $\mathcal{G}_V$ the set of the other sides of $\mathcal{C}_V$. We have $\mathcal{E}_V = \mathcal{F}_V \cup \mathcal{G}_V$, $\mathcal{F}_V \cap \mathcal{G}_V = \emptyset$. Let us now consider the equations (2a) for the basis functions $\mathbf{v}_\gamma$, $\gamma \in \mathcal{F}_V$. We remark that the support of all $\mathbf{v}_\gamma$, $\gamma \in \mathcal{F}_V$, is included in $\mathcal{C}_V$ and that $\mathbf{u}_h|_{\mathcal{C}_V} = \sum_{\sigma \in \mathcal{E}_V} U_\sigma \mathbf{v}_\sigma$. This leads, using also that $p_h|_K = P_K$ and denoting $\mathbf{D}^{-t} = (\mathbf{D}^{-1})^t$,

$$\sum_{\sigma\in\mathcal{E}_V} U_\sigma(\mathbf{v}_\sigma, \mathbf{D}^{-t}\mathbf{v}_\gamma)_{\mathcal{C}_V} - \sum_{K\in\mathcal{C}_V} P_K(\nabla\cdot\mathbf{v}_\gamma, 1)_K = -\langle \mathbf{v}_\gamma\cdot\mathbf{n}, p_D\rangle_{\partial\Omega} \quad \forall\,\gamma\in\mathcal{F}_V\,, \tag{4}$$

i.e. $|\mathcal{F}_V| = \mathrm{card}(\mathcal{F}_V)$ equations. We now notice that the cluster is constructed so that $|\mathcal{C}_V| = |\mathcal{G}_V|$, and hence we can consider the equations (2b) for all ϕ_K, $K\in\mathcal{C}_V$, which gives

$$-\sum_{\sigma\in\mathcal{E}_K} U_\sigma(\nabla\cdot\mathbf{v}_\sigma, 1)_K = -(q,1)_K \qquad \forall K\in\mathcal{C}_V\,, \tag{5}$$

where $\mathcal{E}_K$ stands for the sides of the element K. The matrix problem associated with (4)–(5) reads

$$\begin{pmatrix} \mathbb{A}_V & \mathbb{C}_V \\ \mathbb{D}_V & \mathbb{I}_V \end{pmatrix} \begin{pmatrix} U_V^{\mathcal{F}} \\ U_V^{\mathcal{G}} \end{pmatrix} = \begin{pmatrix} -\mathbb{B}_V^t P_V + F_V \\ G_V \end{pmatrix}, \tag{6}$$

where $U_V^{\mathcal{F}} = \{U_\sigma\}_{\sigma\in\mathcal{F}_V}$, $U_V^{\mathcal{G}} = \{U_\sigma\}_{\sigma\in\mathcal{G}_V}$, and $P_V = \{P_K\}_{K\in\mathcal{C}_V}$. The identity matrix $\mathbb{I}_V$ comes from the fact that, for $\sigma\in\mathcal{G}_V$, there is only one $K\in\mathcal{C}_V$ such that $\sigma\in\mathcal{E}_K$, and using that $(\nabla\cdot\mathbf{v}_\sigma, 1)_K = \pm 1$. We have multiplied the equation for $K\in\mathcal{C}_V$ by -1 whenever it was necessary.

Considering the second equation of (6), we have

$$(\mathbb{A}_V - \mathbb{C}_V\mathbb{D}_V)U_V^{\mathcal{F}} = -\mathbb{B}_V^t P_V + F_V - \mathbb{C}_V G_V \tag{7}$$

for each vertex $V\in\mathcal{V}_h$. Let us call the matrix $\mathbb{M}_V = \mathbb{A}_V - \mathbb{C}_V\mathbb{D}_V$ a *local condensation matrix* associated with V. It is clear that it now suffices to invert $\mathbb{M}_V$ for each $V\in\mathcal{V}_h$ to obtain the flux unknowns as functions of the scalar unknowns, sources, and boundary conditions and to insert this expression into the second equation of (3) to obtain a system for the scalar unknowns only. It appears that in some particular cases, the matrix $\mathbb{M}_V$ is not invertible. The approaches how to modify the proposed technique in order to overcome this difficulty, which resembles the presence of "singular" triangles in the method of [7], are studied in [6]. If all $\mathbb{M}_V$ are invertible, we can associate weights α_σ^i, $1\le i\le d$, $\sum_{i=1}^d \alpha_\sigma^i = 1$, with each $\sigma\in\mathcal{E}_h$ and multiply the expression for U_σ from $\mathcal{C}_{V_i}$ by α_σ^i for the d clusters $\mathcal{C}_{V_i}$ such that $\sigma\in\mathcal{F}_{V_i}$. We finally obtain

$$U = \tilde{\mathbb{A}}^{-1}(-\mathbb{B}^t P + F) + \mathbb{J}G$$

and

$$-\mathbb{B}\tilde{\mathbb{A}}^{-1}\mathbb{B}^t P = G - \mathbb{B}\tilde{\mathbb{A}}^{-1}F - \mathbb{B}\mathbb{J}G\,.$$

We have the following results. We refer to [6] for the proofs.

Theorem 1.1 *Let $\mathbb{M}_V$ be invertible for all $V\in\mathcal{V}_h$. Then on a row of the final matrix $\mathbb{B}\tilde{\mathbb{A}}^{-1}\mathbb{B}^t$ corresponding to an element $K\in\mathcal{T}_h$, the only possible nonzero entries are on columns corresponding to $L\in\mathcal{T}_h$ such that K and L share a common vertex.*

The assertion of this theorem follows from the fact that by (7), the flux across a side σ is expressed only using the scalar unknowns of the elements $K\in\mathcal{T}_h$ such that K and σ share a common vertex.

Theorem 1.2 *Let $\mathbb{M}_V$ be positive definite for all $V\in\mathcal{V}_h$. Then with the choice of the weights $\alpha_\sigma^i = 1/d$, $1\le i\le d$, $\sigma\in\mathcal{E}_h$, the final matrix $\mathbb{B}\tilde{\mathbb{A}}^{-1}\mathbb{B}^t$ is also positive definite.*

A simple sufficient condition for $\mathbb{M}_V$ for all $V\in\mathcal{V}_h$ to be positive definite, for $d=2$, is that

$$2(\mathbf{v}_1 - \mathbf{v}_3, \mathbf{D}^{-t}\mathbf{v}_1)_K > |(\mathbf{v}_2 - \mathbf{v}_3, \mathbf{D}^{-t}\mathbf{v}_1)_K + (\mathbf{v}_1 - \mathbf{v}_3, \mathbf{D}^{-t}\mathbf{v}_2)_K|$$

for all possible ordering of basis functions $\mathbf{v}_i$ associated with the edges of K and oriented outward from K, for all $K\in\mathcal{T}_h$. The right hand side of this inequality equals to zero when $\mathbf{D}|_K$ is constant and scalar and when K is equilateral and grows with deforming K. Other (less restrictive) conditions for $d=2,3$ are given in [6]. The condition for the positive definiteness may allow angles much greater than $\pi/2$.

Theorem 1.3 *Let $\mathbb{M}_V^{-1}$ be symmetric for all $V\in\mathcal{V}_h$. Then with the choice of the weights $\alpha_\sigma^i = 1/d$, $1\le i\le d$, $\sigma\in\mathcal{E}_h$, the final matrix $\mathbb{B}\tilde{\mathbb{A}}^{-1}\mathbb{B}^t$ is also symmetric.*

One can check that $\mathbb{M}_V^{-1}$ are symmetric for equilateral simplices and $\mathbf{D}$ piecewise constant and scalar.

2. Numerical example

Let us consider $\Omega = (0,1) \times (0,1)$, $\mathbf{D} = Id$, $q = -2e^x e^y$, and p_D given by the solution $p(x,y) = e^x e^y$. We perform the computations, using a notebook with Intel Pentium 4-M 1.8 GHz processor, on an unstructured triangular mesh of Ω which we refine regularly. We consider the method proposed in this paper and the hybridization onto Lagrange multipliers. In both cases the system matrices are positive definite but they are symmetric only in the latter case. We compare the number of unknowns, the system matrices condition number, and the CPU time and the number of iterations of the Bi-CGStab method to solve the associated matrix problems. For the hybridization, we consider also the CG method.

Refinements	Unknowns	Cond. no.	Bi-CGStab (sec.)	No. iter.
3	1024	721	0.20	76.5
4	4096	2882	1.43	147.5
5	16384	11523	12.55	295.5
6	65536	46093	117.58	555.5

Table 1
Elimination onto scalar unknowns associated with triangles
Elimination sur les inconnues scalaires associées aux triangles

Refinements	Unknowns	Cond. no.	Bi-CGStab (sec.)	No. iter.	CG (sec.)	No. iter.
3	1504	1397	0.31	118.0	0.22	157
4	6080	5616	2.43	230.5	1.75	316
5	24448	22499	23.40	449.5	16.87	623
6	98048	89995	227.04	864.0	162.09	1226

Table 2
Hybridization onto Lagrange multipliers associated with edges
Hybridisation sur les multiplicateurs de Lagrange associés aux arêtes

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